

Lemma 1 (1) If  $X \subseteq \omega$  s.t.

- (a)  $\emptyset \in X$
- (b) if  $u \in X$  then  $\{u\} \cup u \in X$

then  $X = \omega$

(2) For any property  $\varphi = \varphi(x)$  p.t.

if  $\varphi(u)$  then  $\varphi(u \cup \{u\})$

then  $\varphi(n)$  holds for all  $n \in \omega$   $\rho(n)$

(3) All  $m \in \omega$  is transitive

$\omega = \bigcap \{X \mid \emptyset \in X \text{ for all } x (x \in X \rightarrow x \cup \{x\} \in X)\}$

(7) For any  $n, m \in \omega$  if  $\rho(n) = \rho(m)$  then  $n = m$

(4) For any  $n \in \omega$  if  $n \neq 0$  then there is some  $m \in \omega$  s.t.  $m = n \cup \{n\}$

(5)  $\omega$  is transitive (i.e. if  $m \in n$  and  $u \in m$  then  $u \in n$ )

We work in  $\mathcal{ZF}$  without Axiom of Foundation

Proof We prove  $\varphi$

(7): "for  $m \in \omega$  if  $\rho(m) = \rho(n)$  then  $m = n$ " by induction on  $m$

Suppose  $\rho(m') = \rho(n) = \{\emptyset\}$

$m' \cup \{m'\} = \rho(m') = \{\emptyset\}$

Thus  $m' = \emptyset$  and  $\varphi(\emptyset)$  holds.

Suppose that  $\varphi(n)$  for  $n \in \omega$ , and  $m' \in \omega$  is not.  $\rho(n) = \rho(n \cup \{n\})$

That is,

$m' \cup \{m'\} = (m \cup \{m\}) \cup \{m \cup \{m\}\}$

Suppose, for contradiction, that  $m' \neq \rho(m)$

Since  $m' \in (m \cup \{m\}) \cup \{m \cup \{m\}\}$

$m' \in m \cup \{m\}$ . If  $m' = m$  then  $\rho(m') = \rho(m)$

So by (1),  $\rho(m) = \rho(n) \neq \rho(\rho(m))$  contradicts (3).

If  $m' \in m$  then,  $m' \subseteq m$  by (6). Then  $m \cup \{m\} \subseteq m$ .

$$n \cup \{n\} \subseteq n \subseteq n \cup \{n\} \models A(n(n))$$

$\not\models$

$\vdash_{(1)} \vdash_{(2)}$

A contradiction to (\*).

Thus  $m \models A(n)$ . So we have proved

$$\varphi(A(n)). \quad \square$$

**Lemma 2 (1)** For any  $n, m \in W$ , exactly one of the following (d), (p), (l) holds:

$$(d) n \in m \quad (p) n = m \quad (l) n \notin m$$

$$(2) \quad m \in m \quad (\Leftrightarrow) \quad m \not\models m$$

(3) For any  $n, m \in W$ , exactly one of the following

$$m \not\models m \quad (\Leftrightarrow) \quad m \subseteq m \text{ and } m \neq m$$

$$(d') \quad (p') \quad (l') \text{ holds.} \quad (d') \quad m \not\models m \quad (p') \quad m = m$$

(l')  $m \subseteq m$ .

(4)  $\in$  on  $W$  is a linear ordering

(i.e. for any  $l, m \in W$   $l \in m$  and  $m \in l$  implies  $l \in m$ )

(5) for any  $l, m \in W$ ,  $m \neq l$  either  $m \in l$  or  $l \in m$

Proof (1): (d), (p), (l) are mutually exclusive;

For example if  $m = m$  then  $m \not\models m$  by Lemma 1.(6),

If  $m \in m$  and  $m \in m$  then by Lemma 1.(3)  $m \in m$ . A contradiction to second part of Lemma 1.(6).

"if for any  $m \in W$  if  $m \neq m$  then either  $m \in m$  or  $m \in m$  holds."  $\varphi(n)$

We prove by induction on  $m$  that  $\varphi(m)$  holds for all  $m \in W$ . (Exercise)

(2) If  $m \in m$  then  $M \subseteq m$  by Lemma 1.(3)

Since  $m \neq m$  by the second part of Lemma 1.(6)

$m \not\models m$ . If  $M \not\models m$  then  $M \neq m$ .  $m \in m$  is impossible

(because if  $m \in m$  then  $m \in m$  this is impossible by the second part of Lemma 1.(6).) By (1) it follows,  $M \in m$ .

(3) follows from (2) and (2)

(4) (3) holds as we saw it in the proof of (2)

(3) is a reformulation of (1)

(1) If  $l \in m$  and  $m \in m$  then

$l \not\models m$  and  $m \not\models m$  by (2). Hence

$l \not\models m$ . But then again by (2)  $l \in m$ .  $\square$

For  $m, m \in W$  we write  $m < m$  for  $m \in m$ .

### Lem. 3

(0)  $\emptyset$  is the smallest element of  $\omega$   
 $\omega \neq \emptyset$

(1) For any  $m \in \omega$   $\rho(m)$  is the  
 next element to  $m$  w.r.t.  $<$   
 successor of  $m$ .

(2) For any  $X \subseteq \omega$  there is the  
 minimal element if  $\omega \neq X$

Proof (0): For any  $m \in \omega$   $m \notin \emptyset$ ,

thus by Lem. 2 (1) we have  $m = \emptyset$

or  $\emptyset \subset m$  ( $\emptyset < m$ )

(1):  $\rho(m) = m \cup \{m\} \ni m$  since

$\rho(m) \setminus m = \{m\}$  there can not be a set  $U$

with  $n \subsetneq X \subsetneq \rho(n)$

This shows that  $\rho(n)$  is (immediate) successor of  $n$ .

(2): Suppose that  $X \subseteq \omega \setminus \{\emptyset\}$ .

Suppose toward a contradiction that  $X$  d.h.

contain the minimal element. We show that  
 $X = \emptyset$  contradicting our assumption.

Let  $Y = \omega$ . It is enough to show that

$Y = \omega$ . We prove this by Lem. 1 (1)

$\emptyset \in Y$ : otherwise  $\emptyset \in X$  Hence  $\emptyset$  should be the  
 minimal element of  $X$ . A contradiction to  $X \neq \emptyset$ .

So  $\emptyset \in X$ .  $\rho(\emptyset) = \{\emptyset\} \subseteq \omega \setminus X$  i.e.  $\emptyset \in Y$

$$Y = \{n \in \omega \mid \text{r.t. } \rho(n) \subseteq \omega \setminus X\} \subseteq \omega \setminus X$$

Suppose  $u \in Y$ . We have to show that

$$\rho(u) \subseteq \omega \setminus X$$

If  $\rho(u) \notin Y$  then  $\rho(u) \in X$ .

But (since)  $\rho(u) \cap X = \emptyset$ ,  $\rho(u)$  is the  
 $\{n \in \omega \mid n < \rho(u)\}$

minimal element of  $X$ . A contradiction to  $X \neq \emptyset$ .

Thus  $\rho(u) \in Y$ .

By Lem. 1 (1) it follows that  $Y = \omega$  and  $X = \emptyset$

This is a contradiction to  $X \neq \emptyset$   $\square$