Interpretation of vectors and matrices

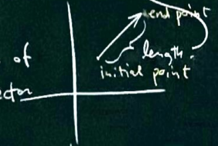
Geometrical

A plane (space, n-dimensional space)

vectors

a plane vector is a quantity with a direction space

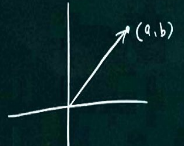
an arrow representation of a plane vector



Two arrows with the same direction and length (parallel) are identified

In particular each plane vector can be represented with an arrow whose initial point is put at the origin of the plain (2 dimensional)

We consider new vector



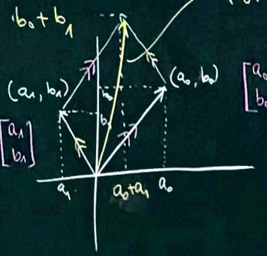
$\begin{bmatrix} a \\ b \end{bmatrix}$ as a representation of the plane vector ← we simply identify them!

We define:

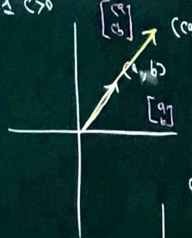
$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_0+a_1 \\ b_0+b_1 \end{bmatrix}$$

$$c\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} ca_0 \\ cb_0 \end{bmatrix}$$

$\begin{bmatrix} a_0+a_1 \\ b_0+b_1 \end{bmatrix}$ — (a_0+a_1, b_0+b_1)

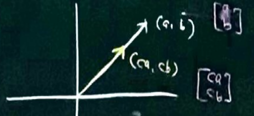


Case 1: $c > 0$

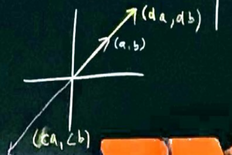


$c \geq 1$

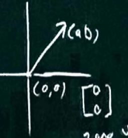
$0 < c < 1$



Case 2: $c < 0$ $\quad c = -d \quad (d > 0)$



Case 3: $c = 0$



plane vectors space vectors etc are also denoted by

$\mathbb{a}$ $\mathbb{b}$ $\mathbb{c}$, ... or $\vec{a}, \vec{b}, \vec{c}$ etc.

We obtain all the calculation rules of row vectors also for plane (space etc) vectors:

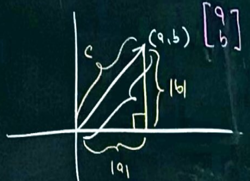
$$\mathbb{a} + \mathbb{b} = \mathbb{b} + \mathbb{a}$$

$$(\mathbb{a} + \mathbb{b}) + \mathbb{c} = \mathbb{a} + (\mathbb{b} + \mathbb{c})$$

$$a(\mathbb{a} + \mathbb{b}) = a\mathbb{a} + a\mathbb{b}$$

$$(a+b)\mathbb{a} = a\mathbb{a} + b\mathbb{a} \quad \cdots \text{ etc.}$$

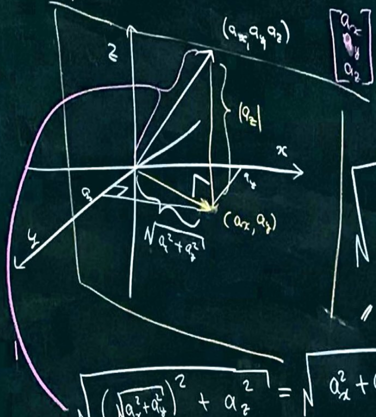
The length of vectors



$$c^2 = |a|^2 + |b|^2$$
$$= a^2 + b^2 \quad \text{(Pythagoras Theorem)}$$

$$c = \sqrt{a^2 + b^2} = \sqrt{[a, b]\begin{bmatrix} a \\ b \end{bmatrix}}$$

In space:



$$\sqrt{\left(\sqrt{a_x^2 + a_y^2}\right)^2 + a_z^2} = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{[a_x\, a_y\, a_z]\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}}$$

For n-dimensional vectors

we define the length $|\mathbb{a}|$ of a vector $\mathbb{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

as

$$\sqrt{a_1^2 + \cdots + a_n^2}$$

$$= \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{\mathbb{a}^t\, \mathbb{a}}$$

(Linear) transformation

A transformation of (plane) vectors is a mapping which sends each plane vector to some (plane) vector.

