Predicate Logic

from the point of view of Metamathematics

Sentential Logic (Propositional Logic)

all objects we treat here are concretely given object! If we say

"for any sequence of symbols t . . . . " or write ($t \in Seq$)

we mean "for any concretely given sequence t ..."

"$n \in \mathbb{N}$.." is an abbreviation of the statement

"for any concretely given number n . . . .

Fixed symbols [redacted] "$\rightarrow$", "$\neg$", "$\exists$"

(Logical symbols:)

equality symbol "$\equiv$"

non. parenthesis "(", ")" ","

Variables, $x_0, x_1, x_2, \ldots\ldots$

Supplementary symbols

Constant symbols $c_0, c_1, \ldots$

function symbols $f_0, \ldots$

relation symbols $r_0 \ldots$

arity (≥ 1) of function and relation symbols are thought to be fixed.

A set of constant, function and relation symbols $\mathcal{L}$ is called a language.

$\mathcal{L} = \phi$

binary relation symbol

$\mathcal{L}_{ZF} = \{\varepsilon\}$

binary function symbol

$\mathcal{L} = \{e, \cdot\}$

constant symbol.

Axioms of a group can be formulated in this logic!

For a given language $\mathcal{L}$ let $Seq^{\mathcal{L}}$ denote the set of all (finite) sequences of symbols which are either fixed symbols or symbols from $\mathcal{L}$

This means if we say

$t \in Seq^{\mathcal{L}}$ it means "t is a sequence of ... "

We define inductively $Term^{\mathcal{L}} \subseteq Seq^{\mathcal{L}}$

the set of $\mathcal{L}$-terms

1) all variables and all constant symbols in $\mathcal{L}$ are $\mathcal{L}$-terms;

2) If f is an n-ary function symbol in $\mathcal{L}$ and if $t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms "(" "," ")" then the sequence obtained by putting "f" $t_0, \ldots, t_{n-1}$ as $f(t_0, \ldots, t_{n-1})$ is an $\mathcal{L}$-term

$(f(t_0, \ldots, t_{n-1}) \in Term^{\mathcal{L}})$

3) nothing else.

If t is a term and all variables appearing in t are among $\underbrace{x_0, \ldots, x_{k-1}}$ then we denote this situation by $t = t(x_0, \ldots, x_{k-1})$

We define now inductively $Fml^{\mathcal{L}} \subseteq Seq^{\mathcal{L}}$

set of $\mathcal{L}$-formulas

k symbols

I usually assume they are different to each other but they may be symbols other than "x_0" etc.

0) If $t_0, t_1 \in Term^{\mathcal{L}}$ then $t_0 \equiv t_1$ is an $\mathcal{L}$-formula $(t_0 \equiv t_1 \in Fml^{\mathcal{L}})$

1) If r is an n-ary relation symbol in $\mathcal{L}$ and $t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms then $r(t_0, \ldots, t_{n-1})$ is an $\mathcal{L}$-formula

atomic formulas

2) If φ, ψ are $\mathcal{L}$-formulas then $(\varphi \to \psi)$, $\neg\varphi$ are also $\mathcal{L}$-formulas

3) If φ is an $\mathcal{L}$-formula and x a variable $\exists x \varphi$ is also an $\mathcal{L}$-formula.

4) nothing else! $\boxed{\exists x (x \equiv x)}$

$\neg x (x \equiv x)$

For each $\mathcal{L}$-formula φ the set $Sub(\varphi)$ of subformulas of φ is defined inductively as follows.

0), 1) For an atomic formula φ $Sub(\varphi) = \{\varphi\}$

2)a) If φ is of the form $(\varphi_0 \to \varphi_1)$ then $Sub(\varphi) = \{\varphi\} \cup Sub(\varphi_0) \cup Sub(\varphi_1)$

b) If φ is of the form $\neg\varphi_0$, then $Sub(\varphi) = \{\varphi\} \cup Sub(\varphi_0)$

3) If φ is of the form $\exists x \varphi_0$ then $Sub(\varphi) = \{\varphi\} \cup Sub(\varphi_0)$

If $\psi \in Sub(\varphi)$ then we say ψ is a subformula of φ.

If ψ is a subformula of φ then $\psi \in Fml^{\mathcal{L}}$ and there are subsequences of φ which are identical with ψ.

a variable (x) is said to be free in an $\mathcal{L}$-formula φ

If x appears at k-th place of φ for some $k <$ length of φ, s.t. there is no subsequence of φ containing the k-th place of φ which is a subformula of φ of the form $\exists x \psi$

If x appears in φ but each appearance of x in φ is not free then x is said to be bound in φ

$$\varphi$$

$$\cdots \underbrace{\exists x(\quad x \quad)}_{\text{a subformula}} \cdots$$

If free variables in φ are among $x_0, \dots, x_{k-1}$ we write this as $\varphi = \varphi(x_0, \dots, x_{k-1})$.

If φ (an $\mathcal{L}$-formula) does not contain any free variable φ is called a $\mathcal{L}$-sentence.

Intended interpretation of symbols $\equiv$ (equality) $R(t_0, \dots, t_{n-1})$ (R holds for $t_0, \dots, t_{n-1}$)

$\rightarrow$ (then) $\quad \neg$ (not) $\quad \exists x \varphi$ (there exists x s.t. φ holds)

Claim Axioms of ZFC can be formulated as $\mathcal{L}_{ZF}$-sentences!

$(\varphi \vee \psi)$ is an abbreviation of $(\neg\varphi \to \psi)$

$(\varphi \wedge \psi)$ " " $\neg(\neg\varphi \vee \neg\psi)$

$(\varphi \leftrightarrow \psi)$ is an abbreviation of $((\varphi \to \psi) \wedge (\psi \to \varphi))$

$\forall x \varphi$ is an abbreviation of $\neg \exists x \neg \varphi$

Axiom of extensionality

$$\forall x \forall y (x \equiv y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$$

Axiom of empty set

$$\exists x \forall y (\neg y \in x)$$

$x \in \emptyset$ can be seen as an abbreviation of (say) $\neg x \equiv x$

$\emptyset \in x$ is an abbreviation of

$$\exists y \forall z (\neg z \in y \wedge y \in x)$$

Pairing Axiom

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \equiv x \vee u \equiv y))$$

Axiom of the Union

$$\forall x \exists y \forall u (u \in y \leftrightarrow \exists v (v \in x \wedge u \in v))$$

Axiom of the Power set

$\forall x \exists y \forall z (z \in y \leftrightarrow \forall u (u \in z \to u \in x))$

Axiom of infinity (Exercise!)

Axiom Scheme of Separation for each $\varphi \in \mathrm{Fml}^{\mathcal{L}_{ZF}}$ with $\varphi = \varphi(x_0, \ldots x_{n-1})$

The following $\mathcal{L}_{ZF}$-formula is an axiom of ZF

$\forall x \forall x_1 \ldots \forall x_{n-1} \exists y \forall u (u \in y \leftrightarrow (u \in x \wedge \varphi(u, x_1 \ldots x_{n-1})))$

$y = \{u \in x \mid \varphi(u, x_1 \ldots x_{n-1})\}$