Exercise. Formulate other axioms of ZFC as $\mathcal{L}_{ZF}$-sentences

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Still in metamathematics

Sentential logic (Propositional Logic)

Let $A_0, A_1, \ldots$ be propositional symbols.

Formulas of Sentential logic (S.L.) are defined inductively by

atomic formulas

(1) The sequence of length 1 consisting of a propositional symbol is a formula

(2) if φ, ψ are formulas then so are $(\varphi \to \psi)$ $\neg\varphi$

(3) nothing else.

Ex. $(\neg((A_0 \to A_1) \to A_4) \to A_5)$

If φ contains (only) propositional symbols among $A_0, \ldots, A_{n-1}$ we denote $\varphi = \varphi(A_0, \ldots, A_{n-1})$

Note if e.g. $\varphi = \varphi(A_0, A_1, A_2)$ then we also have $\varphi = \varphi(A_0, A_1, A_2, A_3, A_4)$

For a formula $\varphi = \varphi(A_0, \ldots, A_{n-1})$ let

$$f_{\varphi(A_0, \ldots, A_{n-1})} : (2)^n \to 2$$

[where $2 = \{0, 1\}$ (false, true)

$$(2)^n = \underbrace{2 \times \cdots \times 2}_{n} = \{\langle i_0, \ldots, i_{n-1}\rangle \mid i_k \in 2\}$$

Actually what I am considering here is a finite table of values of the function.]

be defined recursively as follows:

(1) If φ is "A_k" for some $k < n$ then

$$f_{\varphi(A_0, \ldots, A_{n-1})}(\langle i_0, \ldots, i_{n-1}\rangle) = \begin{cases} 1 & \text{if } i_k = 1 \\ 0 & \text{if } i_k = 0 \end{cases}$$

(2) if $\varphi = (\varphi_0 \to \varphi_1)$,

[note $\varphi_0 = \varphi_0(A_0, \ldots, A_{n-1})$ and $\varphi_1 = \varphi_1(A_0, \ldots, A_{n-1})$]

then we define

$$f_{\varphi(A_0, \ldots, A_{n-1})}(\langle i_0, \ldots, i_{n-1}\rangle) = \begin{cases} 1 & \text{if } f_{\varphi_0(A_0, \ldots, A_{n-1})} = 0 \text{ or } f_{\varphi_1(A_0, \ldots, A_{n-1})} = 1 \\ 0 & \text{otherwise} \end{cases}$$

If φ is $\neg\varphi_0$ then

$$f_{\varphi(A_0,\dots,A_{n-1})}(\langle i_0,\dots,i_{n-1}\rangle) = \begin{cases} 1 & \text{if } f_{\varphi_0(A_i)}(\langle i_j,\dots\rangle)=0 \\ 0 & \text{otherwise.} \end{cases}$$

φ

A formula in p.l. is a tautology if

$\varphi=\varphi(A_0,\dots,A_{n-1})$ and $f_{\varphi(A_0,\dots,A_{n-1})}$ takes only the value 1

Note we can prove that this does not depend on the choice of $A_0,\dots,A_{n-1}$!

For a language $\mathcal{L}$ and an $\mathcal{L}$-formula φ,

φ is said to be a tautology if φ is the form

$\varphi=\psi(\varphi_0,\dots,\varphi_{n-1})$ where $\psi=\psi(A_0,\dots,A_{n-1})$

is a tautology in sentential logic and

$\varphi_0,\dots,\varphi_{n-1}$ are $\mathcal{L}$-formulas.

(we represent with $\psi(\varphi_0,\dots,\varphi_{n-1})$ the $\mathcal{L}$-formula obtained by replacing each occurrence of A_i by φ_i for all $i<n$)

Ex. $(A\to A)$ is a tautology

So $(x\in y\to x\in y)$ is a $\mathcal{L}_{ZF}$-formula which is a tautology!

- For any language $\mathcal{L}$ there is an algorithm to check if any given $\mathcal{L}$-formula is a tautology or not.

proof system

deduction system

(ingannari Logischer Kalkül)

The deduction system K^* consists of

Axioms ...

