$Z_0 \subseteq Z$

weak fragment of Zermelo's set theory

$\mathcal{L}_{\{\}}$ the language with $\in$, $\{\cdot,\cdot\}$, ϕ, $\cup, \cap, \setminus$

binary relation symbol

constant symbol

binary function symbols

$Z_{\{\}}$ the canonical conservative extension of Z obtained by

⑦ adding the axioms telling the expected interpretation of new symbols. Thus,

$$\forall x\,(x \notin \phi) \text{ is an axiom of } Z_{\{\}}.$$

We can express "all" hereditarily finite sets $\underbrace{\text{sets}}_{\text{by closed } \mathcal{L}_{\{\}}\text{-terms.}}$

In particular for any concretely given number n we have a numeral $\underline{n}$ which is a closed $\mathcal{L}_{\{\}}$-term.

$0 : \phi$

$1 : \{\phi,\phi\}$

$2 : \{\phi,\{\phi,\phi\}\}$ ...

$3 : \{\phi,\{\phi,\phi\}\} \cup \{\{\phi,\{\phi,\phi\}\}\}$

⋮

We consider : $Z_0 \subseteq Z_{\{\}}$

Z_0 should correspond to Robinson's $Q \subseteq PA$

in some cases we can take Z_0 to be a finite axiom system

finitely axiomatizable

Thm 15.1 (Representability Thm)

Let $T \supseteq Z_0$ be concretely given. (so we can consider $\ulcorner T \urcorner \subseteq V_\omega$ in Z or T)

"$S \subseteq V_\omega$" is computable (i.e. given some closed $\mathcal{L}_{\{\}}$-term t "$t \in S$" is computable

in metamathematics

$\Leftrightarrow$ there is a formula φ in $\mathcal{L}_{\{\}}$ s.t. for any closed $\mathcal{L}_{\{\}}$-term t

$$t \in S \Rightarrow T \vdash \varphi(\ulcorner t \urcorner)$$

$$t \notin S \Rightarrow T \vdash \neg\varphi(\ulcorner t \urcorner)$$

Thm 15.2 (Fixed-Point Theorem)

For T as above and for any $\mathcal{L}_{\{\}}$-formula ψ there is a formula σ in $\mathcal{L}_{\{\}}$ s.t.

with $x_0 \in \mathrm{free}(\psi)$

(1) $\mathrm{free}(\sigma) \subseteq \mathrm{free}(\psi) \setminus \{x_0\}$

(2) $T \vdash \sigma \leftrightarrow \psi(\ulcorner \sigma \urcorner / x_0)$

Thm 15.3 Let T be as above.
If T is consistent, then T is not decidable, that is, $Th(T) = \{\varphi : \mathcal{L}_{\{\in\}}\text{-sentence } T \vdash \varphi\}$ is not recursive.

Proof Suppose otherwise then by Thm 15.1 there is an $\mathcal{L}_{\{\in\}}$-formula $\psi = \psi(x_0)$ s.t.

(**) (1) $\sigma \in Th(T) \Rightarrow T \vdash \psi(\ulcorner \sigma \urcorner)$

(2) $\sigma \notin Th(T) \Rightarrow T \vdash \neg\psi(\ulcorner \sigma \urcorner)$

Notation for an expression (term, formula, proof etc.) σ:

$\ulcorner \sigma \urcorner$ denotes a closed $\mathcal{L}_{\{\in\}}$-term expressing the set corresponding to the finite sequence σ

By Thm 15.2 there is an $\mathcal{L}_{\{\in\}}$-formula σ_0 s.t.

(*) $T \vdash \sigma_0 \leftrightarrow \neg\psi(\ulcorner \sigma_0 \urcorner)$

We have $\underbrace{T \nvdash \sigma_0}_{\sigma_0 \notin Th(T)}$ [If we had $T \vdash \sigma_0$ then by (*) it follows that $T \vdash \neg\psi(\ulcorner \sigma_0 \urcorner)$

Since T is assumed to be consistent, $T \nvdash \psi(\ulcorner \sigma_0 \urcorner)$. By (**) (1) $\sigma_0 \notin Th(T)$ i.e. $T \nvdash \sigma_0$. A contradiction to]

Then by (**) (2) it follows that $T \vdash \neg\psi(\ulcorner \sigma_0 \urcorner)$. By (*) $T \vdash \sigma_0$

A contradiction to . $\boxtimes$

Let T be as before. For $\mathcal{L}_{\{\in\}}$-formula φ let $W_T(\ulcorner \varphi \urcorner)$ denote the length of the shortest proof of φ from T in K^*

We consider a proof of K^* as a sequence of $\mathcal{L}$-formulas as a long sequence of symbols "shortest" refers to the length of such a sequence.

or undefined if $T \nvdash \varphi$

Predicate $W_T(\ulcorner\varphi\urcorner) < n$ is recursive.

[If φ and n are given we can enumerate all proofs in $\mathcal{L}_{\{\in\}}$ of length $< n$ (there are only finitely many proofs!) We can decide $W_T(\ulcorner\varphi\urcorner) < n$ by checking all these proofs!]

Thm 15.4 (Ehrenfeucht–Mycielski)

For any theory T as above (in $\mathcal{L}_{\{\in\}}$)

If an $\mathcal{L}_{\{\in\}}$-sentence φ_0 is independent from T

Then there is no recursive $S : \mathbb{N} \to \mathbb{N}$

s.t.

$$W_T(\tau) \leq S(W_{T+\varphi_0}(\tau))$$

for all $\mathcal{L}_{\{\in\}}$-sentence τ provable in T

That is, for any recursive $S : \mathbb{N} \to \mathbb{N}$

there is a theorem τ of T ($T \vdash^{K^*} \tau$)

but

$$W_T(\tau) \gg S(W_{T+\varphi_0}(\tau)) \quad \text{speed up}$$

$$\gg W_{T+\varphi_0}(\tau)$$

For any proper extension T' of ZFC (Z_0)

there is some theorem (τ) of ZFC (Z_0) which can be

proved much easier in T' than in ZFC!

$Z_0 \subseteq T$

If $Z_0 \subseteq T$ $\quad T \vdash \mathrm{cons}(Z_0)$

$\quad ZFC \subseteq T$ $\quad T \vdash \mathrm{cons}(ZFC)$

then there is a much stronger speed up

Gödel's speedup theorem