積分の計算例

有理関数の積分

多項式も含む

$\dfrac{\text{多項式}}{\text{多項式}}$ の形の関数を有理関数

という　有理関数はすべて具体的に積分の計算できる!

$= \displaystyle\int \frac{dx}{x^2-1}$ と書くこともある

例1　$\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x+1)(x-1)}\,dx$

$= (*)$

$\dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1}$ と書けないか?

↑部分分数展開
Partial fraction decomposition

$= \dfrac{1}{2}\left(\dfrac{-1}{x+1} + \dfrac{1}{x-1} \right)$ とできる!

$(*) = \dfrac{1}{2}\left(\displaystyle\int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx \right)$

$\log|x|' = \dfrac{1}{x}$

$= \dfrac{1}{2}\left(\log|x-1| - \log|x+1| \right) + C$

$= \dfrac{1}{2}\log\left| \dfrac{x-1}{x+1} \right| + C$

$c\log a + d\log b = \log(a^c \cdot b^d)$

$\log a - \log b = \log \dfrac{a}{b}$

$e^{\text{左辺}} = e^{(c\log a + d\log b)} = e^{c\log a}\, e^{d\log b}$

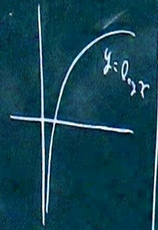
$= (e^{\log a})^c (e^{\log b})^d$

$= a^c\, b^d$

$e^{\text{右辺}} = e^{\log(a^c \cdot b^d)} = a^c \cdot b^d$

関数 $x \mapsto e^x$ は 1-1 だから、

$c\log a + d\log b = \log(a^c \cdot b^d)$



$\log|x| = \begin{cases} \log x & x>0 \text{ のとき} \\ \log(-x) & x<0 \text{ のとき} \end{cases}$

$(\log|x|)' = \begin{cases} \dfrac{1}{x} = \dfrac{1}{|x|} & x>0 \text{ のとき} \\ (\log(-x))' = -\dfrac{1}{-x} = \dfrac{1}{x} & x<0 \text{ のとき} \end{cases}$

$(\log|x|)' = \dfrac{1}{x}$

例12

$$\int \frac{x^3 - x + 4}{x^2 - 3x + 2}\,dx = \int \left(x+3 + \frac{6x-2}{x^2-3x+2} \right) dx = (*)$$

$$\begin{array}{r}
x+3 \\
x^2-3x+2 \,\big)\, \overline{x^3 \quad \boxed{} - x + 4} \\
x^3 - 3x^2 + 2x \\
\hline
3x^2 - 3x + 4 \\
3x^2 - 9x + 6 \\
\hline
6x - 2
\end{array}$$

$$\frac{6x-2}{x^2-3x+2} = \frac{6x-2}{(x-2)(x-1)}$$

$$= \frac{A}{x-2} + \frac{B}{x-1}$$ とおき A, B を求める

$$= \frac{(x-1)A + (x-2)B}{(x-2)(x-1)}$$ $x(A+B) - (A+2B) = 6x-2$

$A+B=6$ $x=1$ とおくと $-B=4$

$A=10$

$$= \frac{10}{x-2} - \frac{4}{x-1}$$

$$(*) = \frac{1}{2}x^2 + 3x + 10\log|x-2| - 4\log|x-1| + C$$

$$= \frac{1}{2}x^2 + 3x + \log\left|\frac{(x-2)^{10}}{(x-1)^4}\right| + C$$

$$\boxed{\begin{aligned} e^{(c\log a + d\log b)} &= e^{c\log a}\, e^{d\log b} \\ &= (e^{\log a})^c (e^{\log b})^d \end{aligned}}$$

例13 $$\int \frac{x-2}{(x-1)^2}\,dx = \int \frac{(x-1)-1}{(x-1)^2}\,dx$$

$$= \int \frac{\cancel{x-1}}{(x-1)^{\cancel{2}}}\,dx - \int \frac{1}{(x-1)^2}\,dx$$

$$= \log|x-1| - -\frac{1}{x-1} + C$$

$$= \log|x-1| + \frac{1}{x-1} + C$$

例4

$$\int \frac{x}{x^2-2x+2}\,dx$$ ← 実数のはんいでは因数分解できない

$$= \int \frac{x}{(x-1)^2+1}\,dx$$

$$= \int \frac{(x-1)+1}{(x-1)^2+1}\,dx$$

$$= \underbrace{\int \frac{x-1}{(x-1)^2+1}\,dx}_{} + \underbrace{\int \frac{1}{(x-1)^2+1}\,dx}_{} = (*)$$

$$\underbrace{\frac{1}{2}\int \frac{2(x-1)}{(x-1)^2+1}}_{}\,dx = \frac{1}{2}\int \frac{\left((x-1)^2+1\right)'}{(x-1)^2+1}\,dx$$

$$= \frac{1}{2}\log\left|(x-1)^2+1\right| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C$$

$$\underbrace{\int \frac{1}{t^2+1}\,dt = \mathrm{Arctan}\,t}_{}$$

$$\left(\mathrm{Arctan}\,t\right)' = \frac{1}{(\tan x)'}\bigg|_{x=\mathrm{Arctan}\,t} = \frac{1}{t^2+1}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$$

$$= \int \frac{1}{(x-1)^2+1}\,dx$$

$$= \mathrm{Arctan}\,(x-1) + C$$

$$(*) = \frac{1}{2}\log\left|(x-1)^2+1\right| + \mathrm{Arctan}\,(x-1) + C$$

[redacted]

例5

$$\int \frac{1}{(x^2+1)^2}\,dx$$

$$= \int \frac{(x^2+1)-x^2}{(x^2+1)^2}\,dx$$

$$= \underbrace{\int \frac{1}{x^2+1}\,dx}\; - \underbrace{\int \frac{x^2}{(x^2+1)^2}\,dx} = (*)$$

$$\left(\frac{1}{x^2+1}\right)' = -1\cdot 2x\,\frac{1}{(x^2+1)^2}$$

$$= -\frac{2x}{(x^2+1)^2}$$

$$\int \frac{x^2}{(x^2+1)^2}\,dx = -\frac{1}{2}\int \frac{-2x\cdot x}{(x^2+1)^2}\,dx$$

$$\boxed{\int \frac{1}{t^2+1}\,dt = \mathrm{Arctan}\, t}$$

$$= -\frac{1}{2}\int \left(\frac{1}{x^2+1}\right)' x\,dx$$

部分積分

$$= -\frac{1}{2}\left(\frac{x}{x^2+1} - \underbrace{\int \frac{1}{x^2+1}\,dx}\right) + C$$

$$(*) = \mathrm{Arctan}\, x + \frac{1}{2}\,\frac{x}{x^2+1} - \frac{1}{2}\mathrm{Arctan}\, x + C$$

$$= \frac{1}{2}\mathrm{Arctan}\, x + \frac{1}{2}\,\frac{x}{x^2+1} + C$$

例

$$\int \frac{1}{1+\sqrt{1+2x}}\,dx = \int \frac{1}{1+t}\,t\,dt$$

$t=\sqrt{1+2x}$ とおく

$$t^2 = 1+2x$$

$$\frac{dt}{dx} = \cancel{2}\cdot\frac{1}{\cancel{2}}(1+2x)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{1+2x}} = \frac{1}{t} \qquad \frac{dx}{dt} = t$$

$$= \int 1 - \frac{1}{1+t}\,dt$$

$$(*) = t - \log|1+t|\Big|_{t=\sqrt{1+2x}} + C$$

$$= \sqrt{1+2x} - \log\left(1+\sqrt{1+2x}\right) + C$$