Prop 14.2 (ZF)

In W_1, let $\mathfrak{A} = \langle A, E, \dots \rangle$ (well-founded in W_1) in countable language L, $L \in W_0$. If φ is an L-sentence with $\mathfrak{A} \models \varphi$ then there is $\mathfrak{B} = \langle B, E', \dots \rangle$, $\mathfrak{B} \in W_0$ s.t. $\mathfrak{B} \models \varphi$ and E' is well founded in W_0.

$W_0 \subseteq W_1$ are inner models(*) of enough portion of ZF (finite fragment of ZF)

(*) $W \subseteq V$ is an inner model if W is transitive (i.e. $\forall x \in W\ \forall y \in x\ (y \in W)$), $On \subseteq W$ and satisfies the amount of set theory mentioned.

Objective: To prove the following Thm:

Thm 14.3 (Lévy–Shoenfield Absoluteness Lemma)

For $W_0 \subseteq W_1$ as above, Σ_1-formula $\varphi(\bar{x})$ and $\bar{c} \in H(\aleph_1)^{W_0} \subseteq H(\aleph_1)^{W_1}$:

$$W_1 \models \varphi(\bar{c}) \iff W_0 \models \varphi(\bar{c})$$

Proof of Prop 14.2 Wlog we may assume $\mathfrak{A}$ is relational and φ is of the form $\forall \bar{x}\, \exists \bar{y}\, \varphi_0(\bar{x}, \bar{y})$ (φ_0 quantifier free)

(Skolem Normal Form Thm w/o AC)

Σ_1-formulas are of the form (formulas)

$\underbrace{\exists \bar{x}}_{= \exists x_1 \exists x_2 \dots \exists x_k} \varphi_0(\bar{x}, \bar{y})$ where φ_0 is Δ_0 (bounded) if all the quantifiers of φ_0 appear in the form $(\forall x \in y)$ $(\exists x \in y)$.

Lemma 17.1 If $\varphi(\bar{x})$ is Δ_0 then for any transitive $M \subseteq N$, $\bar{a} \in M$

$$(M, \in) \models \varphi(\bar{a}) \iff (N, \in) \models \varphi(\bar{a})$$

φ is absolute $\boxtimes$

Cor 17.2 For Σ_1-formula $\varphi = \varphi(\bar{x})$, $M \subseteq N$ transitive and $\bar{a} \in M$

$$M \models \varphi(\bar{a}) \Rightarrow N \models \varphi(\bar{a})$$

φ is upward absolute

In W_0,

$\mathbb{P} = \{ \langle \mathfrak{B}, f \rangle \mid \mathfrak{B} = \langle B, E^{\mathfrak{B}}, \dots \rangle$ is a finite L-structure, $B \in \omega$, $f : B \to On$, (**) for any $a, b \in B$ if $a E^{\mathfrak{B}} b$ then $f(a) < f(b) \}$

(*) well orderable

Still in W_0. For $\langle \mathfrak{B}, f\rangle$ $\langle \mathfrak{B}', f'\rangle$

$\langle \mathfrak{B}', f'\rangle <_{\mathbb{P}} \langle \mathfrak{B}, f\rangle \iff$ ① $\mathfrak{B}' \supseteq \mathfrak{B}$, ② $f' \supseteq f$

③ For any $\bar{a} \in B$ there are $\bar{b} \in B'$ s.t. $\mathfrak{B}' \models \varphi_0(\bar{a}, \bar{b})$.

with $\mathfrak{B} = \langle B, E, \dots\rangle$, $\mathfrak{B}' = \langle B', E', \dots\rangle$

In W_0,

If $\leq_{\mathbb{P}}$ is non-well-founded then by (*) there is an infinite $\leq_{\mathbb{P}}$-decreasing chain, say

$$\langle \mathfrak{B}_0, f_0\rangle \geq_{\mathbb{P}} \langle \mathfrak{B}_1, f_1\rangle \geq_{\mathbb{P}} \langle \mathfrak{B}_2, f_2\rangle \geq_{\mathbb{P}} \cdots$$

If we let $\mathfrak{B} = \bigcup_{n\in\omega} \mathfrak{B}_n$ with $B = \bigcup_{n\in\omega} B_n$ and $f = \bigcup_{n\in\omega} f_n$. We have $f : B \to On$ and f satisfies (**). Thus $E^{\mathfrak{B}}$ is well-founded. $\mathfrak{B} \models \varphi$ by ②. Thus this $\mathfrak{B}$ is as desired.

i.e. by (*) we can claim this without the help of AC!

Lemma 15.3 (ZF) A binary relation E on a set B is well-founded iff there is a mapping $f : B \to On$ s.t., for any $a, b \in E$, if $a E b$ then $f(a) < f(b)$.

See also the corrections in the bbd images of 15th lecture.

