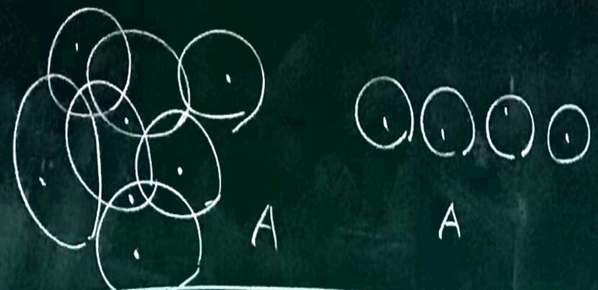
Axiom of Choice (AC)

(AC): For any A with $\emptyset \notin A$ there is a mapping $f: A \to \bigcup A$ s.t. for all $a \in A$ $f(a) \in a$ — a choice function for A

Thm 6.1 (Z, Zermelo) TFAE (over the set theory Z)

(a) AC

(b) For any X there is a well-ordering R on X.



Def. $\langle X, R\rangle$ with $R \subseteq X^2$ is said to be a well-ordered set (or well-ordering, or R is a well-ordering on X) if R is a linear ordering on X and for any nonempty $Y \subseteq X$ Y has the minimal element w.r.t. R.

Examples

(a) $\langle \mathbb{N}, < \rangle$ is a well-ordering; $< \; = \; \in\restriction\mathbb{N}$ (Lemma 5.2)

(b) each $n \in \mathbb{N}$ with $\in\restriction n$ is a well-ordering; $n = \{m \in \mathbb{N} \mid m < n\}$ (Exercise)

Def $f: X \to Y$ is surjection (onto) (to Y)
$\iff f''X = \{y \in Y \mid \text{there is some } x \in X \text{ with } f(x) = y\} = Y$

$f: X \to Y$ is injection (1-1) if $x, x' \in X$ $x \neq x'$ then $f(x) \neq f(x')$

f is bijection (1-1 onto) if f is surjection and injection

$f''X = f[X] = f(X)$

For orderings $\langle X_0, R_0\rangle$ $\langle X_1, R_1\rangle$

$f: X_0 \to X_1$ is an isomorphism if f is a bijection and for any $x_0, x_1 \in X_0$

$x_0 R_0 x_1 \iff f(x_0) R_1 f(x_1)$

If $\langle X_0, R_0\rangle = \langle X_1, R_1\rangle$ then an isomorphism $f: X_0 \to X_1$ ($= X_0$) is called an automorphism.

An isomorphism is often denoted by

$f : X_0 \xrightarrow{\cong} X_1$ or $f : \langle X_0, R_0 \rangle \xrightarrow{\cong} \langle Y_1, R_1 \rangle$

(1) If there is an isomorphism from X_0 to X_1

We say that $\langle X_0, R_0 \rangle$ and $\langle X_1, R_1 \rangle$ are isomorphic if there is an isomorphism from X_0 to X_1

then $f^{-1} = \{ \langle a, b \rangle \mid \langle b, a \rangle \in f \}$ is an isomorphism from X_1 to X_0.

(2) Composition of two isomorphisms

$f : X_0 \xrightarrow{\cong} X_1$ and $g : X_1 \xrightarrow{\cong} X_2$ is an isomorphism where the composition $g \circ f$ of f and g is defined by

$g \circ f = \{ \langle a, c \rangle \in X_0 \times X_2 \mid$ there is some $b \in X_1$ s.t. $\langle a, b \rangle \in f$ and $\langle b, c \rangle \in g \}$

If there is an isomorphism from $\langle X_0, R_0 \rangle$ to $\langle X_1, R_1 \rangle$ we write $\langle X_0, R_0 \rangle \cong \langle X_1, R_1 \rangle$

By (1), (2) and (3) $\cong$ is an equivalence relation

(3) $id_X : X \to X ; x \mapsto x$

i.e. $id_X = \{ \langle x, x \rangle \mid x \in X \}$ is an automorphism on $\langle X, R \rangle$

<u>Lemma 6.3</u> For a well-ordering $\langle X, R \rangle$ id_X is the unique automorphism on $\langle X, R \rangle$.

<u>Proof</u> Let $f : \langle X, R \rangle \xrightarrow{\cong} \langle X, R \rangle$.

Suppose $f \neq id_X$ then $Z = \{ x \in X \mid f(x) \neq x \}$ is non empty. Let x^* be the minimal element of Z.

Let $Y = \{ x \in X \mid x R x^* \}$ Then $f \restriction Y = id_Y$ by the minimality of x^* and x^* is minimal above Y

Thus $f(x^*)$ must be minimal above $f''Y = Y$

Because R is linear we should have $f(x^*) = x^*$.

A contradiction to the choice of x^*. □

For a linearly ordered set $\langle X, R \rangle$, $Y \subseteq X$ is an initial segment if for any $y \in Y$ and $z \in X$ with $z R y$ we have $z \in Y$.

