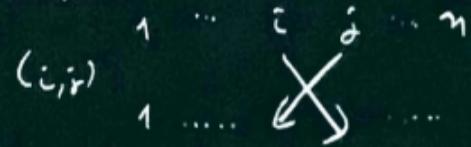


$n \in 1, 2, 3 \dots$
 $S_n = \{ \sigma : \sigma \xrightarrow{\text{bijections}} \{1, \dots, n\} \rightarrow \{1, \dots, n\} \}$
 symmetric group
 permutation of degree n

For $1 \leq i < j \leq n$ (i, j) denotes the element of S_n
transposition



$id \in S_n$
 $id: \{1, \dots, n\} \rightarrow \{1, \dots, n\}; i \mapsto i$

Lemma 4.4 + (4.5)

All $\sigma \in S_n$ can be represented as a product of transpositions.

If $\sigma = \tau_1 \dots \tau_m = \tau'_1 \dots \tau'_{m'}$

(it is possible that $m \neq m'$)

then the parity of them is the same.
 (address on evenness)

in particular, $(-1)^m = (-1)^{m'}$ ← This is decided by $\text{sgn}(\sigma)$
 depends only on σ
 signature of σ
 (parity of σ)

For $n \times n$ matrix $A = [a_{ij}]$, we define

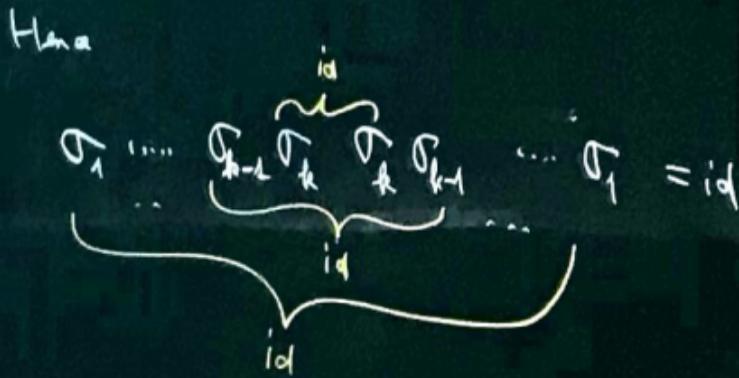
$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

Lemma 5.1 If $\sigma = \sigma_1 \dots \sigma_k$

where $\sigma \in S_n$ $\sigma_1 \dots \sigma_k \in S_n$
 transpositions

then $\sigma^{-1} = \sigma_k \dots \sigma_1$ in particular $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$
 inverse function
 in reverse order!!

proof
 Note that $(i, j)(i, j) = id$.
 Thus $(i, j) = (i, j)^{-1}$



Hence $\sigma_k \dots \sigma_1 = \sigma^{-1}$ \square

Theorem 5.2 For any $n \times n$ -matrix A

$$\det(A) = \det({}^t A)$$

Lemma 5.3

$$I: S_n \rightarrow S_n; \sigma \mapsto \sigma^{-1}$$

is 1-1 onto mapping on S_n

Proof To see that I is 1-1. Suppose $I(\sigma) = I(\tau)$.

$$\begin{aligned} \sigma &= \tau^{-1} \tau \sigma = \tau^{-1} \sigma = \tau \\ &= \underbrace{\tau^{-1}}_{id} \sigma = \tau \underbrace{\sigma^{-1}}_{id} = \tau \end{aligned}$$

S_n is finite (actually S_n has exactly $n!$ elements.)

So $I[S_n] \subseteq S_n$ and (since $I \circ I = 1$)

$I[S_n]$ has $n!$ elements. It follows that $I[S_n] = S_n$.

Proof of Thm 5.2 Note $\{(1, \sigma(1)), \dots, (n, \sigma(n))\}$

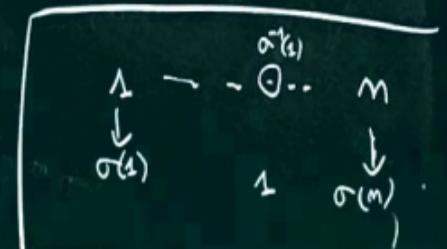
$$= \{(\sigma^{-1}(1), 1), \dots, (\sigma^{-1}(n), n)\}$$

$$\text{For } \sigma \in S_n \quad \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

$$= \text{sgn}(\sigma) a_{\sigma^{-1}(1), 1} a_{\sigma^{-1}(2), 2} \dots a_{\sigma^{-1}(n), n}$$

Lemma 5.1

$$\downarrow \text{sgn}(\sigma^{-1}) a_{\sigma^{-1}(1), 1} \dots a_{\sigma^{-1}(n), n}$$



