

Thm 12 (Lövy-Naughton Reflection Principle)

Let  $\varphi$  be a  $\mathcal{L}_\infty$ -formula

Suppose that  $M$  is a transitive class and  $(M_\alpha \mid \alpha \in \mathcal{O}_n)$  is a continuous increasing sequence of subsets of  $M$  with

$M = \bigcup_{\alpha \in \mathcal{O}_n} M_\alpha$

$\Pi_\gamma = \bigcup_{\alpha < \gamma} M_\alpha$   
for all limit ordinal  $\gamma$

(cf.  $V^* = \bigcup_{\alpha \in \mathcal{O}_n} V_\alpha$ )

Then there is a club unbounded class  $\mathcal{C}_\varphi$  of ordinals s.t. for every  $\delta \in \mathcal{C}_\varphi$  all subformulas of  $\varphi$  are absolute in  $M$  above  $M_\delta$

Proof By induction on  $\varphi$

A class  $\mathcal{C}$  of ordinals is club unbounded if

- ①  $\forall \delta \in \mathcal{O}_n \exists \beta \in \mathcal{C} \delta < \beta$  (unbounded)
- ② For any  $\gamma \in \mathcal{O}_n$  if  $\gamma \cap \mathcal{C}$  is unbounded in  $\gamma$  (closed) then  $\gamma \in \mathcal{C}$

For a class  $M$  and a set  $M \subseteq M$   $\varphi = \varphi(x_1, \dots, x_n)$  is absolute in  $M$  above  $\Pi$  if, for all  $a_1, \dots, a_n \in \Pi$

$M \models \varphi(a_1, \dots, a_n) \Leftrightarrow \varphi^M(a_1, \dots, a_n)$

For an atomic formula  $x \equiv y$   $x \in y$  we may take

$\mathcal{C}_\varphi = \mathcal{O}_n$  If the theorem holds for  $\varphi$  and  $\psi$  with  $\mathcal{C}_\varphi$  and  $\mathcal{C}_\psi$

Then we may take  $\mathcal{C}_{\varphi \wedge \psi} = \mathcal{C}_\varphi \cap \mathcal{C}_\psi$

Fact If  $\mathcal{C}, \mathcal{D} \subseteq \mathcal{O}_n$  are club then  $\mathcal{C} \cap \mathcal{D}$  is also club  
To see that  $\mathcal{C} \cap \mathcal{D}$  unbounded let  $\delta \in \mathcal{O}_n$ . Then let  $\delta < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$  s.t.  $\alpha_i \in \mathcal{C}$   $\beta_i \in \mathcal{D}$  (possible  $\mathcal{C}, \mathcal{D} \neq \emptyset$ ) Let  $\gamma = \sup \{\alpha_i, \beta_i \mid i < \omega\}$   
 $\gamma \in \mathcal{C}$  by ① and  $\gamma \in \mathcal{D}$  by ②. Similarly  $\gamma \in \mathcal{D}$ .

Suppose that  $\varphi = \exists x \psi(x, x_1, \dots, x_n)$  and  $\psi$  satisfies the theorem with the club class  $\mathcal{C}_\psi$ . It is enough to find  $\mathcal{C}_\varphi \subseteq \mathcal{C}_\psi$  for which we should have  $\varphi$  is absolute in  $M$  over each  $M_\delta$   $\delta \in \mathcal{C}_\varphi$   
Let

$\mathcal{C}_\varphi = \{\delta \in \mathcal{C}_\psi \mid \varphi \text{ is absolute in } M \text{ over } M_\delta\}$

Claim 1  $\mathcal{C}_\varphi$  is closed. Suppose that  $\gamma \cap \mathcal{C}_\varphi$  is unbounded in  $\gamma$ . Let  $a_1, \dots, a_n \in M_\gamma$  and suppose that  $M_\delta \models \varphi(a_1, \dots, a_n)$   
Then there is  $\alpha \in M_\delta$  s.t.  $M_\alpha \models \psi(\alpha, a_1, \dots, a_n)$  (CK)

By the induction on  $\mathcal{L}_\infty$  and since  $\gamma \in \mathcal{C}_\psi$  and  $\alpha < \gamma$  we have  $\psi^M(\alpha, a_1, \dots, a_n)$ . Since  $\alpha \in M$  it follows that  $\varphi^M(a_1, \dots, a_n)$ .

