

# Z: Zermelo's set theory

Axiom of Extensionality

Axiom of Empty Set

Pairing Axiom

Axiom of Union

Axiom of Separation

Axiom of Infinity

Axiom of Power set

$$\mathbb{N} = \bigcap \left\{ X \mid \begin{array}{l} \phi \in X, \\ \text{for any } a \in X \\ a \cup \{a\} \in X \end{array} \right\}$$

Lemma 2.2  $\mathbb{N}$  satisfies (a) and (b)

$\mathbb{N}$  satisfies Peano axioms ( $\neq$  Peano Arithmetic)

$$p: \mathbb{N} \rightarrow \mathbb{N}$$

$$p = \{ \langle m, m \rangle \in \mathbb{N}^2 \mid m = m \cup \{m\} \}$$

$\mathbb{N} \times \mathbb{N}$   
= Successor function

A set  $a$  is transitive if, for any  $b, c$ ,  $b \in a$  and  $c \in b$  implies  $c \in a$   
(all elements of  $a$  are also in  $a$ )

Lemma 3.1 (1) For any  $X \subseteq \mathbb{N}$  if

$X \neq \emptyset$  and (a), (b) then  $X = \mathbb{N}$ .

↑  
non-empty

(2) All  $m \in \mathbb{N}$  are transitive.

(3) For any  $m \in \mathbb{N}$ ,  $p(m) \neq m$ , in particular  $m \notin m$ .

(4) For any  $m \in \mathbb{N}$  if  $m \neq \emptyset$  then there is  $n \in m$  with  $m = p(n)$

(5) For any  $m, n \in \mathbb{N}$ , if  $p(m) = p(n)$  then  $m = n$

Proof (1): Suppose  $X \subseteq \mathbb{N}$  and  $X \neq \emptyset$

Then  $X^* \in \{ \emptyset \}$

$\mathbb{N} = \bigcap \{ \emptyset \} \subseteq X^*$ . Then  $\mathbb{N} = X^*$ .

(2): By (3), it is enough to show that  $\emptyset$  is transitive and

if  $a$  is transitive then  $a \cup \{a\}$  is also transitive.

$\phi$  is vacuously transitive!

Suppose  $\alpha$  is transitive and let  $b \in \alpha \cup \{\alpha\}$  and  $c \in b$ . If  $b \in \alpha$  then by transitivity of  $\alpha$   $c \in \alpha \subseteq \alpha \cup \{\alpha\}$ . If  $b \notin \alpha$  then  $b = \alpha$  then  $c \in \alpha \subseteq \alpha \cup \{\alpha\}$ .

(3): Let  $X = \{m \in \mathbb{N} \mid \rho(m) \neq m\}$ .

By (1), it is enough to show that  $X \neq (a), (b)$ .

(a):  $\phi \neq \phi$  by def. of  $\phi$ !

(b): Suppose  $m \in X$  i.e.  $\rho(m) \neq m$  we have to show  $\rho(\rho(m)) \neq \rho(m)$ . Assume otherwise then

$$\rho(m) = \rho(m) \cup \{\rho(m)\}$$

$$\rho(m) \in \rho(m) \text{ i.e. } \rho(m) \in m \cup \{m\}$$

By the assumption  $\rho(m) \neq m$  we must have  $m \in \rho(m) \in m$ . Then by (2) it follows that  $m \in m$ .

$$\text{Hence } m = \underbrace{m \cup \{m\}}_{\rho(m)}. \text{ A contradiction!}$$

(c):  $X' = \{m \in \mathbb{N} \mid \text{There is some } n \in \mathbb{N} \text{ s.t. } m = \rho(n)\}$

$$\text{Let } X = X' \cup \{\phi\}$$

By (1), it is enough to show that  $X \neq (a), (b)$ .

(a):  $\phi \in X$  by definition of  $X$ !

(b): Assume that  $m \in X$  then  $\rho(m) \in X' \subseteq X$ . Thus  $\rho(m) \in X$ .

(d): Suppose that  $\rho(m) = \rho(m')$  but  $m \neq m'$ .

