Descriptive Set-theoretic Context

A. Kanamori: The Higher Infinite

$$\mathcal{A}^2 = \langle \omega \cup {}^\omega\omega, \omega, {}^\omega\omega, ap, +, \times, exp, <, 0, 1 \rangle$$

interpretation of unary relation symbols $\underline{\omega}$ $\underline{{}^\omega\omega}$

[$\mathcal{A}$ is for arithmetic
2 is for second order]

${}^\omega\omega \approx \mathbb{R}$

${}^\omega\omega \cong \mathbb{R}\setminus\mathbb{Q}$ (topologically)

$$ap(f,n) = \begin{cases} f(n) & \text{if } f \in {}^\omega\omega,\ n \in \omega \\ 0 & \text{otherwise} \end{cases}$$

$$n \underset{\times}{+} m = \begin{cases} n \underset{\times}{+} m & \text{if } n, m \in \omega \\ 0 & \end{cases}$$

$$exp(m,n) = \begin{cases} m^n & \text{if } m, n \in \omega \\ 0 & \end{cases}$$

$<$ canonical ordering on ω. $0, 1 \in \omega$

We assume that $ap, +, \times, exp, <, 0, 1$ are interpretations of symbols $\underline{ap}, \underline{+}, \underline{\times}, \underline{exp}, \underline{<}, \underline{0}, \underline{1}$ respectively. $\mathcal{L}_{\mathcal{A}^2}$ the language of $\mathcal{A}^2$.

We want to study the definability of $A \subseteq {}^k\omega \times {}^l({}^\omega\omega)$ in $\mathcal{A}^2$.

A is definable by a formula φ (possibly with parameters $\vec{a}$) $\Leftrightarrow$ $\langle \vec{m}, \vec{f} \rangle \in A \Leftrightarrow \mathcal{A}^2 \models \varphi(\vec{m}, \vec{f}, \vec{a})$

$\omega \cup {}^\omega\omega$

$\Leftrightarrow$ (*) $A = \{ \langle \vec{m}, \vec{f} \rangle \mid \mathcal{A}^2 \models \varphi(\vec{m}, \vec{f}, \vec{a}) \}$

If M an inner model of some fragment of ZF and if the fragment proves (*) and $\vec{a} \in M$

Then we write.

$$A^M = \{ \langle \vec{m}, \vec{f} \rangle \in ({}^k\omega \times {}^l({}^\omega\omega))^M \mid M \models \mathcal{A}^2 \models \varphi(\vec{m}, \vec{f}, \vec{a}) \}$$

In first order logic in $\mathcal{L}_{\mathcal{A}^2}$ $(\forall x)(\underline{\omega}(x) \to \ldots)$

$(\exists x)(\underline{\omega}(x) \wedge \ldots)$ are abbreviated as $\underbrace{\forall^0 x(\ldots) \text{ and } \exists^0 x(\ldots)}_{\text{number quantifiers}}$

$\forall x(\underbrace{\psi_\omega}(x) \to \dots)$ $\exists x(\underbrace{\psi_\omega}(x) \wedge \dots)$

are abbreviated as $\underbrace{\forall^1 x(\dots)\ \exists^1 x(\dots)}_{\text{function quantifiers}}$

An $\mathcal{L}_{\mathcal{A}}$-formula φ is said to be bounded if φ is in a prenex form only with quantifiers of the form $(\forall^0 x < t)$ or $(\exists^0 x < t)$ where t is any $\mathcal{L}_{\mathcal{A}}$-term.

$A \subseteq {}^k\omega \times {}^l({}^\omega\omega)$ is arithmetical if it is defined by a formula φ in prenex form where all quantifiers are number quantifiers. (We call such φ also arithmetical)

If A is definable by a formula φ in prenex form only with number quantifiers with parameters $\vec{a} \in {}^\omega\omega$ we say A is arithmetical in $\vec{a}$. (We call such φ also arithmetical in $\vec{a}$)

$A \subseteq {}^k\omega \times {}^l({}^\omega\omega)$ is Δ^0_0 if A is definable by a bounded formula. A is Δ^0_0 in $\vec{a}$ … is also defined similarly.

A is Σ^1_0 / Π^1_0 (in $\bar{a}$) if there is a bounded formula $\varphi(\vec{x}, \vec{y}, \vec{X})$ s.t. A is defined by $\exists \vec{x}\, \varphi(\vec{x}, \vec{y}, \vec{X})$ $\frown \bar{a}$

$\forall \vec{x}\, \varphi(\vec{x}, \vec{y}, \vec{X})$ $\frown \bar{a}$

$\vec{a} \in A$ is an abbreviation of $\underbrace{a_0 \dots a_{n-1}}_{\vec{a}} \in A$

If $\vec{y}$ is $y_0, \dots, y_{n-1}$ then $\forall \vec{y}$ denotes $\forall y_0 \dots \forall y_{n-1}$