Assume now that  $\varphi^M(a_1, \dots, a_n)$  holds. Then there is  $\gamma_0 \in \gamma \cap \mathcal{C}_\psi$  s.t.  $a_1, \dots, a_n \in M_{\gamma_0}$ . Since  $\gamma_0 \in \mathcal{C}_\psi$  we have

$M_{\gamma_0} \models \varphi(a_1, \dots, a_n)$  Then there is  $\alpha \in M_{\gamma_0}$  s.t.  $M_{\gamma_0} \models \psi(\alpha, a_1, \dots, a_n) \Leftrightarrow \psi^M(\alpha, a_1, \dots, a_n)$

$\Leftrightarrow M_{\gamma_0} \models \psi(\alpha, a_1, \dots, a_n)$   
 $\Rightarrow M_{\gamma_0} \models \varphi(a_1, \dots, a_n)$   
Thus  $\gamma \in \mathcal{C}_\varphi$

Q:  $\mathcal{L}_\psi$  is unbounded.

Let  $\alpha \in \mathcal{O}_n$

Let  $\alpha = \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \dots$

be a sequence of ordinals  $\in \mathcal{L}_\psi$

$\gamma_{\delta+1} = \min \{ \beta \in \mathcal{O}_n \mid \text{for } a, \gamma_1, \dots, \gamma_n \in M_\beta, \text{ if } \psi^m(a, a_1, \dots, a_n) \text{ for some } a \text{ then there is } a' \in M_\beta \text{ s.t. } \psi^m(a', a_1, \dots, a_n) \}$

Let  $Y = \bigcup_{\alpha \in \mathcal{O}_n} Y_\alpha, Y \in \mathcal{L}_\psi$

Subcl:  $Y \in \mathcal{L}_\psi$

It is enough to show that  $\psi$  is absolute in  $M$  over  $M_\gamma$

Suppose  $M_\gamma \models \psi(a_1, \dots, a_n), a_1, \dots, a_n \in M_\gamma$

Then there is  $\alpha \in M_\gamma$  s.t.  $M_\gamma \models \psi(a, a_1, \dots, a_n)$

Since  $Y \in \mathcal{L}_\psi$ , it follows that  $\psi^m(a, a_1, \dots, a_n)$  holds. Since  $\alpha \in M_\gamma \subseteq M$ ,  $\psi^m(a, a_1, \dots, a_n)$

Suppose now  $\psi^m(a_1, \dots, a_n)$  for  $a_1, \dots, a_n \in M_\gamma$

Then there is some  $\beta \in \mathcal{O}_n$  s.t.  $a_1, \dots, a_n \in M_{\gamma_\beta}$

By definition of  $\gamma_{\delta+1}$  there is some  $\alpha \in M_{\gamma_{\delta+1}}$  s.t.  $\psi^m(a, a_1, \dots, a_n)$

Since  $Y \in \mathcal{L}_\psi$ , it follows that  $M_\gamma \models \psi(a, a_1, \dots, a_n)$  so  $M_\gamma \models \psi(a_1, \dots, a_n)$   $\square$

Prop 1 Suppose the following axiom systems

- (T<sub>0</sub>) ZFC
- (T<sub>1</sub>) ZFC +  $\exists m (m \models \ulcorner \text{ZFC} \urcorner)$
- (T<sub>2</sub>) ZFC +  $\exists m (m \models \ulcorner \text{ZFC} \urcorner \wedge m \text{ is an } \omega\text{-model})$
- (T<sub>3</sub>) ZFC +  $\exists m (m \models \ulcorner \text{ZFC} \urcorner \wedge m \text{ is transitive and } \epsilon\text{-model})$
- (T<sub>4</sub>) ZFC +  $\exists \alpha \in \mathcal{O}_n (\forall \delta \models \ulcorner \text{ZFC} \urcorner)$
- (T<sub>5</sub>) ZFC +  $\exists \kappa (\kappa \text{ is an inaccessible cardinal})$

For all  $i < 5$

$(T_{i+1} \vdash \text{consis}(\ulcorner T_i \urcorner))$

$T_{i+1} \vdash T_i$  and

By completeness Thm  $M \models \exists m (m \models \ulcorner \text{ZFC} \urcorner)$   
 Thus  $M \models \ulcorner T_1 \urcorner$

Proof

$T_1 \vdash \text{consis}(T_0)$ : trivial since if there is a model of theory then the theory is consistent

$T_2 \vdash \text{consis}(\ulcorner T_1 \urcorner)$ : Work in  $T_2$

Let  $\mathcal{M}$  be a model of  $\ulcorner \text{ZFC} \urcorner$  s.t.  $\mathcal{M}$  is an  $\omega$ -model. We may assume that

$(V_\omega)^\mathcal{M} = V_\omega$  and  $\epsilon^\mathcal{M} \cap (V_\omega)^2 = \epsilon_\omega(V_\omega)^2$

In particular  $(\ulcorner T_1 \urcorner)^\mathcal{M} = \ulcorner T_1 \urcorner$

$M \models \text{consis}(\ulcorner \text{ZFC} \urcorner)$  [otherwise there would be  $\underline{P} \in M$

of inconsistency from  $\ulcorner \text{ZFC} \urcorner$ . But  $\underline{P}$  is really a proof of inconsistency]