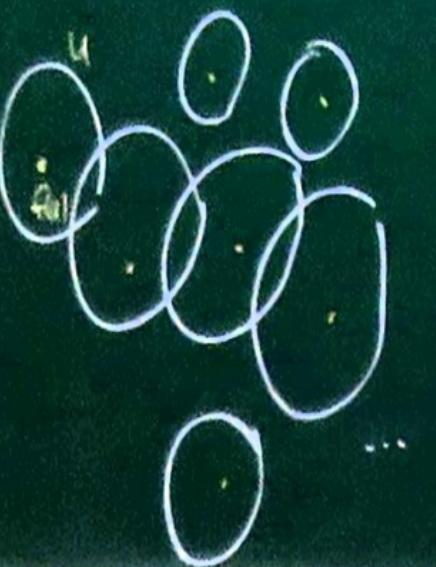


Axiom of Choice

For $x \in \mathcal{X}$ s.t. $\emptyset \notin x$ there is a function $f: x \rightarrow \bigcup x$ s.t. for all $u \in x$ $f(u) \in u$



ZC (Zermelo-Fraenkel Axiom System with Axiom of Choice)

Z (Zermelo-Fraenkel Axiom System without Axiom of Choice)

Z^- (Zermelo-Fraenkel Axiom System without Axiom of Choice without Foundation)

For x_1, \dots, x_n we may consider

$\langle\langle x_1, x_2, \dots, x_n \rangle, x\rangle$ as ordered n -tuple and denote it as $\langle x_1, x_2, \dots, x_n \rangle$.

But with this definition of n -tuple we cannot treat

$w> X = \{ \langle u_1, \dots, u_n \rangle \mid m \in w, u_1, \dots, u_n \in X \} \quad \begin{matrix} \text{set of finite sequences} \\ \text{of elements of } X \end{matrix}$

So we define that a n -tuple is a function from \mathbb{N}_0^n to X of elements of X

$f: m \rightarrow X$ then we interpret f as

the n -tuple $\langle f(0), f(1), \dots, f(n-1) \rangle$

$$m = \{0, 1, \dots, n-1\}$$

$$\begin{aligned} w> X &= \left\{ f \mid f: m \rightarrow X \text{ for some } m \in w \right\} \\ &\quad - \left\{ f \in P(w \times X) \mid f: m \rightarrow X \text{ for some } m \in w \right\} \end{aligned}$$

Explain If $m=3$ then $f: 3 \rightarrow X$ is of the form $\{ \langle 0, q_0 \rangle, \langle 1, q_1 \rangle, \langle 2, q_2 \rangle \}$. This set is considered to be the triple $\langle q_0, q_1, q_2 \rangle$.

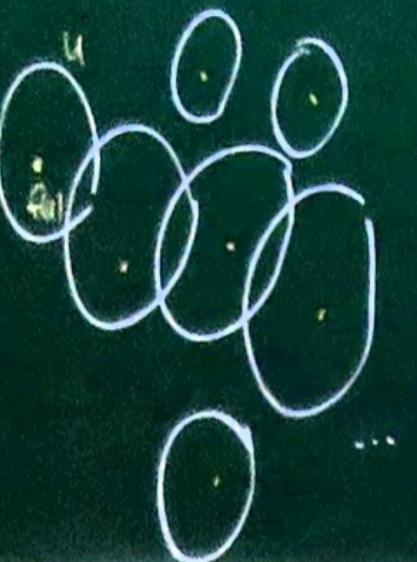
$$f(0) = q_0$$

$$f(1) = q_1$$

$$f(2) = q_2$$

Axiom of Choice

From a set X n.t. $\emptyset \notin X$ there is a function $f: X \rightarrow \bigcup X$ n.t. for all $x \in X$ $f(x) \in x$



ZC (Zermelo-Fraenkel Axiom System with Axiom of Choice)

Z (Zermelo-Fraenkel Axiom System without Axiom of Choice)

Z^+ (Zermelo-Fraenkel Axiom System without Axiom of Choice without Foundation)

For x_1, \dots, x_n we may consider

$\langle\langle x_1, x_2, x_3, \dots, x_n \rangle\rangle$ as above n-tuple and denote it as $\langle x_1, x_2, x_3, \dots, x_n \rangle$.

