Recap:

AF: Axiom of Foundation

Lemma 5 ZF w.o. AF

(1) For all $\alpha \in On$ V_α transitive

(2) $V_\alpha \subseteq V_\beta$ for $\alpha < \beta$, $\alpha, \beta \in On$

(3) $\alpha \in On$ $V_\alpha \cap On = \alpha$

$\boxed{V^* = \bigcup_{\alpha \in On} V_\alpha}$

(4) Cor 7

(5) $V = \bigcup_{\alpha \in On} V_\alpha \Leftrightarrow$ AF holds

AF: $\forall x (\underbrace{\exists y (y \in x) \to \exists y (y \in x \wedge y \cap x = \phi)}_{\Delta_0})$

$\boxed{V^* \text{ is a transitive class}}$

Th 8 For any transitive class (i.e. such class M that $\forall x \forall y ((x \in M \wedge y \in x) \to y \in M)$)

If φ is Δ_0-formula $(\varphi = \varphi(x_1 \dots x_n))$ then for any $a_1 \dots a_n \in M$

(*) $\varphi^M(a_1 \dots a_n) \leftrightarrow \varphi(a_1 \dots a_n)$

$(\forall x \in u) \varphi$: $\forall x (x \in u \to \varphi)$ $\Leftrightarrow \neg (\exists x \in u) \neg \varphi$

$(\exists x \in u) \varphi$: $\exists x (x \in u \wedge \varphi)$

$(\forall x \varphi)^M = \forall x (\underbrace{x \in M}_{\psi(x)} \to \varphi^M)$ where ψ defines the class M

$(\exists x \varphi)^M = \exists x (x \in M \wedge \varphi^M)$

$(\varphi \to \psi)^M = (\varphi^M \to \psi^M)$ $(\neg \varphi)^M = \neg \varphi^M$

Proof of Th 8 By induction on the construction of the formula φ in meta-mathematics.

If φ, ψ are Δ_0-formula and satisfies (*) then it is easy to see that $(\varphi \to \psi)$ $\neg \varphi$ also satisfy (*)

Suppose that φ is of the form $(\exists x \in x_0) \psi$ where $\psi = \psi(x, x_0, \dots x_{m-1})$

Suppose $\varphi^M(a_0, \dots a_{m-1})$ for some $a_0, \dots, a_{m-1} \in M$. Then

$\varphi = \exists x (x \in x_0 \wedge \psi)$

$\varphi^M = \exists x (x \in M \wedge (x \in x_0 \wedge \psi)^M)$

$= \exists x (x \in M \wedge x \in x_0 \wedge \psi^M)$

b) If φ is of the form $\exists x \psi$ where ψ is (logically) equivalent to a Δ_0-formula and there is $a \in M$ s.t. $\psi(a)$, then we have φ^M,

Thus we have

for $a_0, \dots, a_{m-1} \in M$

$\exists x (x \in M \wedge x \in a_0 \wedge \psi^M(x, a_0, \dots, a_{m-1}))$

$\Leftrightarrow \exists x (x \in M \wedge x \in a_0 \wedge \psi(x, a_0, \dots a_{m-1}))$

$\to \exists x (x \in a_0 \wedge \psi(x, a_0 \dots a_{m-1})) \Leftrightarrow \varphi(a_0, \dots a_{m-1})$

Suppose now $\varphi(a_0, \dots a_{m-1})$ hold. This means

$\exists x (x \in a_0 \wedge \psi(x, a_0, \dots a_{m-1}))$ Let $b \in a_0$ s.t.

$\psi(b, a_0, \dots, a_{m-1})$ holds

Then, since M is transitive, $b \in M$. So by the assumption of the induction, we have

$\exists x (x \in M \wedge x \in a_0 \wedge \psi^M(x, a_0 \dots a_{m-1}))$

$= \varphi^M(a_0 \dots a_{m-1})$ ∎

Lemma 9 If M is transitive and

a) φ is a $\forall$-closure of Δ_0-formula of the form $\forall x_0 \dots \forall x_{k-1} \psi$ where ψ is Δ_0.

If $\psi(a_0, \dots, a_{k-1})$ holds for all $a_0, \dots a_{k-1} \in M$ then φ^M holds

Proof a):

$\varphi^M = \forall x_0 \dots \forall x_{k-1} (x_0, \dots x_{k-1} \in M \to \psi^M)$

$\Leftrightarrow \forall x_0 \dots \forall x_{k-1} (x_0, \dots x_{k-1} \in M \to \psi)$ (by Th 8)

Thus φ^M holds. ∎

Cor 10 AF^{V^*} Proof By Lemma 9 + the proof of Lemma 5 (4) and (5)

Thm 11 In ZF(C) w.o. AF,

for any formula φ in ZF(C) (including AF) we have φ^{V^*}.

Proof AF^{V^*} is already seen in Cor 10.

"$x = \phi$" is Δ_0: $\quad \forall y\,(y \notin x)$

$\Leftrightarrow \forall y\,(y \in x \to x \neq x)$

$(\forall y \in x)\,(x \neq x)$

Since $\phi \in V^*$

$(\text{Axiom of Emptyset})^{V^*}$ by Lemma 10 (b).