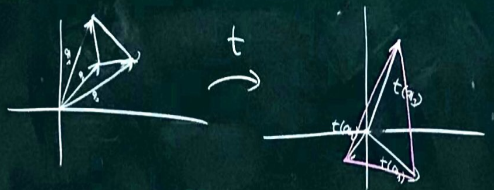
We denote with $\mathbb{R}^2$ the set of all plane vectors.

Thus a transformation of plane vectors (t) is simply a

mapping (or function) $t : \mathbb{R}^2 \to \mathbb{R}^2$

$\mathbb{R}^2$ the set of all plane vectors, or

$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$



A 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ induces a transformation φ_A

defined by

$$\underset{\mathbf{a}}{\begin{bmatrix} u \\ v \end{bmatrix}} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\mathbf{a}$$

Such φ_A has the following properties:

(1) $\varphi_A(\mathbf{a}+\mathbf{b}) = \varphi_A(\mathbf{a}) + \varphi_A(\mathbf{b})$

(2) $\varphi_A(c\mathbf{a}) = c\,\varphi_A(\mathbf{a})$

$(\mathbf{a}, \mathbf{b} \in \mathbb{R}^2,\ c \in \mathbb{R})$

$A(\mathbf{a}+\mathbf{b}) = A\mathbf{a} + A\mathbf{b}$

$A(c\mathbf{a}) = c\,A\mathbf{a}$

A transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ with the properties:

(1) $t(\mathbf{a}+\mathbf{b}) = t(\mathbf{a}) + t(\mathbf{b})$

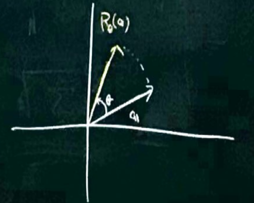
(2) $t(c\mathbf{a}) = c\,t(\mathbf{a})$

for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ and $c \in \mathbb{R}$

is called a linear transformation (or linear mapping)

Example Rotations (around the origin) are linear transformations R_θ

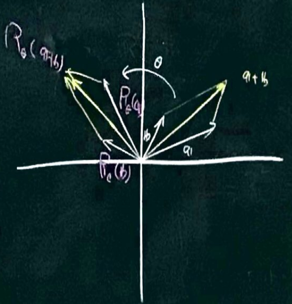
Consider the rotation of plane vectors (with the initial point at the origin of the plane) by angle θ anticlockwise:



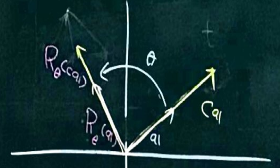
Theorem 3.1

R_θ is a linear transf:

$$R_\theta(a+b) = R_\theta(a) + R_\theta(b):$$



$$R_\theta(ca) = cR_\theta(a)$$



We saw:

Theorem 3.0 For any 2×2 matrix A

$\varphi_A : \mathbb{R}^2 \to \mathbb{R}^2$; $a \mapsto Aa$

is a linear transformation

$\varphi_A(a+b) = \varphi_A(a) + \varphi_A(b)$

$\varphi_A(ca) = c\,\varphi_A(a)$

Theorem 3.2

For any linear transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$, there is a 2×2 matrix A s.t.

$$t = \varphi_A$$

i.e. for any plane vector a we have.

$$t(a) = Aa.$$

We will prove this next time:

with the proof, we can also calculate the 2×2 matrix R_θ which induces the rotation of vectors by θ!

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example rotation (around the origin) is a linear transf R_θ

Consider the rotation of plain vectors (with the initial point at the origin of the plain) by angle θ anticlockwise: