

Proof as in Supplement

\mathcal{U} is Grothendieck universe $\Leftrightarrow \mathcal{U} = H(\kappa)$
uncountable $= \{ \alpha \mid \alpha \text{ is hereditarily of cardinality } < \kappa \}$

for $\kappa = \aleph_n \cap \mathcal{U}$
and κ is inaccessible cardinal

IF \mathcal{U} is an uncountable Grothendieck universe
 $\mathcal{U} \models \text{ZFC}$
 $\mathcal{U} \models \text{ZFC}$ $\mathcal{U} \models \text{ZC}$

Question How similar is \mathcal{U} to V ?

The answer depends on Axioms of Large Cardinals?

Limitations of Zermelo's axioms under Z

\aleph is also denoted by ω (in particular if we would like to see \aleph as a transfinite ordinal number. Existence of ω is a theorem in Z.

In Z, for a set a and $m \in \omega$ let

$\mathcal{P}^m(a)$ be defined by $\mathcal{P}^0(a) = a$, $\mathcal{P}^{m+1}(a) = \mathcal{P}(\mathcal{P}^m(a))$

Thm 8.1 If Z is consistent then

(1) Z does not prove the existence of

$\{ \mathcal{P}^m(\aleph) \mid m \in \omega \}$

(2) Z does not prove the existence of

$V_{\omega+\omega}$

We can prove by induction

$V_m \subseteq V_n$ for $m \in m \in \omega$

$V_{\omega+m} \subseteq V_{\omega+n}$

For $m \in \omega$ let V_m be defined by

$V_0 = \emptyset$ $V_{m+1} = \mathcal{P}(V_m)$.

Then let $V_\omega = \bigcup_{n \in \omega} \{ V_n \mid n \in \omega \}$

Assuming V_ω exists let

$V_{\omega+m}$ be defined by

$V_{\omega+0} = V_\omega$ $V_{\omega+n+1} = \mathcal{P}(V_{\omega+n})$

Let $V_{\omega+\omega} = \bigcup_{n \in \omega} V_{\omega+n}$

Remark 'Z is consistent' is what we believe firmly but what cannot be "proved" completely due to The ? Incompleteness Theorem.

Sketch of proof

(2): In Z, suppose that there is the set $V_{\omega+\omega}$ then $\langle V_{\omega+\omega}, \in \upharpoonright V_{\omega+\omega} \rangle \models \ulcorner Z \urcorner$.
 By "Soundness Theorem" of predicate logic, it follows that " $\ulcorner Z \urcorner$ " is consistent!
 (This is often denoted as $\text{Consis}(\ulcorner Z \urcorner)$)

is a theorem in Z. But then by the ? Incompleteness Theorem Z is inconsistent.

A contradiction to the assumption. P^*
 (1)! We can prove $V_{\omega} \subseteq \bigcup \{P^n(N) \mid n \in \omega\}$
 (In Z, suppose toward a contradiction that the set $\{P^n(N) \mid n \in \omega\}$ (by induction on $n \in \omega$)

Thus V_{ω} exists by the assumption and Axiom of Separation.
 Also we can prove that V_{ω} is countable i.e. there is a 1-1 onto mapping from \mathbb{N} to V_{ω}
 t

Let $R_0 \subseteq \mathbb{N} \times \mathbb{N}$ be defined by $\langle m, n \rangle \in R_0 \Leftrightarrow t(m) \in t(n)$ Let $t_0 = t$.

Thus $t: \langle \mathbb{N}, R_0 \rangle \cong \langle V_{\omega}, \in \rangle$
 For $n \in \omega$, assuming $R_n \subseteq P^n(N) \times P^n(N)$ and $t_n: P^n(N) \rightarrow V_{\omega+n}$ have been defined,

we let $t_{n+1}: P^{n+1}(N) \rightarrow V_{\omega+n+1}$
 \parallel $P(P^n(N))$ \parallel $P(V_{\omega+n})$
 and $R_{n+1} \subseteq P^{n+1}(N) \times P^{n+1}(N)$

by $t_{n+1}(a) = \{t_n(b) \mid b \in a\}$ for $a \in P^{n+1}(N)$

and $\langle a, b \rangle \in R_{n+1} \Leftrightarrow t_{n+1}(a) \in t_{n+1}(b)$

for $a, b \in P^{n+1}(N)$
 We also obtain $t_n: \langle P^n(N), R_n \rangle \cong \langle V_{\omega+n}, \in \upharpoonright V_{\omega+n} \rangle$

Let $R^* = \bigcup_{\text{new}} R_m$
 R^* exists since $R^* \subseteq P^* \times P^*$
 Since (intuitively) we have
 $\langle P^*, E^* \rangle \cong \langle V_{\text{new}}, \in \rangle$ Hence
 $\langle P^*, E^* \rangle \models \mathbb{Z}^+$. As before
 we obtain a contradiction. \square

Axiom of Replacement (AR)

For any property $\varphi(x, y, z, \dots)$ about sets
 and sets a and a_1, \dots, a_n if

For any $b \in a$ there is a unique c s.t.

$\varphi(b, c, a_1, \dots, a_n)$ holds. Then

$\{c \mid \varphi(b, c, a_1, \dots, a_n) \text{ for some } b \in a\}$ exists.

$\mathbb{Z} + \text{AR}$ proves $\{P^m(\omega) \mid m \in \omega\}$, $V_\omega, V_{\omega+\omega}$
 are sets!

Axiom of Foundation (AF)

For any nonempty a there is $b \in a$
 s.t. $a \cap b = \emptyset$

b is \in -minimal w.r.t. the relation \in

Lemma 8.2 AF implies there is no

\in chain $a_0 \ni a_1 \ni \dots \ni a_n$ s.t.
 $a_0 = a_n$

proof Suppose that $a_0 \ni a_1 \ni \dots \ni a_n = a_0$
 Then $\{a_1, \dots, a_n\}$ contradicts AF.

Remark
 We can extend $V_0, V_1, \dots, V_\omega, V_{\omega+1}, \dots$
 to a sequence V_α for all "ordinal numbers α "
 in $\mathbb{Z} + \text{AR}$

We can show in $\mathbb{Z} + \text{AR}$ that

$$\text{AF} \Leftrightarrow \bigcup_{d \in \omega} V_d = V$$

the class of all ordinal numbers

$\mathbb{Z} + \text{AR} + \text{AF}$ is called Zermelo-Fraenkel
 axiom system of set theory
Fraenkel von Neumann and denoted by ZF

ZF + AC is denoted by ZFC