But with this definition of m-tuple we cannot treat

$w> X = \{ (u_1, \dots, u_m) \mid m \in w, u_1, \dots, u_m \in \chi \}$ set of finite sequences of elements of χ

So we define that a m-tuple is a function from $\overset{\text{m}}{\underset{\wedge}{\wedge}}$ to χ

$f: m \rightarrow \chi$ then we interpret f as

the m-tuple $\langle f(0), f(1), \dots, f(m-1) \rangle$

$$m = \{0, 1, \dots, m-1\}$$

$$\begin{aligned} w> X &= \{ f \mid f: m \rightarrow \chi \text{ for some } m \in w \} \\ &\quad - \{ f \in P(w \times \chi) \mid f: m \rightarrow \chi \text{ for some } m \in w \} \end{aligned}$$

Graph If $m=3$ then $f: S \rightarrow \chi$ is of the form $\{ \langle 0, g \rangle, \langle 1, g_1 \rangle, \langle 2, g_2 \rangle \}$. This set is considered $S = \{0, 1, 2\}$ to be the triple $\langle g_0, g_1, g_2 \rangle$

$$f(0) = g_0$$

$$f(1) = g_1$$

$$f(2) = g_2$$

Number Theory inside \mathbb{Z}

w is the smallest set ($w \in \subseteq$) s.t.
 $\phi \in w$

for any u if $u \in w$ $u \cup \{u\} \in w$
 \Downarrow
 $P(u)$

Lemma 1 (1) If $X \subseteq w$ s.t.

(a) $\phi \in X$ (b) for any u if $u \in X$ then $u \cup \{u\} \in X$
 then $X = w$.

(2) (Induction) For some property φ if ϕ satisfies
 φ and "for any $u \in w$ if m satisfies φ then
 $m \cup \{m\}$ satisfies $\varphi"$. Then for all $m \in w$ m satisfies
 φ ". Then for all $m \in w$ m satisfies
 φ .
 ↑ proof of φ by induction on m .

(3) All $m \in w$ are transitive

(4) For any $n \in w$ if $n \notin \phi$ then there is
 $m_0 \in w$ s.t. $n = m_0 \cup \{m_0\}$

(5) w is transitive

(6) For any $m \in w$ $m \neq m \cup \{m\}$
 In particular $m \notin m$ (we can prove this
 without Axiom of Foundation!)

(7) For any $m, m' \in w$

If $P(m) = P(m')$ then $m = m'$

X is transitive \Leftrightarrow for all $u \in X$ $u \cup \{u\} \in X$
 we have $N \in X$ for all N $u \subseteq N$

\Leftrightarrow for all $u \in X$ $u \subseteq u$

Proof (a): By def of w we have $w \subseteq X$.

Thus $X = w$.

(2): Consider $X = \{m \in w \mid \varphi(m)\}$

(3): Let $X = \{m \in w \mid m \text{ is transitive}\}$ Then
 X satisfies (a) (b)

ϕ is transitive

If $u \in w$ is transitive then if $a \in u \cup \{u\}$ then
 either $\overset{\circ}{a} \in u$ or $\overset{\circ}{a} = u$. If $\overset{\circ}{a} \in u$ holds then, since
 u is transitive $b \in u \subseteq u \cup \{u\}$, if $\overset{\circ}{a} = u$ holds then
 $b \in u \subseteq u \cup \{u\}$. In both cases $b \in u \cup \{u\}$. Thus
 $u \cup \{u\}$ is also transitive.

By (1), $X = w$. So all $m \in w$ are transitive.

(4): We prove by induction on m that

$m = \phi$ or there is $m_0 \in w$ with $m = m_0 \cup \{m_0\}$. (Exercise!)

(5): w_1 prove by induction on m
 $"n \subseteq w"$

For $\phi = n$ this is trivial
 $(\phi \subseteq X \text{ for any } X!)$

Suppose that $m \in w$ and $m \subseteq w$.
 Then $m \cup \{m\} \subseteq w$.

(6): We prove by induction on m that
 $m \neq m \cup \{m\}$

For $m = \phi$, $\phi \cup \{\phi\} = \{\phi\} \Rightarrow \phi \neq \phi \cup \{\phi\}$

Suppose that $m \neq m \cup \{m\}$.

To prove:
 $m \cup \{m\} \neq (m \cup \{m\}) \cup \{m \cup \{m\}\}$ no ind. hyp
 Suppose $m \cup \{m\} = (m \cup \{m\}) \cup \{m \cup \{m\}\}$
 Then $m \cup \{m\} \in m \cup \{m\}$ $\Rightarrow m \cup \{m\} = m$
 \Rightarrow ① $m \cup \{m\} \in m$ or ② $m \cup \{m\} \in \{m\}$

If (1) holds $m \cup \{n\} \in \mathcal{M}$

By (2) it follows that $n \in \mathcal{M}$

It follows that $\mathcal{M} \cup \{n\} = \mathcal{M}$

A contradiction.