$f : \mathbb{R} \to \mathbb{R}$ $x \mapsto -x$

$f : \mathbb{R} \to \mathbb{R}$ $x \mapsto x + a$

X

$R \to$

<u>Lemma 6.2</u> If $\langle X, R \rangle$ is a well-ordering then for all initial segment Y of X, either $Y = X$ or there is some $x^* \in X$ s.t. $Y = \{ x \in X \mid x R x^* \}$.

We denote this set by $I_R(x^*)$

<u>Proof</u> Suppose $Y \neq X$. Let x^* be the minimal element of $X \setminus Y$ Then $\boxed{Y = \{ x \in X \mid x R x^* \}}$ (Exercise). □

Proof of Thm 6.1 (b) $\Rightarrow$ (a): Assume (b) and suppose that A is nt. $\phi \notin A$.

Let R be a well-ordering on $\bigcup A$.

Let $f : A \to \bigcup A$ be defined by

$$f = \{ \langle a, b \rangle \in A \times \bigcup A \mid b \text{ is the minimal element of } \underset{\substack{\ast \\ \phi}}{a} \subseteq \bigcup A \text{ wrt } R \}$$

Then f is a choice function for A.

(a) $\Rightarrow$ (b): Assume (AC) and X is an arbitrary set.

If $X = \phi$ then $R = \phi$ is a well-ordering on X. So let us assume that $X \neq \phi$.

Let $A = \mathcal{P}(X) \setminus \{\phi\}$ and let $f : A \to \bigcup \underset{X}{A}$ be a choice function.

$$\mathcal{Y} = \{ \langle u, r \rangle \mid u \subseteq X,\ r \subseteq u^2,\ r \text{ is a well ordering on } u \text{ s.t. for any proper initial segment } I \text{ of } u \text{ wrt } r,\ f(X \setminus I) \text{ is the minimal element of } u \setminus I \}$$

Idea: we prove that

$\bigcup \{ r \mid \langle u, r \rangle \in \mathcal{Y} \text{ for some } u \}$ is a well ordering on X

Claim 1 $\mathcal{Y} \neq \phi$ $\vdash \langle \phi, \phi \rangle \in \mathcal{Y}$ $\dashv$

Claim 2 If $\langle u_0, r_0 \rangle, \langle u_1, r_1 \rangle \in \mathcal{Y}$ then

either $u_0 \subseteq u_1$ and $r_0 = r_1 \upharpoonright u_0$ or

$u_1 \subseteq u_0$ and $r_1 = r_0 \upharpoonright u_1$.

$\vdash$ Suppose that $\langle u_0, r_0 \rangle$ and $\langle u_1, r_1 \rangle$ are counter examples for the claim above.

$$\mathcal{Y}_0 = \{ x \in u_0 \cap u_1 \mid I_{r_0}(x) = I_{r_1}(x) \text{ and } r_0 \upharpoonright I_{r_0}(x) = r \upharpoonright I_{r_1}(x) \}$$

$\mathcal{Y}_0 \neq \phi$ Let $x_0 = f(X)$ then

$I_{r_0}(x) = I_{r_1}(x) = \phi$ Hence $x_0 \in \mathcal{Y}_0$

Let $u^* = \underbrace{\bigcup \{ I_{r_0}(x) \mid x \in \mathcal{Y}_0 \}}_{\in \mathcal{P}(X)}$

u^* is an initial segment of u_0 and u_1 wrt r_0 and r_1

Then $r_0 \upharpoonright u^* = r_1 \upharpoonright u^*$

Thus $u^* \subsetneq u_0$ and $u^* \subsetneq u_1$

by the choice of $\langle u_0, r_0 \rangle$ and $\langle u_1, r_1 \rangle$

The minimal element of $u_0 \setminus u^*$ wrt $r_0 = \overset{x^*}{\overset{\parallel}{f(X \setminus u^*)}}$

$=$ The minimal element of $u_1 \setminus u^*$ wrt r_1

$u^* \cup \{x^*\} \subseteq u_0 \cap u_1$ and

$r_0 \upharpoonright u^* \cup \{x^*\} = r_1 \upharpoonright u^* \cup \{x^*\}$

Hence $u_0 \neq u^* \cup \{x^*\}$ and $u_1 \neq u^* \cup \{x^*\}$

The minimal element of u_0 above x^* wrt r_0

$= f(X \setminus u^* \cup \{x^*\}) =$ the minimal element of u_1 above x^* wrt r_1 x^{**}

Then $x^{**} \in \mathcal{Y}_0$ This is a contradiction to the choice of u^* because we should have

$$I_{r_0}(x^{**}) \subseteq u^* \quad \dashv$$

$\phantom{I_{r_0}(}\Downarrow \qquad \not\Downarrow$

$\phantom{I_{r_0}(}x^* \qquad x^*$

Let

$X_0 = \bigcup \{ u \mid \langle u, R \rangle \in \mathcal{M} \text{ for some } R \}$

$R_0 = \bigcup \{ R \mid \langle u, R \rangle \in \mathcal{M} \text{ for some } u \}$

$X_0 \subseteq X$ and R_0 is a well ordering on X_0.

<u>Claim 3</u> $X_0 = X$

$\square$