

$L = \{c_i \mid i \in I\} \cup \{f_i \mid i \in J\} \cup \{r_i \mid i \in K\}$   
 language signature  
 $\uparrow$   $\uparrow$   $\uparrow$   
 n-ary function symbols    n-ary relation symbols

$\mathcal{A} = \langle A, c_i^{\mathcal{A}}, f_i^{\mathcal{A}}, r_i^{\mathcal{A}} \rangle$   
 $\uparrow$   $\uparrow$   $\uparrow$   $\uparrow$   
 underlying set of  $\mathcal{A}$      $\{c_i, f_i, r_i\}$ ,  $\text{rel } K$   
 $\mathcal{A}$ -structure (notation  $|\mathcal{A}|$ )  
 $|A|$  cardinality of  $A$

Given  $X \xrightarrow{\text{with certain structure}} \overline{X}$   
 $\uparrow$   $\uparrow$   
 not after forgetting the structure    "Nichtigkeit" consistency

$\mathcal{L}$ -term  $t = t(x_0, \dots, x_{m-1})$   $x_0, \dots, x_{m-1} \in \text{Var}$   
 For  $\mathcal{A}$ -structure  $\mathcal{A} = \langle A, \dots \rangle$   $a_0, \dots, a_{m-1} \in A$   $\leftarrow$  pairwise distinct variables  
 $t_{(a_0, \dots, a_{m-1})}^{\mathcal{A}} \in A$  is defined recursively / countable  
 Symbols:  $\neg, \wedge, \vee, \exists, \forall$  (', ') and infinitesimal Var of variables

$\mathcal{L}$ -formulas are defined recursively by:  
 (0) If  $t$  and  $t'$  are  $\mathcal{L}$ -terms then " $t = t'$ " is a  $\mathcal{L}$ -formula  
 (1) If  $r$  is  $n$ -ary relation symbol of  $\mathcal{L}$  and  $t_0, \dots, t_{m-1}$  are  $\mathcal{L}$ -terms then " $r(t_0, \dots, t_{m-1})$ " a  $\mathcal{L}$ -formula  
 (2) If  $\varphi$  and  $\psi$  are  $\mathcal{L}$ -formulas then " $(\varphi \wedge \psi)$ ", " $(\varphi \vee \psi)$ ", " $\neg \varphi$ " are  $\mathcal{L}$ -formulas.

(3) If  $\varphi$  is an  $\mathcal{L}$ -formula and  $x \in \text{Var}$  then  $\exists x \varphi$ ,  $\forall x \varphi$  are  $\mathcal{L}$ -formulas  
 (4) nothing else!  
 For a formula  $\varphi$  the set of free variables in  $\varphi$   $\text{Free}(\varphi)$  is defined inductively as follows:  
 (0)(1) For an atomic formula  $\varphi$   $\text{Free}(\varphi)$  is the set of all variables appearing in  $\varphi$   
 (A) (a) If  $\varphi$  is of the form  $(\varphi_0 \wedge \varphi_1)$  or  $(\varphi_0 \vee \varphi_1)$

② Then  $\text{Free}(\varphi) = \text{Free}(\varphi_0) \cup \text{Free}(\varphi_1)$

(1)  $\text{Free}(\varphi_0) = \text{Free}(\varphi_1)$

(2) If  $\varphi$  is of the form  $\exists x \varphi_0$  or  $\forall x \varphi_0$

then  $\text{Free}(\varphi) = \text{Free}(\varphi_0) \setminus \{x\}$

If  $\text{Free}(\varphi) \subseteq \{x_0, \dots, x_{n-1}\}$  then we express this restriction with  $\varphi = \varphi(x_0, \dots, x_{n-1})$

For  $\mathcal{A}$  structure  $\varphi = \varphi(x_0, \dots, x_{n-1})$ ,  $\mathcal{A}$ -structure

$\mathcal{A} = \langle A, \dots \rangle$  and  $a_0, \dots, a_{n-1} \in A$

We define  $\mathcal{A} \models \varphi(a_0, \dots, a_{n-1})$   
 $\downarrow$  models  $(\mathcal{A}$  is a model of  $\varphi$  with  $a_0, \dots, a_{n-1})$

recursively as follow:

(0) If  $\varphi$  is of the form  $t_0 = t_1$   
 then we have  $t_0 = t_0(x_0, \dots, x_{n-1})$ ,  $t_1 = t_1(x_0, \dots, x_{n-1})$

In this case  $\mathcal{A} \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow t_0^{a_0, \dots, a_{n-1}}(a_0, \dots, a_{n-1}) = t_1^{a_0, \dots, a_{n-1}}(a_0, \dots, a_{n-1})$

(1) If  $\varphi$  is of the form  $\forall (t_0, \dots, t_{n-1})$  where  $\forall$  is an many relation symbol in  $\mathcal{L}$

$\mathcal{A} \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow$   
 $\mathcal{A} \models \langle t_0^{a_0, \dots, a_{n-1}}(a_0, \dots, a_{n-1}) ; \dots ; t_{n-1}^{a_0, \dots, a_{n-1}}(a_0, \dots, a_{n-1}) \rangle$   
 $\mathcal{A}^m$

(2) (a)  $\mathcal{A} \models (\varphi_0 \wedge \varphi_1)(a_0, \dots, a_{n-1})$   
 $\Leftrightarrow \mathcal{A} \models \varphi_0(a_0, \dots, a_{n-1})$  and  $\mathcal{A} \models \varphi_1(a_0, \dots, a_{n-1})$   
 or

(b)  $\mathcal{A} \models \neg \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow$  not  $\mathcal{A} \models \varphi(a_0, \dots, a_{n-1})$

(3) If  $\varphi$  is of the form  $\exists x \varphi_0$  and  $x$  is not among  $x_0, \dots, x_{n-1}$

Then  $\varphi_0 = \varphi_0(x, x_0, \dots, x_{n-1})$

In this case  $\mathcal{A} \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow$  there is  $a \in A$  s.t.  $\mathcal{A} \models \varphi_0(a, a_0, \dots, a_{n-1})$

If  $x$  is among  $x_0, \dots, x_{n-1}$  then  $\varphi_0 = \varphi_0(x_0, \dots, x_{n-1})$   
 (may  $x$  is  $x_i$ )

$\mathcal{A} \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow$  there is  $a \in A$   $\mathcal{A} \models \varphi_0(a_0, \dots, a_{n-1})$

If  $\mathcal{A}, \mathcal{B}$  are  $\mathcal{L}$ -structures  
 with  $\mathcal{B} \subseteq \mathcal{A}$  in the sense  $\mathcal{B} = \langle B, \dots \rangle$   
 is an  $\mathcal{L}$ -substructure  
 (or simply "substructure") of  $\mathcal{A}$

If for any  $\mathcal{L}$ -formula  $\varphi = \varphi(x_0, \dots, x_{n-1})$   
 and  $b_0, \dots, b_{n-1} \in B$

$$\mathcal{B} \models \varphi(b_0, \dots, b_{n-1}) \Leftrightarrow \mathcal{A} \models \varphi(b_0, \dots, b_{n-1})$$

Notation:  $\mathcal{B} \prec \mathcal{A}$

Exempl. 1

$$\mathcal{A} = \langle \mathbb{R}, < \rangle$$

$$\mathcal{B} = \langle \mathbb{Z}, < \rangle$$

Then  $\mathcal{B} \subseteq \mathcal{A}$  but  $\mathcal{B} \not\prec \mathcal{A}$

$$\mathcal{A} \models \exists y (x_0 < y \wedge y < x_1) \quad (0/x_0, 1/x_1) \text{ but}$$

$$\mathcal{B} \not\models \exists y ( \quad ) \quad (0/x_0, 1/x_1)$$

Exempl. 2

$$\mathcal{A} = \langle \mathbb{R}, < \rangle$$

$$\mathcal{B} = \langle \mathbb{Q}, < \rangle$$

Then  $\mathcal{B} \prec \mathcal{A}$

(explained later)