We have to show that $\leq_{\mathbb{P}}$ is non-well founded (in W_0). By (*) it is enough to prove that $\leq_{\mathbb{P}}$ is non-well founded in W_1.

In W_1, let $f^* : A \to On$ be s.t. for any $a, b \in A$ if $a E b$ then $f^*(a) < f^*(b)$ (there is such f^* by Lemma 15.3.)

Let $Q \subseteq \mathbb{P}$ be defined by (†)

Note that $\mathbb{P}$ is also a (definable) class in W_1

(†)

$$Q = \{\langle \mathfrak{B}, f\rangle \in \mathbb{P} \mid \text{there is an embedding } i : \mathfrak{B} \to \mathfrak{A} \text{ s.t. } f \circ i^{-1} = f^* \restriction i''B\}$$

Then $Q \subseteq \mathbb{P}$ doesn't have minimal element wrt $\leq_{\mathbb{P}}$. Thus $W_1 \models \leq_{\mathbb{P}}$ is not well founded. ⊠

Proof of Lemma 15.3 (without AC).

Suppose that E is a wellfounded relation on B.

For each $\alpha \in On$ let B_α be defined recursively s.t.

$$B_\alpha = \text{minimal elements of } B \setminus \bigcup_{\beta < \alpha} B_\beta .$$

$\subseteq B$

The $f : B \to On$ defined by

$f(b) =$ the α s.t. $b \in B_\alpha$

is well-defined since $B_\alpha, \alpha \in On$ are pairwise disjoint (if $\alpha < \beta$ then $B_\beta \subseteq B \setminus \bigcup_{\gamma < \beta} B_\gamma$ ← disjoint from B_α)

For any $b \in B$ there is some $\alpha \in On$ s.t. $b \in B_\alpha$

[Otherwise $\underbrace{B \setminus \bigcup_{\alpha \in On} B_\alpha}_{\text{this set}} \neq \emptyset$ and has $\underbrace{\text{no minimal}}_{E\text{-}}$ element.

A contradiction to the assumption that E is well-founded.]

Suppose now there is $f : B \to On$ s.t. $\forall a, b \in B$ if $a E b$ then $f(a) < f(b)$.

For any $B_0 \subseteq B$, $B_0 \neq \emptyset$, let $S = f''B_0$ and let $\alpha_0 = \min S$. Then any element (b) of B_0 with $f(b) = \alpha_0$ is minimal wrt E.

Thus E is well-founded on B. ☒

Prop 17.3 (A slight improvement of Prop 14.2)

Let $W_0 \subseteq W_1$ as before.

In W_1, let $\mathfrak{A} = \langle A, E \dots \rangle$ an L-structure, L countable, $L \in W_0$,

E is well founded in W_1

Let $\langle \varphi_n \mid n \in \omega \rangle \in W_0$ be a sequence of L-sentences and $\mathfrak{A} \models \{\varphi_n \mid n \in \omega\}$

Then there is $\mathfrak{B} = \langle B, E' \dots \rangle \in W_0$ s.t.

$W_0 \models E'$ is well-founded and $\mathfrak{B} \models \{\varphi_n \mid n \in \omega\}$

Cor 17.4

For $W_0 \subseteq W_1$ as before

$C \in W_0$, $W_0 \models C$ is countable and transitive

If $M \in W_1$, $M \supseteq C$, φ an $\mathcal{L}_\in$-formula,

$M \models$ extensionality $+ \varphi(c_0, \dots, c_{n-1})$ for some $c_0, \dots, c_{n-1} \in C$

Then there is $N \in W_0$ s.t. $C \subseteq N$

$N \models$ extensionality $+ \varphi(c_0, \dots, c_{n-1})$

N is transitive

Proof of Theorem 14.3 from Cor 17.4

Suppose $a_0, \ldots, a_{m-1} \in H(\aleph_1)^{W_0}$ and let $C = \mathrm{tcl}(a_0, \ldots, a_{m-1})$. Then C is countable.

If $W_0 \models \varphi(a_0, \ldots, a_n)$ then $W_1 \models \varphi(a_0, \ldots, a_n)$ by Lemma 17.2. $(W_1 \in \Gamma(W_0)^2)$

Suppose $W_1 \models \varphi(a_0, \ldots, a_n)$ then by Lévy-Montague Reflection Theorem there is $\alpha \in \mathrm{On}$ (limit) s.t. $V_\alpha \models \varphi(a_0, \ldots, a_n)$ + extensionality

By Cor 17.4 There is (transitive) $N \in W_0$ s.t. $C \subseteq N$ and $N \models \varphi(a_0, \ldots, a_m)$ + extensionality

By Lemma 17.2 it follows that $W_0 \models \varphi(a_0 \ldots a_{m-1})$ ($\subseteq N$)

$\boxtimes$