$\mathcal{Z}$: Zermelo Set Theory

Axiom of Extensionality

Axiom of Empty Set

Pairing Axiom

Axiom of Union

Axiom of Separation

Axiom of Infinity

Axiom of Powerset

$$\mathbb{N} = \bigcap \{ X \mid \overset{(a)}{\phi \in X}, \overset{(b)}{\text{for any } a \in X,}\ a \cup \{a\} \in X \}$$

Lemma 2.2 $\mathbb{N}$ satisfies (a) and (b)

$\mathbb{N}$ satisfies Peano axioms ($\neq$ Peano Arithmetic)

$s : \mathbb{N} \to \mathbb{N}$

$$s = \{ \langle n, m \rangle \in \underset{\underset{\mathbb{N}\times\mathbb{N}}{\parallel}}{\mathbb{N}^2} \mid m = n \cup \{n\} \}$$

$\underline{\underline{s}}$uccessor function

A set a is transitive if, for any b, c, $b \in a$ and $c \in b$ implies $c \in a$

(all elements of elements are elements!)

Lemma 3.1 (1) For any $X \subseteq \mathbb{N}$ if $X \models (a), (b)$ then $X = \mathbb{N}$.

satisfies

(2) All $n \in \mathbb{N}$ are transitive.

(3) For any $n \in \mathbb{N}$, $s(n) \neq n$, in particular $n \notin n$.

(4) For any $n \in \mathbb{N}$ if $n \neq 0$ then there is $m \in \mathbb{N}$ with $n = s(m)$

(5) For any $n, n' \in \mathbb{N}$, if $s(n) = s(n')$ then $n = n'$

Proof (1): Suppose $X^* \subseteq \mathbb{N}$ and $X^* \models (a)(b)$

Then $X^* \in \{ \quad \mid \quad \}$

$\mathbb{N} = \bigcap \{ \quad \mid \quad \} \subseteq X^*$. Then $\mathbb{N} = X^*$.

(2): By (1), it is enough to show that ϕ is transitive and if x is transitive then $x \cup \{x\}$ is also transitive.

$\emptyset$ is vacuously transitive!

Suppose x is transitive and let $b \in x \cup \{x\}$ and $c \in b$. If $b \in x$ then by transitivity of x $c \in x \subseteq x \cup \{x\}$. If $b \notin x$ then $b = x$ then $c \in x \subseteq x \cup \{x\}$.

(3): Let $X = \{n \in \mathbb{N} \mid \rho(n) \neq n\}$.

By (1), it is enough to show that $X \models (a),(b)$.

(a): $\emptyset \notin \emptyset$ by def. of $\emptyset$!

(b): Suppose $n \in X$ i.e. $\rho(n) \neq n$ we have to show $\rho(\rho(n)) \neq \rho(n)$. Assume otherwise then

$\rho(n) = \rho(n) \cup \{\rho(n)\}$. In particular

$\rho(n) \in \rho(n)$ i.e. $\rho(n) \in n \cup \{n\}$

By the assumption $\rho(n) \neq n$ we must have $n \in \rho(n) \in n$. Then by (2) it follows that $n \in n$

Hence $n = \underbrace{n \cup \{n\}}_{\rho(n)}$. A contradiction!

(2): $X' = \{n \in \mathbb{N} \mid$ There is some $m \in \mathbb{N}$ s.t. $n = \rho(m)\}$

Let $X = X' \cup \{\emptyset\}$.

By (1), it is enough to show that $X \models (a),(b)$

(a): $\emptyset \in X$ by definition of X!

(b): Assume that $n \in X$ then $\rho(n) \in X' \subseteq X$

Thus $\rho(n) \in X$.

(†): Suppose that $\overset{(1)}{\rho(n) = \rho(n')}$ but $\overset{(2)}{n \neq n'}$

By (2) $n \cup \{n\} = n' \cup \{n'\}$ But since $n \neq n'$

$n \in n'$ $\quad n' \in n$

By (2) it follows that $n \in n$.

