

Recap  $\exists \subset T \subseteq T'$  in  $\delta^*$

If  $T' \vdash \text{comin}(\vdash T')$

then for a recursive  $f: N \rightarrow N$

there is a  $\delta^*$ -formula  $\varphi(x_1)$  s.t.

for any  $m \in N$   $T \vdash \varphi(m)$  but

all proofs  $P$  of  $\varphi(n)$  from  $T$  are

s.t.  $Z \vdash \text{rank}(P) > f(n)$

But  $T' \vdash V_{\text{new}}(\varphi(n))$

and the proof for this is feasible.

Then we may natural pairs of theories  $T, T'$  with

$T' \vdash \text{comin}(\vdash T')$

$(T = Z; T' = ZF)$  is one of such examples!

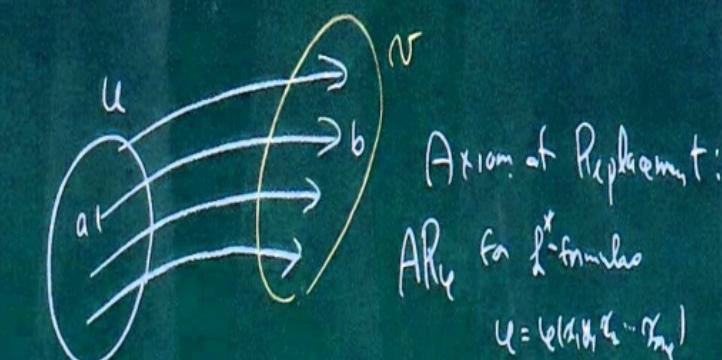
In particular  $ZF \vdash \text{comin}(\vdash Z')$ .

Axiom of Replacement

For each  $\delta^*$ -formula  $\varphi(x_1, y_1, \dots, x_m)$ , let  $AR_\varphi$  be the

$\delta^*$ -sentence asserting:

$AR_\varphi$ : For any  $a, a_1, \dots, a_m$ , if for any  $a \in u$  there is the unique  $b$  s.t.  $\varphi(a, b, a_1, \dots, a_m)$ , then there is no  $a \in u$  s.t.  $\varphi(a, b, a_1, \dots, a_m)$ ,



In  $Z$  we cannot prove that  $V_\omega = \bigcup_{n \in \omega} V_n$  is a  $\delta^*$  (on the function  $\omega \rightarrow V_\omega$  is  $n \mapsto V_n$ )

$$\text{where } V_\omega = \bigcup_{n \in \omega} V_n = V_\omega \cup P(V_n)$$

We can define the predicate "x = V\_m" ( $\delta^*$ -formula)

by expressing "there is a set X and mapping  $f: m+1 \rightarrow X$ "  
s.t.  $f(0) = k$ ,  $f(k+1) = f(k) \cup P(f(k))$  and  $x = f(m)"$

ZF:  $\mathcal{Z} + \text{AR}_\beta$

The idea of the proof of

$\text{ZF} \vdash \text{cons}(\mathcal{Z})$ : In ZF

① Introduce the notion of ordinal numbers  
(a generalization of the notion of natural numbers)  $0, 1, 2, \dots, \omega, \omega+1, \omega+2, \dots$

The class of ordinal numbers  $\text{On}$

② We prove that induction and recursion definitions are possible for  $\text{On}$

③ We define the von Neumann hierarchy:

$V_\alpha = V$ $V_{\alpha+1} = V_\alpha \cup P(V_\alpha) \quad \alpha \in \text{On}$ $V_\gamma = \bigcup_{\alpha < \gamma} V_\alpha \quad \text{for a limit } \gamma$	(i.e. it contains enough ordinals $\alpha$ for which there are no $\beta \in \text{On}$ with $\alpha = \beta + \beta''$ ) $\beta \in \beta$
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④ We prove for any limit ordinal  $\gamma > \omega$   $(\text{ZF} \subseteq^{\omega} (\text{On} \times \text{On}))$   
 $(V_\gamma, \in) \models \text{ZF}$

⑤ By completeness theorem  $\text{ZF}$  is consistent.

