$\ulcorner \cdot \urcorner$ corner quote of $\cdot$

$\llcorner \cdot \lrcorner$ reverse corner quote of $\cdot$.

metamathematical closed term in $\mathcal{L}_{\{\}}$ — closed $\mathcal{L}_{\{\}}$-term expressing the sequence t

$$t \longmapsto \ulcorner t \urcorner$$

$Z_0 \vdash t \in V_\omega$

$Z_0 \vdash \ulcorner t \urcorner \in {}^{\omega>}(\omega^2)$

$Z_0 \vdash \ulcorner t \urcorner^{V_\omega} \equiv t$

$\mathcal{L}_{\{\}}$ all braces $\in, \phi, \{..\}, \cup, \ldots$

In Z_0

$${}^{\omega>}(\omega^2) \supseteq ClTerm_{\mathcal{L}_{\{\}}} \qquad V_\omega$$

$$\tau \longmapsto (\tau)^{V_\omega}$$

$$\llcorner a \lrcorner \longleftarrow a$$

Fix $\sqsubset$ ordering on ${}^{\omega>}(\omega^2)$ order type ω in Z_0 but definable concretely calculable.

$\llcorner a \lrcorner = \sqsubset$-minimal closed $\mathcal{L}_{\{\}}$-term τ s.t. $\tau^{V_\omega} = a$

modify the def of $\ulcorner t \urcorner$ for closed $\mathcal{L}_{\{\}}$-term t and $\underline{n}$ for $n \in \mathbb{N}$ s.t. they are also "minimal" wrt metamathematical counterpart of $\sqsubset$ (on sequences of symbols)

Thus we obtain:

Lemma 20.1

(1) $Z_0 \vdash \llcorner t \lrcorner \equiv \ulcorner t \urcorner$

(2) $Z_0 \vdash \llcorner \ulcorner s \urcorner \lrcorner = \ulcorner \ulcorner s \urcorner \urcorner$

(3) $Z_0 \vdash (\ulcorner \underline{n} \urcorner)^{V_\omega} \equiv \underline{n}$

(4) $Z_0 \vdash \llcorner \underline{n} \lrcorner \equiv \ulcorner \underline{n} \urcorner$

note that they are closed $\mathcal{L}_{\{\}}$ terms

For the modified definition of $\underline{n}$ we have

(1) $Z_0 \vdash \underline{0} \equiv \phi$

(2) $Z_0 \vdash \underline{n+1} \equiv \underline{n} \cup \{\underline{n}, \underline{n}\}$

for all $n \in \mathbb{N}$

(5) $Z_0 \vdash \mathrm{Subst}(\ulcorner \varphi \urcorner, \underline{m}, \llcorner \ulcorner p \urcorner \lrcorner)$
$\equiv \ulcorner \varphi(\ulcorner p \urcorner / x_m) \urcorner$

(6) $Z_0 \vdash \mathrm{Subst}(\ulcorner \varphi \urcorner, m \llcorner \underline{k} \lrcorner)$
$\equiv \ulcorner \varphi(\underline{k} / x_m) \urcorner$

proof of (6)

$$Z_0 \vdash \mathrm{Subst}(\ulcorner \varphi \urcorner, \underline{m}, \ulcorner p \urcorner) \equiv \ulcorner \varphi(p / x_m) \urcorner$$

by def of Subst. Thus with (4) implies

$$Z_0 \vdash \underline{\mathrm{Subst}(\ulcorner \varphi \urcorner, n, \llcorner \underline{k} \lrcorner)}$$
$$\equiv \mathrm{Subst}(\ulcorner \varphi \urcorner, n, \ulcorner \underline{k} \urcorner)$$
$$\equiv \underline{\ulcorner \varphi(\underline{k} / x_m) \urcorner}$$

$\boxtimes$

Thm 20.2 (A variant of Gödel's speedup Thm)

Let $T \supseteq Z_0$ be a concretely given consistent theory (In a finite language $\mathcal{L} \supseteq \mathcal{L}_{\{\varepsilon\}}$). For any recursive (computable) function $f : \mathbb{N} \to \mathbb{N}$ there is an $\mathcal{L}_{\{\varepsilon\}}$-formula $\varphi = \varphi(x_1)$ satisfying

(α) For any $n \in \mathbb{N}$ $T \vdash \varphi(\underline{n}/x_1)$

But for any proof P of this we have $\mathrm{ln}(P) > f(n)$

(β) $T + \mathrm{consis}(\ulcorner T \urcorner) \vdash^{P^*} (\forall x_1 \varepsilon \omega)\varphi$

Note (β) implies that $T + \mathrm{consis}(\ulcorner T \urcorner)$ gives P_n, $n \in \mathbb{N}$ s.t.

$$T + \mathrm{consis}(\ulcorner T \urcorner) \vdash^{P_n} \varphi(\underline{n}/x_1)$$

and $\mathrm{ln}(P_n) \leq$ some polynomial $(\mathrm{ln}(P^*), n)$

So for fast growing $f : \mathbb{N} \to \mathbb{N}$

$\mathrm{ln}(P_n)$ is much less than the length of the shortest proof of $\varphi(\underline{n}/x_1)$ in T

