$\mathcal{L}'_{\{\}} = \{ \underbrace{\in, \{\cdot,\cdot\}, \phi, \cup, \cap, \setminus}_{6 \text{ symbols}} \}$

$Z_{\{\}}$ The canonical and conservative extension of $\underset{\substack{\uparrow \\ \text{Zermelo's Axiom System}}}{Z}$ by adding the supposed definitions of new symbols.

We can denote all concrete elements of V_ω as a closed $\mathcal{L}_{\{\}}$-term

$1 = \{\phi\} = \{\phi, \phi\}$

$2 = \{\phi, \{\phi\}\}$

$\vdots$

$\{1\} \cup \{2\} = \{\{\phi,\phi\}, \{\phi, \{\phi\}\}\}$

$= \{\{\phi,\phi\}\} \cup \{\{\phi,\{\phi\}\}\}$

$\vdots$

In the following

$T \supseteq Z_0$

$\uparrow$

minimal subtheory of $Z_{\{\}}$ strong enough to have Fixed Point Th (Th 15.2)

since T is concretely given so that we have

$\ulcorner T \urcorner$ in Z_0 (or in T)

$\rotatebox{90}{$\subseteq$}$

V_ω

<u>Thm 15.1</u> (Representability)

"$S \subseteq V_\omega$" is recursive (i.e. "$a \in S$" is computable)

$\Leftrightarrow$ there is an $\mathcal{L}_{\{\}}$ formula $\psi = \psi(a)$ s.t.

"$a \in S$" $\Rightarrow Z_0 \vdash \psi(a)$

"$a \notin S$" $\Rightarrow Z_0 \vdash \neg\psi(a)$ $\boxtimes$

<u>Thm 15.2</u> (Fixed Point Theorem, Diagonal Lemma, R. Carnap. 1934)

For any $\mathcal{L}_{\{\}}$-formula ψ (with $x_0 \in \mathrm{free}(\psi)$) there is an $\mathcal{L}_{\{\}}$-formula σ s.t.

(1) $\mathrm{free}(\sigma) \subseteq \mathrm{free}(\psi) \setminus \{x_0\}$

(2) $T \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner / x_0)$

$\boxed{\text{Numeral of } \sigma}$ $\uparrow$ the $\mathcal{L}_{\{\}}$-term of the element in V_ω corresponding to σ

Thm 15.3 (If T is consistent, then)

T is not decidable (i.e. $Th(T)$ is not recursive) $\Leftrightarrow$ there is no $\mathcal{L}_{\{\}}$-formula ψ s.t.

Thm 15.1

$\varphi \in Th(T) \Rightarrow Z_0 \vdash \psi(\ulcorner \varphi \urcorner)$

$\varphi \notin Th(T) \Rightarrow Z_0 \vdash \neg\psi(\ulcorner \varphi \urcorner)$

Thm 15.4 (Speedup Theorem, Ehrenfeucht and Mycielski, 1971)

Suppose (T is consistent and) φ_0 ($\mathcal{L}_{\{\}}$-sentence) is independent from T. Then there is no recursive $S: \mathbb{N} \to \mathbb{N}$ s.t.

① $W_T(\tau) \leq S(W_{T+\varphi_0}(\tau))$

for all $\mathcal{L}_{\{\}}$-sentence τ with $T \vdash \tau$

---

for any $T \supseteq Z_0$ we denote

$W_T(\tau)$ = the smallest (possible) length of a proof (of τ) as a sequence of symbols!

Note that for each $n > 12$ there are only less than n^n proofs!

$\underbrace{\forall, \neg, \exists, (,),}_{6 \text{ symbols}}$

Lemma 16.1 If $Th(T+\neg\varphi_0)$ is undecidable (*) Then there is no recursive $F: \mathbb{N} \to \mathbb{N}$ s.t. (+)

② $W_T(\tau) \leq F(W_T(\varphi_0 \to \tau))$

for all $\underbrace{\tau}_{\mathcal{L}_{\{\}}\text{-sentence}}$ s.t. $T \vdash \tau$

$\boxed{T' \vdash \mathrm{Consis}(\ulcorner T \urcorner)}$

(*) By Th 15.3, This is the case if and only if $T+\neg\varphi_0$ is consistent.

---

proof of the Lemma 16.1

Note that $T+\neg\varphi_0 \vdash^{k^*} \sigma$

$\Leftrightarrow T \vdash^{k^*} \varphi_0 \vee \sigma$

$\Updownarrow$

$\neg\varphi_0 \to \sigma$

by the Deduction Theorem. (Theorem 11.2)

Suppose, toward a contradiction, that there is rec. $F : \mathbb{N} \to \mathbb{N}$ with ②

Wlog we may assume that F is increasing

(F can be replaced by $n \longmapsto \max\{F(k) \mid k \leq n\}$)

$$W_T(\varphi_0 \vee \sigma) \leq F(W_T(\varphi_0 \to (\varphi_0 \vee \sigma)))$$

holds for any σ with $T \vdash \sigma$. (*)

We have a recursive function $g : \mathbb{N} \to \mathbb{N}$

$$\underbrace{W_T(\varphi_0 \to (\varphi_0 \vee \sigma))}_{\text{tautology}} \leq g(\ell(\sigma))$$

length of the sequence

$$(*) \leq F(\underbrace{g(\ell(\sigma))})$$

For each $\mathcal{L}_{\{\}}$-sentence σ we can search in the finite collection of the proofs in $\mathcal{L}_{\{\}}$ of length $\leq F(g(\ell(\sigma)))$ to decide if $(\varphi_0 \vee \sigma)$ has a proof in T. This is a contradiction to the assumption of the Lemma. $\boxtimes$

Lemma 16.2 There is a rec. function $R : \mathbb{N} \to \mathbb{N}$ and (+) s.t.

$$W_{T+\varphi_0}(\tau) \leq R(W_T(\varphi_0 \to \tau))$$

Proof. By the proof of the Deduction Theorem. $\boxtimes$

Proof of Thm 15.4

Suppose, toward a contradiction, that there is a rec. $S : \mathbb{N} \to \mathbb{N}$ with ①

Wlog S is increasing.

$$W_T(\tau) \leq S(W_{T+\varphi_0}(\tau))$$

$$\leq S(R(\underbrace{W_T(\varphi_0 \to \tau)}_{\text{recursive!}}))$$

Lemma 16.2

A contradiction to Lemma 16.1.

Note that $T + \neg\varphi_0$ is consistent (and), hence undecidable by Theorem 15.3. $\boxtimes$