

1. official part of the lecture

Wet 15<sup>th</sup>

3:30PM — Seminar room inside the students room next to the presentation room

$\text{On}$  is a proper class

Theorem 1 For any class  $C \subseteq \text{On}$ , if

(\*) for any  $d \in \text{On}$  if  $d \in C$  then  $d \in C$

Then  $C = \text{On}$

Proof Suppose otherwise. Then  $\mathcal{X} = \text{On} \setminus C \neq \emptyset$ . Let  $d_0 = \min \mathcal{X}$ . Then  $d_0 \in \mathcal{C}$  Hence by (\*)  $d_0 \in C$ . A contradiction.

Cor 2 For any  $C \subseteq \text{On}$  if

- ①  $\emptyset \in C$
- ② if  $d \in C$  then  $d+1 \in C$

③ if  $d$  is a limit and  $d \in C$  then  $\delta \in C$  Then  $C = \text{On}$

Proof We claim that  $C$  satisfies (3)!

Suppose  $d \in C$  if  $d = \emptyset$  then  $\emptyset \in C$  by (2).

If  $d = \beta + 1$  then since  $\beta \in C$   $\beta \in C$  hence  $d \in C$  by (2)

We write  $d+1$

$$\text{fun}(d) = d \cup \{d\}$$
$$d \notin d \cup \{d\}$$

We also write  
 $\delta < \beta$  for  $\delta \in \beta$   
( $\exists \delta \in \beta$ )

$$d \text{ is limit} \Leftrightarrow \forall d = \delta$$
$$\Leftrightarrow d \text{ is closed limit} + 1$$

If  $d$  is limit then (3)  $\forall \delta < d \in C$ ,  
 $\text{fun}_d, \delta \in \text{On}$  by Thm 1.

Thm 3 (Transfinite Recursion)

Suppose  $\mathcal{M}_1 = \{f : f \text{ is function with } \text{dom}(f) \in \text{On}\}$

and  $g$  is a class function on  $\mathcal{M}_1$

Then there is a class function  $H$  on  $\text{On}$

$$\text{for } d \in \text{On} \quad H(d) = g(H|_d)$$

Proof Uniqueness: Suppose that  $H$  and  $H'$  satisfy (\*), and we show that  $H = H'$ . Suppose not. Then let  $d_0$  be the smallest ordinal s.t.  $H(d) \neq H'(d)$ .

$$H(d) = g(H|_d) = g(H'|_d) = H'(d)$$

$$H|_{d_0} = H'|_{d_0}$$

by min. w.r.t.

A contradiction of  $d_0$ !  
to the choice of  $d_0$ !

Existence: (\*\*\*)

$$\mathcal{M}_0 = \{f \in \mathcal{M}_1 \mid \forall_{\beta \in \text{dom}(f)} f(\beta) = g(f|_\beta)\}$$

Claim (1) For any  $d \in \text{On}$  there is at most one  $f \in \mathcal{M}_0$  with  $d = \text{dom}(f)$

(2) For any  $d \in \text{On}$  there is  $f \in \mathcal{M}_0$  s.t  $\text{dom}(f) = d$ .

(2) For any  $f_1, f_2 \in \mathcal{F}_\alpha$ , if  
 $\text{dom}(f_1) < \text{dom}(f_2)$  then  $f_1 \not\subseteq f_2$

→ (1) can be proved just as in previous part  
 & the uniqueness  $\leftarrow \text{Cn}^2$

(3) we prove by induction on  $d$  that  
 "there is  $f \in \mathcal{M}_d$  with  $\text{dom}(f) = d$ "

① For  $d = \emptyset$  then  $f = \emptyset$  will do

② If there is  $f \in \mathcal{M}_d$  with  $\text{dom}(f) = d$  then

let  $\hat{f} = f \cup \{\langle d, Q(f) \rangle\}$ . Then  $\hat{f} \in \mathcal{M}_{d+1}$   
 and  $\text{dom}(\hat{f}) = d+1 = \text{dom}(f) \cup \{d\}$ .

③ Suppose that  $\gamma \in \text{cf}(\mu_\alpha, \delta)$  and for each  $d \in \gamma$   
 there is  $f \in \mathcal{F}_d$  with  $\text{dom}(f) = d$ . By (2)  $\langle f_d | d < \gamma \rangle$   
 is an increasing chain. Hence  
 $f = \bigcup_{d \in \gamma} f_d$  is a function with  $\text{dom}(f) = \gamma$  ④

(2): Suppose  $f_1, f_2 \in \mathcal{M}_d$  and  $\text{dom}(f_1) < \text{dom}(f_2)$

Then  $f_2 \upharpoonright d \in \mathcal{M}_d$ . By the uniqueness  $f_2 \upharpoonright d = f_1$

Hence  $f_1 \not\subseteq f_2$ .