Summarizing the arguments above we translated into a proof  $\text{ZF} \vdash \text{cons}(\mathcal{Z})$

In ZF: We don't need AF for the following:  
(Axiom of Foundation)

A set  $X$  is transitive if and only if  
for any  $y, y' \in X$  with  $y \in y'$ ,  $y \in X$  we have  $y' \in X$   
(equivalently, for any  $x \in X$  we have  $x \subseteq X$ )

A set  $d$  is an ordinal if  $d$  is transitive  
and  $\in$  on  $d$  (i.e.  $\{(\beta, \gamma) \in d^2 \mid \beta \in \gamma\}$ ) is a  
well ordering (i.e.  $\in$  is a linear ordering on  $d$   
and every subset  $A \subseteq d$  has the minimal element  
w.r.t  $\in$  (non empty))

Lemma 1 (0)  $\phi$  is an ordinal.

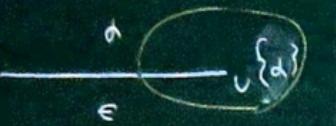
- (1) If  $d$  is an ordinal then  $d+1 = d \cup \{d\}$  is also an ordinal
- (2) all  $n \in \omega$  are ordinals ( $\omega \subseteq \text{On}$ )
- (3) If  $S$  is a set of ordinals then  $\bigcup S$  is an ordinal
- (4)  $\bigcup \omega = \omega$  is an ordinal

(2.2) For an ordinal  $d$ ,  $\{d\}$  exactly one of the following holds:  
 $\alpha \in \beta \wedge d = \beta \wedge \beta \in d$ .

Proof (0): trivial!

(1): Since  $d$  is transitive  $d \cup \{d\}$  is also transitive.  
Since  $d$  is "bigger" than all other elements of  $d$  w.r.t  $\in$

$d \neq d$ , otherwise  $\{d\} \subseteq d$ .  $R, + \{d\}$   
 does not have a) minimal element!  
 A contradiction!



Also it is clear by the picture above  
 that  $\in$  on  $d \cup \{d\}$  is well-ordering!

(2) : By induction principle for natural numbers.

(3) : Suppose  $S \subseteq \text{On}$ .

(a)  $\text{US}$  is transitive  $d \in \text{US}$  and  $p \in d$   
 Let  $T \in S$  be s.t.  $d \in T$  since  $T$  is transitive  
 and  $p \in d$  we have  $p \in T$ . Thus  $p \in \text{US}$

(b)  $\in$  is a linear ordering on  $\text{US}$

Let  $a, b \in \text{US}$  then there are  $r, f' \in S$  s.t.

$a \in r, b \in f'$  w.l.g.  $r \subseteq f'$  (by (2))  $r \subseteq f'$  and  $f = f' \cup \{r\}$

Then  $a, b \in f'$  Then exactly one of  $a \in b$  or  $a = b$  or  $b \in a$   
 holds since  $\in$  on  $f'$  is linear. Suppose that

$a, b, r \in \text{US}$ ,  $a \in b$  and  $b \in r$  Let  $s \in S$  s.t.

$r \subseteq s$  it follows by transitivity  $a \in s$ ,  $b \in s$

Since  $\in$  on  $s$  is a linear ordering it follows that  $a \in b$ .

Let  $\phi \neq T \subseteq \text{US}$ . We have to show that  $T$  has the minimal element.

Let  $Y \in S$  s.t.  $T \cap Y$  is not empty Let  $d$  be the minimal element of  $T \cap Y$

$d$  is minimal in  $T$ : Let  $p \in T$  then  $p \in d$  or  $p = d$

or  $d \in p$ , But  $p \in d$  is not possible because

If  $\beta \in d$ , since  $d \in T \cap Y$  it follows that

$\beta \in Y$ . So  $\beta \in T \cap Y$ . This is

a contradiction to the minimality of  $d$   
 in  $T \cap Y$ .  $\square$