Deduction Rules:

(A) $$\frac{\varphi,\ (\varphi\to\psi)}{\psi}$$

(B) $$\frac{(\varphi\to\psi)}{(\exists x\varphi\to\psi)}$$

if φ_0 and φ_1 are $\mathcal{L}$-formulas then so is $(\varphi_0\to\varphi_1)$.

We assume that the predicate logic is introduced by using symbols $\neg\ \to\ \exists$

$(\varphi\vee\psi)$: $(\neg\varphi\to\psi)$

$(\varphi\wedge\psi)$: $\neg(\neg\varphi\vee\neg\psi)$

$\forall x\varphi$: $\neg\exists x\neg\varphi$

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where x does not appear as a free variable in ψ

If Γ is a set of $\mathcal{L}$-formulas then $\mathcal{L}$-formula φ is provable in K^* from Γ (notation: $\Gamma \vdash^{K^*} \varphi$) iff there is a sequence of $\mathcal{L}$ formulas $\varphi_0, \dots, \varphi_{m-1}, \varphi_m$ where $\varphi_m = \varphi$ each φ_i is either an axiom or a formula from Γ or there are $\varphi_{i_0}, \varphi_{i_1}$ $i_0, i_1 < i$ s.t.

$$\frac{\varphi_{i_0}, \varphi_{i_1}}{\varphi_i}$$ is of the form (A)

or there's $i_0 < i$ s.t. $\frac{\varphi_{i_0}}{\varphi_i}$ is of the form (B).

Axioms of K^*:

all $\mathcal{L}$-formulas which are tautologies

(Axiom of Equality) All formulas of the form

$$\boxed{\text{we write } A \to B \to \dots \to D \text{ for } (A \to (B \to (\dots \to D))\dots)))}$$

$x \equiv x$;

$x \equiv y \to y \equiv x$

$x \equiv y \to y \equiv z \to x \equiv z$

n-ary function symbol

$x_0 \equiv y_0 \to x_1 \equiv y_1 \to \dots \to x_{n-1} \equiv y_{n-1} \to f(x_0, \dots, x_{n-1}) \equiv f(y_0, \dots, y_{n-1})$

$x_0 \equiv y_0 \to x_1 \equiv y_1 \to \dots \to x_{n-1} \equiv y_{n-1} \to r(x_0, \dots, x_{n-1}) \to r(y_0, \dots, y_{n-1})$

n-ary relation symbol

Axiom of Substitution of variables by terms

all formulas of the form

$$\varphi(t/x) \to \exists x \varphi$$

the formula obtained by φ by substituting t in all free appearances of x in φ

where t is substitutable for x in φ

this means:

there is no subsequence of φ which is a subformula of φ of the form $\exists y \psi$ where x appears freely inside ψ and y appears in t.

$$\overbrace{- - \exists y - - x - - - -}^{\varphi}$$

Translation of K^* into (a weak) fragment of Zermelo's set theory Z

Z_0

We consider an $\mathcal{L}_{ZF}$-sentence φ a theorem in ZFC if and only if

$$ZFC \vdash^{K^*} \varphi$$

Treatment of sequences in Z_0

Working in Z_0

a (finite) sequence is a function f with $\mathrm{dom}(f) \in \omega$

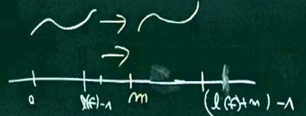
$\boxed{\text{idea: if } \mathrm{dom}(f) = n \text{ then } f \text{ is the sequence } \underline{f(0), f(1) \ldots f(n-1)}}$

For a sequence f the length of f is

$l(f) = \mathrm{dom}(f)$

If f is a sequence $(m \in \omega)$ then $\mathrm{shift}(f, m)$ is the function on $\mathrm{dom}(f)+m \setminus m$ s.t.

$\mathrm{shift}(f, m)(l) = f(k)$ where $\underbrace{l = k+m}_{k \in \mathrm{dom}(f) \text{ and}}$



For sequences f, g

$f^\frown g = f \cup \mathrm{shift}(g, l(f))$.

$\vdots$