[P_n can be constructed by editing P^* and $Z_0 \vdash^{P_n'} \underline{n} \varepsilon \omega$]

Proof of Theorem 21.2

Let $\psi(x_0, x_1)$ be the $\mathcal{L}_{\{\}}$-formula:

(a) $x_0 \in \mathrm{Fml}_{\mathcal{L}_{\{\}}} \wedge \forall p \,(l_n(p) < f(x_1) \rightarrow \neg \mathrm{proof}(\ulcorner T \urcorner, p, \mathrm{Subst}(x_0, 1, \llcorner x_1 \lrcorner)))$

By Diagonal Lemma there is an $\mathcal{L}_{\{\}}$-formula $\varphi = \varphi(x_1)$ s.t.

(b) $Z_0 \vdash \varphi \leftrightarrow \psi(\ulcorner \varphi \urcorner / x_0)$

We claim that this φ is as desired!

Claim 1 For each $n \in \mathbb{N}$ we have $T \vdash \varphi(\underline{n})$.

$\vdash$ Suppose otherwise and $T \nvdash \varphi(\underline{n}/x_1)$ for some $n \in \mathbb{N}$. Then for any proof P $T \nvdash^{P} \varphi(\underline{n}/x_1)$ in particular this holds for all proofs P with $l_n(P) < f(n)$.

Thus $Z_0 \vdash \forall p\,(l_n(p) < f(\underline{n}) \rightarrow \neg \mathrm{proof}(\ulcorner T \urcorner, p, \underbrace{\mathrm{Subst}(\ulcorner \varphi \urcorner, 1, \llcorner \underline{n} \lrcorner)}))$

$\equiv \ulcorner \varphi(\underline{n}/x_1) \urcorner$ by L 20.1(6)

By (a) and (b), $\underbrace{Z_0 \vdash \varphi(\underline{n}/x_1)}$ $\ominus$ contradiction
$\supseteq$
T

Claim 2 For each $n \in \mathbb{N}$ there is no proof P with $l_n(P) < f(n)$ s.t. $T \vdash^{P} \varphi(\underline{n}/x_1)$

$\vdash$ Suppose there is some $n \in \mathbb{N}$ and a proof P s.t. $l_n(P) < f(n)$ but $\boxed{T \vdash^{P} \varphi(\underline{n}/x_1)}$

$T \supseteq Z_0 \vdash l(\ulcorner P \urcorner) < f(n) \wedge \mathrm{proof}(\ulcorner T \urcorner, \ulcorner P \urcorner, \underbrace{\ulcorner \varphi(\underline{n}) \urcorner})$

$\equiv \mathrm{Subst}(\ulcorner \varphi \urcorner, 1, \llcorner \underline{n} \lrcorner)$ by L 20.1(6)

$\boxed{} + (b) \Rightarrow T \vdash \psi(\ulcorner \varphi \urcorner / x_0, n/x_1)$

this is in contradiction to

It follows that T proves a contradiction

$\ominus$ contradiction to the assumption that T is consistent. $\dashv$

Claim 3 $T + \mathrm{cons}_{\ulcorner T \urcorner} \vdash \forall x_1 \in \omega\, \varphi(x_1)$

$\vdash$ In $T + \mathrm{cons}_{\ulcorner T \urcorner}$ we can translate the proof of claim 1 into a formal proof. $\dashv$

$\boxtimes$

Thm 20.3 (The 2nd Incompleteness Thm)

Let $T \supseteq Z_0$ be a concretely given consistent theory. Then

$$T \nvdash \mathrm{Consis}(\ulcorner T \urcorner)$$

Proof Let $f : \mathbb{N} \to \mathbb{N}$ be recursive function growing faster than any poly. function. Let φ be as in Thm 20.2 for this f. If $T \vdash \mathrm{Consis}(\ulcorner T \urcorner)$

Then T (i.e. $T + \mathrm{Consis}(\ulcorner T \urcorner)$)

would give the situation $\oplus$

A contradiction to the choice of f! ☒

In contrast to Thm 20.2 we have:

Thm 20.4 For any concretely given theory T there is a concretely given theory T^* s.t. T and T^* are equivalent (they prove the same sentences) and each theorem φ of T (or T^*) there is a proof of φ in 3 formulas! (from T^*)

Proof Let $\mathcal{L}$ be the language of T and let $\langle \psi_n : n \in \mathbb{N} \rangle$ enumerate all theorems of T (recursively)

Let $\eta_n = \underbrace{\psi_n \wedge \cdots \wedge \psi_n}_{n \text{ times}}$.

Let

$$T^* = \{\eta_n : n \in \mathbb{N}\} \cup \{(\eta_n \to \psi_n) \mid n \in \mathbb{N}\}$$

T^* is as desired! (Exercise)

By modifying this definition T^* can be taken to be primitive recursive! ☒