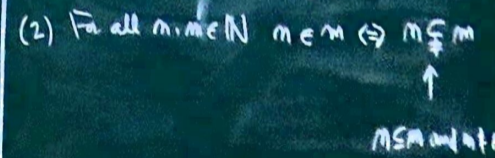
By (2)  $m \cup \{m\} = m' \cup \{m'\}$ . But since  $m \neq m'$

$$\underbrace{m \in m'}_{\text{true}} \quad m' \in m$$

By (2) it follows that  $m \in m$ .

This is a contradiction to (3).  $\square$

Lemma 3.2 (1) For any  $m, m' \in \mathbb{N}$  exactly one of  $m \in m$ ,  $m \in m'$  or  $m' \in m$  holds.



(3) for any  $m, m \in \mathbb{N}$  (one of <sup>exactly</sup>  $m \subsetneq m, m = m, m \supsetneq m$ ) holds.

(4)  $\in$  (or equivalently  $\subseteq$ ) is a linear ordering on  $\mathbb{N}$  (transitive reflexive relation  $\subseteq$  with  $m \in m, m = m \wedge m \in m$ )

proof (1): That (a) (b) (c) are pairwise exclusive, follows from Lemma 3.1 (2) (3). If  $m \in m$  and  $m = m$  then we have  $m \in m$  a contradiction to L 3.1 (3). If  $m \in m$   $m \in m$  then by L 3.1 (2)  $m \in m$ . This is again a contradiction!

To prove that one of (a) (b) (c) always holds we let  $X = \{m \in \mathbb{N} \mid \text{for all } n \in \mathbb{N} \text{ if } m \neq n \text{ then } m \in n\}$

(\*) or  $m \in m$  } and show that  $X = \mathbb{N}$  by showing  $X \neq (a), (b)$

(a):  $\phi \in X$ : if  $m \neq \phi$  for any  $m \in \mathbb{N}$  then  $\phi \in m$ : To show this let

$$Y = \{m \in \mathbb{N} \mid m = \phi, \phi \in m\}$$

$$Y = \mathbb{N} \left[ \phi \in Y, \text{ if } m \in Y, \text{ then } \phi \in \rho(m) = m \cup \{m\} \right]$$

(i.e.  $\phi \in m$ )

(b) Suppose  $m \in X$  have to show that  $\rho(m) \in X$

$$Z = \{m \in \mathbb{N} \mid m \in \rho(m) \text{ or } m = \rho(m) \text{ or } \rho(m) \in m\}$$

We have to show that  $Z = \mathbb{N}$   
For this we show that  $Z \neq (a), (b)$

For (a): since  $\rho(m) \neq \phi$   $\rho(m) \ni \phi$  by  $\square$ . Thus  $\phi \in Z$

For (b): Suppose  $m \in Z$  and we show  $\rho(m) \in Z$ .

Since  $m \in Z$  we have (A)  $m \in \rho(m)$  or (B)  $m = \rho(m)$  or (C)  $\rho(m) \in m$

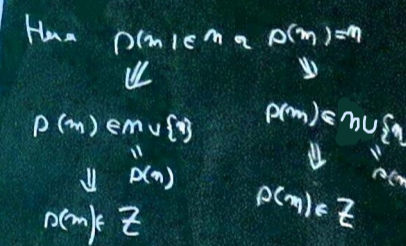
If (A)  $m \in m \cup \{m\} \Leftrightarrow m \in m$  or  $m = m$   
If  $m = m$   $\rho(m) = \rho(m)$  Thus (B) holds for  $\rho(m)$

If  $m \in m$  then since  $m \in X$  [ $\rho(m) \in m \stackrel{\text{on}}{\rho(m)} \rho(m) = m$  or  $m \in \rho(m)$ ] holds.  $m \neq \rho(m)$  otherwise  $m \in \rho(m)$

$m \subseteq \rho(m)$  since  $m \in m$   $\rho(m) = m \cup \{m\} \subseteq m$   
 $\uparrow$  by transitivity

a is transitive  $\Leftrightarrow$  for any  $b \in a$   $b \subseteq a$

$\rho(m) \subseteq m \subseteq \rho(m)$   
 $m = \rho(m)$  a contradiction to Lemma 3.1 (3)



If (B) holds, then  $\rho(m) = m \in \rho(m)$ . Thus  $\rho(m) \in Z$ .  
If (C) holds, then  $\rho(m) \in m \subseteq \rho(m)$ . Thus we again have  $\rho(m) \in Z$ .