Sentential Logic (S.L.)

PropVar inf. many symbols of propositional variable

(1) If $A \in PropVar$ then A as a sequence of length 1 is a formula (of S.L.)

(2) If ψ and φ are formulas of S.L. then $(\varphi \to \psi)$, $\neg\varphi$ are also formulas

(3) Nothing else.

Let $\mathbb{Z} = \{0, 1\}$

For $\varphi = \varphi(A_0, \dots, A_{n-1})$ (all prop. variable in φ are among $A_0, \dots, A_{n-1}$)

we define $f_{\varphi(A_0, \dots, A_{n-1})} : \mathbb{Z}^n \to \mathbb{Z}$ recursively as follows

(1) If φ is A_i then $f_{\varphi(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = t_i$

(2) (a) If φ is $(\psi_0 \to \psi_1)$ then we have

$\psi_0 = \psi_0(A_0, \dots, A_{n-1})$, $\psi_1 = \psi_1(A_0, \dots, A_{n-1})$

We define

$$f_{\varphi(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = \begin{cases} 1 & \text{if } f_{\psi_0(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = 0 \text{ or} \\ & f_{\psi_1(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) If φ is of the form $\neg\psi$ then

$$f_{\varphi(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = \begin{cases} 1 & \text{if } f_{\psi(A_0, \dots, A_{n-1})}(t_0, \dots, t_{n-1}) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma The value of $f_{\varphi(A_0, \dots, A_{n-1})}$ d.n. depend on the choice of the first of the variable $A_0 \dots A_{n-1}$

In particular

If $\{A_0, \dots A_{n-1}\} \subseteq \{B_0 \dots B_{m-1}\}$ and $A_i = B_{k_i}$ then

$f_{\varphi(A_0,\dots,A_{n-1})}(t_0,\dots,t_{n-1})$

$= f_{\varphi(B_0,\dots,B_{n-1})}(t'_0,\dots,t'_{n-1})$

where $t'_{j_i} = t_i$ for all $i<n$.

Proof By induction on the construction of φ ☒

A formula $\varphi=\varphi(A_0,\dots,A_{n-1})$ is tautology

if $f_{\varphi(A_0,\dots,A_{n-1})}(t_0,\dots,t_{n-1}) = 1$

for any $t_0,\dots,t_{n-1}\in 1$

Cor 4 The definition of the tautology d.n. depend on the choice of $A_0,\dots,A_{n-1}$. ☒

We check if a given formula φ of S.L. is a tautology e.g. by making up a truth value table.

The last clause of the definition of the axioms in K^*

A L-formula φ (in predicate logic) is said to be tautology if there is a tautology $\psi=\psi(A_0,\dots,A_{n-1})$ of sentential logic and φ is obtained by replacing $A_0,\dots,A_{n-1}$ by some L-formulas $\varphi_0,\dots,\varphi_{n-1}$ respectively ($\varphi=\psi(\varphi_0,\dots,\varphi_{n-1})$)

All tautologies in L are axioms of K^*

The deduction rules of K^*

$$\frac{\varphi,\ (\varphi\to\psi)}{\psi}$$ (syllogism modes ponens)

$$\frac{(\varphi\to\psi)}{(\exists x\,\varphi\to\psi)}$$ (existential deduction)

A sequence $\langle\varphi_0,\dots,\varphi_{l-1}\rangle$ of L-formulas is a proof in K^* if for all $i<l$ either

(1) φ_i is one of the axioms of K^*; or

(2) there are $i_0, i_1 < i$ s.t. $\dfrac{\varphi_{i_0}\quad\varphi_{i_1}}{\varphi_i}$ is syllogism modes ponens; or

(3) there is $i_0<i$ s.t. $\dfrac{\varphi_{i_0}}{\varphi_i}$ is an existential deduction.

φ_{l-1} is conclusion of the proof.

If there is a proof with the conclusion φ in T we write $T \vdash^{K^*} \varphi$

[It can be shown later that the system K^* is in a sense complete!]