(b) If φ is of the form $\exists x \psi$ and ψ is equivalent to a Δ_0-formula
(logically)

and if there is $a \in M$ s.t. $\psi(a)$

then ψ^M holds

proof of b): If $\psi(a)$ for $a \in M$ then $\psi^M(a)$

Thus $\exists x\,(x \in M \wedge \psi^M)$ holds

$\Updownarrow$

φ^M

Axiom of Extensionality

$\forall x \forall y\,(\forall u\,(u \in x \leftrightarrow u \in y) \to x \equiv y)$

$$\Leftrightarrow \forall x \forall y\,\Big(\underbrace{\underbrace{(\forall u \in x)(u \in y)}_{\Delta_0} \wedge \underbrace{(\forall u \in y)(u \in x)}_{\Delta_0}}_{\Delta_0} \to x \equiv y\Big)$$

$\psi(x, y)$

But since Axiom Extensionality holds we have $\psi(a, b)$ for all $a, b \in V^*$. Thus Lemma 9 a) implies that $(\text{Axiom of Extensionality})^{V^*}$

⋮

Axiom of Replacement

$$(\forall x \exists! y\,(\varphi(x, y, p)) \to \forall u \exists v\,\overbrace{(\forall x \in u\, \exists y \in v\, \varphi(x, y, p))}^{M}$$

Suppose that for some $a \in M$

$\forall x \exists! y\, \varphi(x, y, a)^{V^*}$ holds

Let $\tilde{\varphi}$ be the formula

$(x \notin V^* \to y \equiv 0) \wedge (x \in V^* \to \varphi^M(x, y, p))$

Then we have

$\forall x \exists! y\, \tilde{\varphi}(x, y, a)$

By the Axiom of Replacement "in the real V".

For any $b \in M$ there is C s.t.

$C = \{y \mid \tilde{\varphi}(x, y, a) \text{ for some } x \in b\}$

By definition of $\tilde{\varphi}$ $C \subseteq V^*$. There is some $\alpha \in On$ s.t. $C \subseteq V_\alpha$. Thus $C \in V_{\alpha+1} \subseteq V^*$

$\eta^{V^*}(b, c)$. This shows that ξ^{V^*} …

Thm 11 In ZF(C) w.o. AF,
for any formula φ in ZF(C) (including AF) we have φ^{V^*}.

Thm 12 (Lévy-Montague Reflection Principle)
For every $\mathcal{L}_\in$-formula $\varphi = \varphi(x_1 .. x_n)$ there are club many $\alpha \in On$ s.t. φ is absolute between V_α and V^*

M is a transitive class and m is a transitive set with $m \subseteq M$. A formula $\varphi = \varphi(x_1 .. x_n)$ to be absolute between M and m if for any $a_1 \dots a_n \in m$ $m \models \ulcorner\varphi\urcorner(a_1 .. a_n)$ $\Leftrightarrow$ $\varphi^M(a_1 .. a_n)$ (Note M can be also V!)

Thm 13 If ZF(C) w.o. AF is consistent then so is ZF(C) (with AF)

Proof Assume that ZF(C) w.o. AF is consistent but there is a proof P of a contradiction from ZF(C). Let $\varphi_0 \dots \varphi_{n-1}$ be the axioms of ZF(C) which appear in P. Let $\varphi = \varphi_0 \wedge \dots \wedge \varphi_{n-1}$.
In ZF(C) w.o. AF.

Let α be an ordinal s.t. φ is absolute between V_α and V^*
Then since $\varphi^{V^*} = (\varphi_0^{V^*} \wedge \dots \wedge \varphi_{n-1}^{V^*})$
φ^{V^*} holds by Thm 11.
Hence we have $V_\alpha \models \varphi$ but because P proves a contradiction $\phi \not\equiv \phi$ can be proved from φ
Hence $V_\alpha \models \phi \not\equiv \phi$ A contradiction.
(This contradiction is driven in ZF(C) w.o. AF)
This is a contradiction to the assumption that ZF(C) w.o. AF is consistent. $\boxtimes$

Suppose that for some $a \in M$
$\forall x \exists! y\, \varphi(x, y, a)^{V^*}$ holds
Let $\tilde{\varphi}$ be the formula
$(x \notin V^* \to y \equiv 0) \wedge (x \in V^* \to \varphi^M(x, y, p))$
Then we have
$\forall x \exists! y\, \tilde{\varphi}(x, y, a)$
By the Axiom of Replacement " in the real V".
For any $b \in M$ there is C s.t.
$C = \{ y \mid \tilde{\varphi}(x, y, a) \text{ for some } x \in b \}$
By definition of $\tilde{\varphi}$ $C \subseteq V^*$ There is some $\alpha \in On$ s.t. $C \subseteq V_\alpha$ Thus $C \in V_{\alpha+1} \subseteq V^*$
$\eta^{V^*}(b, c)$. This shows that $\{ ^{V^*} \dots$ (5)