$Z_0 \subseteq Z_{\{\}}$ minimal $\mathcal{L}_{\{\}}$-theory containing everything we need in the following discussion.

$T \supseteq Z_0$ concretely given $\mathcal{L}_{\{\}}$-theory

we have the definable set

$$\ulcorner T \urcorner \subseteq \underbrace{{}^{\omega>}(\omega^2)}_{Z_0 \text{ proves this is a set}} \subseteq V_\omega$$

In T (or in Z_0)

V_ω is a definable class defined by

"$x \in V_\omega$" $\leftrightarrow (\exists n \in \omega)($"$x \in V_n$"$)$

$Term_{\mathcal{L}_{\{\}}}$ the $\underbrace{\text{set}}_{\text{definable}}$ of all $\underbrace{\text{terms}}_{\mathcal{L}_{\{\}}}$

$Fml_{\mathcal{L}_{\{\}}}$ $\mathcal{L}_{\{\}}$-formulas

$ClTerm_{\mathcal{L}_{\{\}}}$ the definable set of all closed $\mathcal{L}_{\{\}}$-terms

Note, for all (concretely given) closed $\mathcal{L}_{\{\}}$-term t

$$Z_0 \vdash t \in V_\omega$$

Lemma 17.1

In Z_0 we have that letting $\mathcal{V}_\omega$ be the $\mathcal{L}_{\{\}}$-structure

$$\mathcal{V}_\omega = \langle V_\omega, \in, \emptyset, \{\cdot,\cdot\}, \dots \rangle$$

$$Z_0 \vdash \forall t \in ClTerm_{\mathcal{L}_{\{\}}} \left(t^{\mathcal{V}_\omega} \in V_\omega \right)$$

(trivial)

$$Z_0 \vdash (\forall x \in V_\omega)(\exists t \in ClTerm_{\mathcal{L}_{\{\}}})(t^{\mathcal{V}_\omega} \equiv x)$$

For a metamathematical object O we denote with $\ulcorner O \urcorner$ $\mathcal{L}_{\{\}}$-term of the corresponding object in set theory

For example, if variables $x_0, x_1, \dots$ are "coded" as elements of $\omega \times \{0\}$

see http://fuchino.ddo.jp/kobe/bbd/logic-11-2019-05-20.pdf

then

$$\ulcorner x_5 \urcorner = \{\{\underline{5}, \underline{5}\}, \{\underline{5}, \underline{0}\}\}$$

where for $n \in \mathbb{N}$ $\underline{n}$ denotes the numeral for n which is a closed $\mathcal{L}_{\{\}}$-term defined inductively by $\underline{0} = \emptyset$ $\underline{n+1} = \underline{n} \cup \{\underline{n}, \underline{n}\}$

$\ulcorner \ulcorner T \urcorner \urcorner$ not an $\mathcal{L}_{\{\}}$-term but rather a definition of a subset of ω^2 (ω^2) (in $\mathcal{L}_{\{\}}$)

In Z_0 $(\subseteq T)$,

for $x \in V_\omega$ let $\llcorner x \lrcorner$ be the standard element of $\mathrm{ClTerm}_{\mathcal{L}_{\{\}}}$ s.t.

$$\llcorner x \lrcorner^{V_\omega} = x$$

Subst: $\mathrm{Fml}_{\mathcal{L}_{\{\}}} \times \omega \times \mathrm{Trm}_{\mathcal{L}_{\{\}}} \longrightarrow \mathrm{Fml}_{\mathcal{L}_{\{\}}}$

the s.t.

$\mathrm{Subst}(\varphi, n, t)$ = the formula obtained from φ by replacing all free appearances of x_n $(= \langle n, 5\rangle)$ with the term t

Lemma 17.2
For any concretely given $\mathcal{L}_{\{\}}$-formula φ and sequence t of symbols and number k

note that $\ulcorner t \urcorner$, $\underline{k}$ are $\mathcal{L}_{\{\}}$-terms.

We have

(1) $Z_0 \vdash \ulcorner \varphi(\ulcorner t \urcorner / x_n) \urcorner \equiv \mathrm{Subst}(\ulcorner \varphi \urcorner, \underline{n}, \llcorner \ulcorner t \urcorner \lrcorner)$

(2) $Z_0 \vdash \ulcorner \varphi(\underline{k} / x_n) \urcorner \equiv \mathrm{Subst}(\ulcorner \varphi \urcorner, \underline{n}, \llcorner \underline{k} \lrcorner)$

Theorem 15.2 (Diagonal Lemma)

actually we don't need this restriction!

For any $\mathcal{L}_{\{\}}$-formula ψ (with $x_0 \in \mathrm{free}(\psi)$) there is an $\mathcal{L}_{\{\}}$-formula σ s.t.

(1) $\mathrm{free}(\sigma) \subseteq \mathrm{free}(\psi) \setminus \{x_0\}$

(2) $Z_0 \vdash \sigma \leftrightarrow \psi(\ulcorner \sigma \urcorner / x_0)$.

Proof If $x_0 \notin \mathrm{free}(\psi)$ Then $\sigma = \psi$ will do.

So assume $x_0 \in \mathrm{free}(\psi)$.

In Z_0, let

$$f^* : ({}^{\omega>}(\omega^2))^2 \to {}^{\omega>}(\omega^2)$$ be defined

by

(†) $$f^*(s,t) = \begin{cases} u & \text{if } s \in \mathrm{Fml}_{\mathcal{L}_{\{\}}} \text{ and} \\ & u \equiv \mathrm{Sbst}(s, \underline{0}, \llcorner t \lrcorner) \\ \emptyset & \text{otherwise} \end{cases}$$

(such in meta math)

Let k be the first index s.t. x_k does not appear in ψ

Let s^* be the (concretely given) (closed) $\mathcal{L}_{\{\}}$-term

(*) $\ulcorner \forall x_k \, (f^*(x_0, x_0) \equiv x_k \to \psi(x_k/x_0)) \urcorner$

Let σ be the $\mathcal{L}_{\{\}}$-sentence

(**) $\forall x_k \, (f^*(s^*, s^*) \equiv x_k \to \psi(x_k/x_0))$

σ satisfies (1)

we show that σ is as desired.

We have, by (†),

(‡) $$Z_0 \vdash f^*(s^*, s^*) \equiv \underbrace{\ulcorner \forall x_k \, (f^*(s^*, s^*) \equiv x_k \to \psi(x_k/x_0)) \urcorner}_{=\ulcorner \sigma \urcorner \text{ by } (**)}$$

Thus

$$Z_0 \vdash (\sigma \to \psi(\ulcorner \sigma \urcorner / x_0))$$

On the other hand.

noting that

$$(\psi(\ulcorner \sigma \urcorner / x_0) \to (f^*(s^*, s^*) \equiv x_k \to \psi(\ulcorner \sigma \urcorner / x_0)))$$

is a tautology $(A \to (B \to A))$

Since x_k does not appear in ψ

$$Z_0 \vdash (\psi(\ulcorner \sigma \urcorner / x_0) \to \forall x_k (f^*(s^*, s^*) \equiv x_k \to \psi(\ulcorner \sigma \urcorner / x_0)))$$

By (‡) it follows that

$$Z_0 \vdash (\psi(\ulcorner \sigma \urcorner / x_0) \to \underbrace{\forall x_k (f^*(s^*, s^*) \equiv x_k \to \psi(x_k / x_0))}_{\sigma})$$

Thus

$$Z_0 \vdash (\psi(\ulcorner \sigma \urcorner / x_0) \to \sigma).$$ ⊠