This is a contradiction to (3). ⊠

Lemma 3.2 (1) For any $m, n \in \mathbb{N}$ exactly one of $m \in n$, $m = n$ or $n \in m$ holds.

(2) For all $m, n \in \mathbb{N}$ $m \in n \Leftrightarrow m \subsetneq n$

$m \subseteq n$ and $n \neq m$

(3) for any $n, m \in \mathbb{N}$ exactly one of

$n \subsetneq m$, $n = m$, $m \subsetneq n$

holds.

(4) $\in$ (or equivalently $\subseteq$) is a linear ordering on $\mathbb{N}$ (transitive irreflexive relation $\sqsubset$ with $n \sqsubset m$, $n = m$ or $m \sqsubset n$)

proof (1): That (α) (β) (γ) are pairwise exclusive, follows from Lemma 3.1 (2) (3):

If $n \in m$ (α) and $n = m$ (β) then we have $m \in m$ a contradiction to L 3.1, (3). If $n \in m$ (α) $m \in n$ (γ) then by L 3.1 (2) $n \in n$. This is again a contradiction!

To prove that one of (α) (β) (γ) always holds,

we let $X = \{ n \in \mathbb{N} \mid$ for all $m \in \mathbb{N}$ if $n \neq m$ then $m \in n$ (γ) or $n \in m \}$ and show that $X = \mathbb{N}$ by showing

$X \models$ (a), (b)

(a): $\emptyset \in X$: if $m \neq \emptyset$ for any $m \in \mathbb{N}$ then $\emptyset \in m$: To show this let

$\boxed{Y = \{ m \in \mathbb{N} \mid m = \emptyset,\ \emptyset \in m \}}$ and we show

$\boxed{Y = \mathbb{N}}$ [$\emptyset \in Y$, if $\underbrace{m \in Y}_{\text{i.e. } \emptyset \in m}$, then $\emptyset \in s(m) = m \cup \{m\}$]

(b) Suppose $n \in X$ have to show that $s(n) \in X$

$Z = \{ m \in \mathbb{N} \mid m \in s(n)$, $m = s(n)$ or $s(n) \in m \}$

We have to show that $Z = \mathbb{N}$

For this we show that $Z \models$ (a), (b)

For (a): since $s(n) \neq \emptyset$, $s(n) \ni \emptyset$ by $\boxed{}$. Thus $\emptyset \in Z$

For (b): Suppose $m \in Z$ and we show $s(m) \in Z$.

Since $m \in Z$ we have $\underline{m \in s(n)}$ (A) or $m = s(n)$ (B) or $s(n) \in m$ (C)

If (A) $m \in n \cup \{n\} \Leftrightarrow m \in n$ or $m = n$

If $m = n$: $s(m) = s(n)$. Hence (B) holds for $s(m)$

If $\underline{m \in n}$ then since $n \in X$ [$s(m) \in n$ or $s(m) = n$ or $n \in s(m)$] holds. $n \notin s(m)$ otherwise [i.e. if $n \in s(m)$]

$n \subseteq s(m)$ (by transitivity) Since $m \in n$ $s(m) = m \cup \{m\} \subseteq n$

a is transitive $\Leftrightarrow$ for any $b \in a$ $b \subseteq a$

$s(m) \subseteq n \subseteq s(m)$

$n = s(m)$ a contradiction $\in m$ to Lemma 3.1, (3)

Hence $s(m) \in n$ or $s(m) = n$

$\Downarrow$ $s(m) \in n \cup \{n\} = s(n)$ $\Downarrow$ $s(m) \in Z$

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If (B) holds, then $s(n) = m \in s(m)$. Thus $s(m) \in Z$.

If (C) holds, then $s(n) \in m \subseteq s(m)$. Thus we again have $s(m) \in Z$. ☒