$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma^{-1}) a_{\sigma^{-1}(1), 1} \dots a_{\sigma^{-1}(n), n}$$

$$\uparrow \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1), 1} \dots a_{\sigma(n), n}$$

$$\text{Lemma 5.3} = \det({}^t A) \quad \square$$

Lemma 5.4

$$\det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \dots & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \dots & a_{nn} \end{bmatrix}$$

proof

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

$$= \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$$= a_{11} \left(\sum_{\substack{\sigma: \{2, \dots, n\} \rightarrow \{2, \dots, n\}}} \text{sgn}(\sigma) a_{2\sigma(2)} \dots a_{n\sigma(n)} \right) = a_{11} \det(A_{11}) \quad \square$$

Theorem 5.2 For any $n \times n$ -matrix A

$$\det(A) = \det(A^t)$$

Cor 5.5

$$\det \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ \vdots & \vdots \\ a_{n2} & a_{nn} \end{pmatrix}$$

Proof By Lemma 5.4 and Theorem 5.2

Theorem 5.6 (1) $\det(E) = 1$

(2) If $A = [a_1 \dots a_n]$ then
 $\det([c a_1 \dots c a_i \dots a_n]) = c \det(A)$

(3) $\det([a_1 \dots a_i + b \dots a_n]) = \det(A) + \det([a_1 \dots b \dots a_n])$

(3) $\det([a_1 \dots a_j \dots a_i \dots a_n]) = -\det([a_1 \dots a_i \dots a_j \dots a_n])$

multi-linearity of determinant

Note the properties above characterize totally the determinant.

proof (1):

$$\det(E) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \delta_{1\sigma(1)} \dots \delta_{n\sigma(n)} = \sum_{\sigma=1d} \overset{1}{\dots} = 1$$

if $\sigma \neq \text{id}$ then $\delta_{k\sigma(k)} = 0$ for k with $\sigma(k) \neq k$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

(2):

$$\begin{aligned} \det([c a_1 \dots c a_i \dots a_n]) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) c a_{1\sigma(1)} \dots c a_{i\sigma(i)} \dots a_{n\sigma(n)} \\ &= c \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)} \\ &= c \det(A) \end{aligned}$$

(3): Let $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$\begin{aligned} \det([a_1 \dots a_i + b \dots a_n]) &= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots (a_{i\sigma(i)} + b_{\sigma(i)}) \dots a_{n\sigma(n)} \\ &= \underbrace{\sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{i\sigma(i)} \dots a_{n\sigma(n)}}_{\det(A)} + \underbrace{\sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots b_{\sigma(i)} \dots a_{n\sigma(n)}}_{\det([a_1 \dots b \dots a_n])} \end{aligned}$$

(4):

$$\det([a_1 \dots a_i \dots a_j \dots a_n])$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{i\sigma(i)} \dots a_{j\sigma(j)} \dots a_{n\sigma(n)}$$

the transposition (i, j)

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(i)(1)} \dots a_{i\sigma(i)(i)} \dots a_{j\sigma(i)(j)} \dots a_{n\sigma(i)(n)}$$

$$= \sum_{\sigma \in S_n} -\text{sgn}(\sigma(i, j)) \dots$$

$$= - \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} \dots a_{i\sigma(i)} \dots a_{j\sigma(j)} \dots a_{n\sigma(n)}$$

$\det(A)$

Note for $k \notin \{i, j\}$
 $\sigma(i, j)(k) = \sigma(k)$
 $f: S_n \rightarrow S_n; \sigma \mapsto \sigma(i, j)$
 is also 1-1 onto

- Theorem 5.6 (1) $\det(E) = 1$
- (2) If $A = [a_1 \dots a_n]$ then $\det([c a_1 \dots c a_n]) = c \det(A)$ multi: linearity of determinant
- (3) $\det([a_1 \dots a_i + b \dots a_n]) = \det(A) + \det([a_1 \dots b \dots a_n])$
- (4) $\det([a_1 \dots a_i \dots a_j \dots a_n]) = -\det([a_1 \dots a_j \dots a_i \dots a_n])$

Note the properties above characterize totally the determinant.

proof (1):

$$\det(E) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \delta_{1\sigma(1)} \dots \delta_{n\sigma(n)} = \sum_{\sigma=id} 1 = 1$$

if $\sigma \neq id$ then $\delta_{k\sigma(k)} = 0$ for k with $\sigma(k) \neq k$

Cor 5.7 Thm 5.6 with rows and columns swapped.

Proof By Thm 5.6 and Thm 5.2. \square