⑤  $f$  satisfies (\*\*\*) For  $\lambda \in \gamma$  let  $\beta \in \gamma$  be s.t.  $d < \beta < \gamma$   
 $d \in \text{dom}(f_\beta)$  and  $f_\beta \upharpoonright \gamma$  satisfies (\*\*\*) Hence  $f(\lambda) = f_\beta(\lambda) = Q(f_\beta \upharpoonright d)$

$= Q(f \upharpoonright d)$ . Thus  $f \in \mathcal{M}_\gamma$ .

Let  $H = \bigcup \mathcal{F}_\alpha$ . By Claim (2),  $H$  is  
 a class function with domain  $\text{On}$

$H$  satisfies (\*\*\*) (This is seen similarly to the  
 last part of the proof of Claim).  $\square$

Von Neumann hierarchy : (in ZF without Axiom of Foundation)

$$V_\emptyset = \emptyset$$

$$V_{d+1} = P(V_d)$$

$$V_\gamma = \bigcup_{d \in \gamma} V_d \text{ for a limit } \gamma \quad (\text{an application-}t) \quad \text{Cn4}$$

Var 4 Let  $a$  be a set and  
 $Q_1$  class function on  $V$   
 $Q_2$  class function on  $Q_1$  in  $\text{Th}_3$   
 Then there is a unique class function on  $\text{On}$   
 p.t.

$$\begin{cases} H(\emptyset) = a \\ H(d+1) = Q_1(H(d)) \\ H(\gamma) = Q_2(H \upharpoonright \gamma) \text{ for a limit } \gamma \end{cases}$$

Lemma 5 (Also without Axiom of Foundation)

(1) For all  $\alpha \in \Omega$ ,  $V_\alpha$  is transitive.

(2) For any  $\alpha, \beta \in \Omega$  with  $\alpha < \beta$

$$V_\alpha \subseteq V_\beta$$

(3) For  $d \in \Omega$ ,  $V_d \cap \Omega = d$

(4) If Axiom of Foundation holds Then

$$V = \bigcup_{d \in \Omega} V_d \quad (5) \text{ If } V = \bigcup_{d \in \Omega} V_d \text{ Then Axiom of Foundation holds}$$

Proof (1): By induction (by Con 2)

①  $V_\alpha$  is transitive

② If  $V_\alpha$  is transitive then  $V_{\alpha+1} = P(V_\alpha)$  is transitive

If  $u \in P(V_\alpha)$  and  $v \in u$  then  $v \in V_\alpha$ . Hence

$N \in V_\alpha$ . By transitivity of  $V_\alpha$  it follows that

$\alpha \in V_\alpha$  Hence  $\alpha \in P(V_\alpha) = V_{\alpha+1}$

③ If  $V_\alpha$  is all transitive then  $\bigcup V_\alpha$  is also transitive:  
 $\text{if } u \in V_\beta, \alpha \in u$

then  $u \in V_\alpha$  for some  $\alpha < \beta$ . It follows that

$$\alpha \in V_\alpha \text{ Thus } \alpha \in \bigcup_{d < \beta} V_d = V_\beta.$$

(2): ...

$ZF^-$

Theorem 6 If  $ZF$  without Axiom of Foundation is consistent then so is  $ZF$  (with Axiom of Foundation)

Sketch of the proof In  $ZF$  without Axiom of Foundation

$$V^1 = \bigcup_{d \in \Omega} V_d$$

Including  
Axiom of Foundation

We prove (metamathematically!) that for any axiom  $\varphi$  of  $ZF$

that

$$ZF^- \vdash \varphi^{V^1}$$

For a class  $\ell$

$\varphi^\ell$  is defined inductively by

$$(\alpha \in \ell)^\ell = \alpha \in y$$

$$(\varphi \rightarrow \psi)^\ell = (\varphi^\ell \rightarrow \psi^\ell)$$

$$(\forall \ell)^\ell = \forall \ell^\ell$$

$$(\exists x \varphi)^\ell = \exists x (x \in \ell \wedge \varphi^\ell)$$

Suppose that  $ZF$  is inconsistent

then there is a proof when only finitely many axioms of  $ZF$  are involved

$$\varphi_0, \dots, \varphi_{n-1} \vdash x \neq x$$

The proof can be translated to the proof relativized to  $V^1$  to get

$$\varphi_0^{V^1}, \dots, \varphi_{n-1}^{V^1} \vdash x \neq x$$

$$\vdash x \neq x$$

$$ZF \vdash x \neq x$$

$$ZF^- \vdash x \neq x$$

$$ZF^- \vdash \alpha \neq \alpha$$

In  $ZF$  we can prove

$$ZF \vdash V_{\omega+\omega} \models \neg \varphi^{V^1}$$

$ZF$  has higher consistency strength than  $Z$

(By Completeness theorem it follows that

$$ZF \vdash \text{Con}_n(\neg \varphi^{V^1})$$

$ZFC \vdash V_\kappa \models ZFC$  if  $\kappa$  is an inaccessible cardinal

Thus  $ZFC + \text{"there is an inaccessible cardinal"}$

has higher consistency strength than  $ZFC$ .