L-theory

For a set of L-sentences T

A sequence of L-formulas $\langle \varphi_0, \dots \varphi_{l-1} \rangle$ is a proof in T, if, for all $i < l$, either

(1') φ_i is in T or φ_i is one of the axioms in K^*

or

(2) or (3)

If φ is the conclusion of some proof in T

Then we say that φ is a theorem in T and write

$T \vdash^{K^*} \varphi$

We consider that Theorems of Z or ZC are just such sentences φ that $Z \vdash^{K^*} \varphi$ (ZC) ┘

Thm 6 (Deduction Theorem of K^*) For any L-theory T and L-sentence φ and L-formula ψ if:

$T \cup \{\varphi\} \vdash^{K^*} \psi \iff T \vdash (\varphi \to \psi)$

Proof By induction on the length of the proof. □

A Theory T is inconsistent if $T \vdash^{K^*} \exists x (\neg x \equiv x)$

Lemma: T is inconsistent iff T proves every L-formula.

Cor 7 For any theory T and L-sentence φ if:

(a) $T \vdash^{K^*} \varphi$

(b) there is some finitely many $\varphi_0, \dots \varphi_{m-1} \in T$ s.t. $\{\varphi_0, \dots, \varphi_{m-1}\} \vdash^{K^*} \varphi$

(c) $\vdash^{K^*} \varphi_0 \to \varphi_1 \to \dots \to \varphi_{m-1} \to \varphi$

Translation of Logic in $\mathbb{Z}$

We interpret symbols as sets of the following form:

Variables: $\langle \langle 0,0 \rangle, 1 \rangle, \langle \langle 0,0 \rangle, 2 \rangle \dots \in \mathbb{1} \times (\mathbb{2} \times \omega)$

constant symbols $\langle 4, \langle 0,0 \rangle \rangle$ $\langle 4, \langle 0,1 \rangle \rangle \dots \in \{4\} \times (\mathbb{2} \times \omega)$

n-ary function symbols: $\langle 2, \langle n, 0 \rangle \rangle$ $\langle 2, \langle n, 1 \rangle \rangle \dots$

$\in \{2\} \times (\mathbb{n} \times \omega)$

n-ary relation symbols

$\langle 3, \langle n, 0 \rangle \rangle$ $\langle 3, \langle n, 1 \rangle \rangle \dots$

$\in \{3\} \times (\mathbb{n} \times \omega)$

In particular we consider

$\langle 3, \langle 2, 0 \rangle \rangle$ is the symbol "$\in$"

$\equiv, (,), \wedge, \neg, \to, \exists$: $\langle 4, \langle 0,0 \rangle \rangle$ $\langle 4, \langle 0,1 \rangle \rangle$

$\dots \langle 4, \langle 0,6 \rangle \rangle$

In $\mathbb{Z}$

For $L \subseteq \{1\} \times (\{0\} \times \omega) \cup \{2\} \times (\omega \times \omega)$

We can define the set of L-terms and L-formulas (L-sentences)

$\mathrm{Tm}_L \subseteq \omega^{<\omega}(\omega \times (\omega \times \omega))$

$\mathrm{Fml}_L \subseteq \mathrm{Sat}_L$

$L_\in = \{\langle 3, \langle 2, 0\rangle\rangle\}$

By the definition corresponding to the definition of corresponding notions in metamathematics.

The notions to define the system $\mathbb{K}^*$ can be also translated into the language of set theory. For example, we let

$\mathrm{Proof}_L = \{ \pi \in {}^{<\omega}\mathrm{Fml}_L : \dots \}$

Each single axiom φ of $\mathbb{Z}$ can be translated into a term in $L_\in$ which represents an element of $\mathrm{Fml}_{L_\in}$ corresponding to the axiom φ this term is denoted by $\ulcorner \varphi \urcorner$

$\ulcorner \mathbb{Z} \urcorner = \{ \ulcorner$ Extension Axiom $\urcorner, \ulcorner$ Axiom of Empty Set $\urcorner, \dots \}$

$\cup \{ \ulcorner$ Separation axiom for $\varphi \urcorner : \varphi \in \mathrm{Fml}_{L_\in} \}$

$\ulcorner \mathbb{Z} \urcorner \subseteq \mathrm{Snt}_{L_\in}$.