$\exists \vec{y}$ " $\exists y_0 \dots \exists y_{n-1}$

This lazy notation was possibly introduced in Shelah: Classification Theory

A is Σ^1_n $(n \geq 1)$ / Π^1_n $(n \geq 1)$ (in $\bar{a}$) if there is an arithmetical φ (in $\bar{a}$) s.t. $\exists^1 \vec{x}_1 \forall^1 \vec{x}_2 \cdots Q \vec{x}_n \varphi$ defines A

$\forall^1 \vec{x}_1 \exists^1 \vec{x}_2 \cdots Q \vec{x}_n \varphi$

Theorem 18.1 (ZF) For $n \in \mathbb{N}$, for $\vec{a} \in {}^\omega\omega$ and $A \subseteq {}^k\omega \times {}^l({}^\omega\omega)$ tfae:

(1) A is $\Sigma^1_{n+1}(\vec{a})$

(2) A is definable by a Σ_n-formula φ (Lévy hierarchy of formulas) in $\mathcal{L}_\in$ s.t. $A = \{\langle \vec{m}, \vec{f}\rangle \in {}^k\omega \times {}^l({}^\omega\omega) \mid \mathcal{H}(\omega_1) \models \varphi(\vec{m}, \vec{f}, \bar{a})\}$

$\mathcal{H}(\omega_1) = \langle \mathcal{H}(\omega_1), \in \rangle$

Cor 17.4 For $W_0 \subseteq W_1$ (transitive models of (fragment) of ZF)

$C \in W_0$, $W_0 \models C$ is countable

$c_1 \dots c_{m+1} \in C$

If $M \in W_1$, $M \models$ Extensionality

$W_1 \models M \models \varphi(c_1, \dots, c_{m+1})$

$\mathcal{L}_\in$-formula

Then there is

$N \in W_0$ s.t. N is transitive

(countable) $C \subseteq N$

$N \models \varphi(c_1, \dots, c_{m+1})$

extensionality +

Thm 18.2 (ZF) (Shoenfield's Absoluteness Thm)

For any transitive (set or class) models $M_0 \subseteq M_1$ of (large enough fragment of) ZF $\vec{a} \in ({}^\omega\omega) \cap M_0$, $M_0, M_1 \models AC_\omega({}^\omega\omega)$

Countable choice for (countable) sequence of subsets of ${}^\omega\omega$

If A is $\Sigma^1_2(a)$ then A is absolute between M_0 and M_1

(i.e. $A^{M_0} = A^{M_1} \cap M_0$)

Proof. Let φ be the Σ_1 formula s.t.

$$A = \{\langle \vec{m}, \vec{f} \rangle \in {}^k\omega \times {}^l({}^\omega\omega) \mid \langle H(\aleph_1), \in \rangle \models \varphi(\vec{m}, \vec{f}, \vec{a})\}$$

(We use Theorem 18.1 and assume that the fragment of the ZF valid in M_0 and M_1 contains all the axioms to prove Theorem 18.1 for this setting)

Let $\varphi = \exists x\, \varphi_0(\vec{x}, \vec{y}, \vec{z}, x)$ (φ_0 bounded.)

If $\langle \vec{m}, \vec{f} \rangle \in A^{M_0}$, then

$M_0 \models \langle H(\aleph_1), \in \rangle \models \varphi(\vec{m}, \vec{f}, \vec{a})$

$\varphi(\vec{m}, \vec{f}, \vec{a}) = \exists x\, \varphi_0(\vec{m}, \vec{f}, \vec{a}, x)$

Then there is some $b \in H(\aleph_1)^{M_0}$ s.t.

$M_0 \models \langle \quad \rangle \models \varphi_0(\vec{m}, \vec{f}, \vec{a}, b)$

$M_1 \models \langle \quad \rangle \models$ " "

Hence $\langle \vec{m}, \vec{f} \rangle \in A^{M_1}$.

If $\langle \vec{m}, \vec{f} \rangle \in A^{M_1}$ then

$M_1 \models \langle H(\aleph_1), \in \rangle \models \exists x\, \varphi_0(\vec{m}, \vec{f}, \vec{a}, x)$

Suppose $\langle \vec{m}, \vec{f} \rangle \in M_0$

Then (in M_1) there is $b \in H(\aleph_1)$ s.t.

$\langle b, \in \rangle \prec \langle H(\aleph_1), \in \rangle$ (Löwenheim-Skolem)

$\vec{a} \in b$, $\vec{m}, \vec{f} \in b$.

Thus $b \models \exists x\, \varphi_0(\vec{m}, \vec{f}, \vec{a}, x)$.

Apply Cor 17.4

We find $N \in W_0$ s.t. N is transitive countable extensional

$N \ni \vec{a}, \vec{m}, \vec{f}$ and

$M_0 \models N \models \exists x\, \varphi_0(\vec{m}, \vec{f}, \vec{a}, x)$

In M_0, $N \subseteq H(\aleph_1)$. Since φ_0 is Δ_0

we have $H(\aleph_1) \models \exists x\, \varphi_0(\vec{m}, \vec{f}, \vec{a}, x)$

Thus $\langle \vec{m}, \vec{f} \rangle \in A^{M_0}$. ☒