Theorem 1 (Tarski-Vaught-test)  $\mathcal{B} = \langle B, \dots \rangle \quad \mathcal{A} = \langle A, \dots \rangle$

Suppose that  $\mathcal{A}, \mathcal{B}$  are  $\mathcal{L}$ -structures with  $\mathcal{B} \subseteq \mathcal{A}$

Then  $\mathcal{B} \prec \mathcal{A}$  iff

For any  $\mathcal{L}$ -formula  $\varphi = \varphi(x_0, \dots, x_{n-1})$

(\*)  $b_0, \dots, b_{n-1} \in B$

If  $\mathcal{A} \models \exists x \varphi(x, b_0, \dots, b_{n-1})$  then

there is  $b \in B$  s.t.

$$\mathcal{A} \models \varphi(b, b_0, \dots, b_{n-1})$$

Lemma 2 For any  $\mathcal{L}$ -structure  $\mathcal{A} = \langle A, \dots \rangle$

$\mathcal{L}$ -formula  $\varphi = \varphi(x_0, \dots, x_{n-1})$  and  $a_0, \dots, a_{n-1} \in A$

$$\mathcal{A} \models \forall x \varphi(x, a_0, \dots, a_{n-1}) \Leftrightarrow \mathcal{A} \models \exists x \exists y \varphi(x, a_0, \dots, a_{n-1})$$

Then we can treat  $\forall x$  as an abbreviation of the relation  $\neg \exists x \neg$ . In this way we may omit arguments about " $\forall x$ ".

Proof of Theorem 1  
(Assume  $\mathcal{L} \rightarrow \mathcal{A}$ )

$\Rightarrow$ : Suppose that  $\mathcal{M} \models \exists x \varphi(x, b_0, \dots, b_{k-1})$  with  $b_0, \dots, b_{k-1} \in B$ .

By assumption,  $\mathcal{L} \models \exists x \varphi(x, b_0, \dots, b_{k-1})$

Hence, there is some  $b \in B$  s.t.

$\mathcal{L} \models \varphi(b, b_0, \dots, b_{k-1})$

Again by assumption:

$\mathcal{M} \models \varphi(b, b_0, \dots, b_{k-1})$

$\Leftarrow$ : Assume that (\*) holds

We have to show:

For any  $\mathcal{L}$ -formula  $\varphi = \varphi(x_0, \dots, x_{k-1})$

we have

(\*\*) For any  $b_0, \dots, b_{k-1} \in B$

$\mathcal{L} \models \varphi(b_0, \dots, b_{k-1})$  iff  $\mathcal{M} \models \varphi(b_0, \dots, b_{k-1})$

We prove this by induction on  $\varphi$

(1), (2) For atomic formula this is trivial

(2) is also a problem!

(3) Assume  $\varphi = \exists x \varphi_0(x, x_0, \dots, x_{k-1})$

(\*\*) holds for  $\varphi_0$

If  $\mathcal{L} \models \varphi(b_0, \dots, b_{k-1})$  then is  $b \in B$  s.t.

$\mathcal{L} \models \varphi_0(b, b_0, \dots, b_{k-1})$ . By the assumption

it follows that  $\mathcal{M} \models \varphi_0(b, b_0, \dots, b_{k-1})$

Thus  $\mathcal{M} \models \exists x \varphi_0(x, b_0, \dots, b_{k-1})$  i.e.  $\mathcal{M} \models \varphi(b_0, \dots, b_{k-1})$

Assume now that

$\mathcal{M} \models \varphi(b_0, \dots, b_{k-1})$

i.e.  $\mathcal{M} \models \exists x \varphi_0(x, b_0, \dots, b_{k-1})$

By (\*) there is  $b \in B$  s.t.

$\mathcal{M} \models \varphi_0(b, b_0, \dots, b_{k-1})$

By assumption

$\mathcal{L} \models \varphi_0(b, b_0, \dots, b_{k-1})$  Thus

$\mathcal{L} \models \exists x \varphi_0(x, b_0, \dots, b_{k-1})$  i.e.  $\mathcal{L} \models \varphi(b_0, \dots, b_{k-1})$

