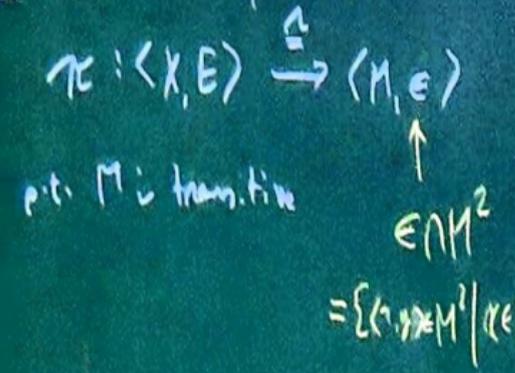


Mostowski's Collapsing Lemma

Suppose E is extensional on well founded binary relation on X .

Then there is a unique



Def $E \subseteq X^2$ is extensional

$(\Rightarrow) \langle X, E \rangle \neq$ extensionality

$(\Leftarrow) \forall x, x' (\forall z (z E x \leftrightarrow z E x') \rightarrow x = x')$

$E \subseteq X^2$ is well-founded $(\Leftrightarrow) \forall Y \subseteq X (Y \neq \emptyset \rightarrow \exists y \in Y (\forall y' \in Y (y' \notin y)))$

$R \subseteq X^2$ is set-like if for all $x \in X$ $\{y \in X \mid y R x\}$ is a set

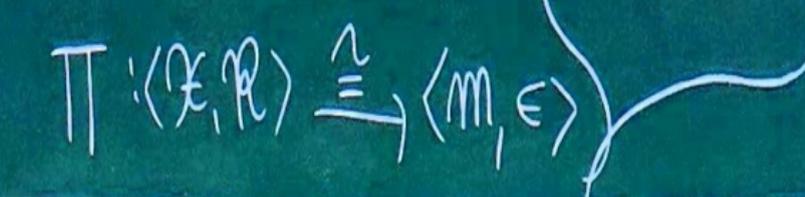
$ext_R(x) = \{y \in X \mid y R x\}$
is a set

A generalization Suppose that R is a (class) relation over a class X ($R \subseteq X^2$) p.t. R is set-like extensional

$\{ \langle x, y \rangle \mid x, y \in X \}$
[i.e. $\forall x, x' \in X (\forall z \in X (z R x \leftrightarrow z R x') \rightarrow x = x')$]

and well-founded
[i.e. $\forall Y \subseteq X (Y \neq \emptyset \rightarrow \exists y \in Y (\forall y' \in Y (y' \notin y)))$]

then there is a unique



Lemma 1 If R is well-founded

then $x \notin x$ for all $x \in X$

Proof Otherwise $\{x\} \subseteq X$ would be a counter example to the well-foundedness of R . \square

For $m \in U$ and $x \in X$ let $ext_R^m(x)$ be defined by:

$ext_R^0(x) = \{x\}$

$ext_R^{n+1}(x) = \{y \in X \mid y R z \text{ for some } z \in ext_R^n(x)\}$
 $= \bigcup \{ ext_R(z) \mid z \in ext_R^n(x) \}$

If R is ext-like then $\text{ext}_R^n(x)$ is a set $\forall x \in X$ is transitive w.r.t R for all $n \in \mathbb{N}$.

$$\text{trcl}_R(x) = \bigcup_{n \in \mathbb{N}} \text{ext}_R^n(x)$$

$$\text{trcl}_R^-(x) = \bigcup_{n \in \mathbb{N}} \text{ext}_R^{n+1}(x)$$

↑
 retr if R is ext-like
 By Lemma 2 if R is well-founded then

$$\text{trcl}_R^-(x) = \text{trcl}_R(x) \setminus \{x\}$$

if for any $x \in Y$ and $y \in X$ with $y R x$ we have $y \in Y$

Lemma 2 Suppose R is a ~~well~~-founded relation on X , then

(1) $\text{trcl}_R(x)$ and $\text{trcl}_R^-(x)$ are transitive.

$\text{trcl}_R(x)$ is the minimal subclass C of X with the property that $x \in C$ and C is transitive. w.r.t R

$\text{trcl}_R^-(x)$ is the minimal subclass C of X with the property that $x \in C$ and C is transitive. w.r.t R

$$\text{trcl}_R(x) = \bigcup \{ \text{trcl}_R(y) \mid y R x \} \cup \{x\}$$

$$\text{trcl}_R^-(x) = \bigcup \{ \text{trcl}_R^-(y) \mid y R x \}$$

(3) $A \subseteq X$ is transitive $\Leftrightarrow \text{trcl}_R(x) \subseteq A$ for all $x \in A$

Proof (1): we show the transitivity of $\text{trcl}_R(x)$

Suppose $y \in \text{trcl}_R(x)$ then there is $n \in \mathbb{N}$ s.t.

$y \in \text{ext}_R^n(x)$, if $z R y$ then

$z \in \text{ext}_R^{n+1}(x)$. Hence $z \in \text{trcl}_R(x)$.

If $Y \subseteq X$ is transitive and $x \in Y$ then by induction on $n \in \mathbb{N}$ we can show

$$\text{ext}_R^n(x) \subseteq Y$$

Thus $\text{trcl}_R(x) \subseteq Y$

The assertions on $\text{trcl}_R^-(x)$ can be proved similarly

(2): Let t be the right side of \subseteq

t is transitive and $x \in t$. Hence

$$\text{trcl}_R(x) \subseteq t \quad \text{by (1)}$$

$$\text{trcl}_R^-(x) \subseteq \text{trcl}_R(x) \quad \text{by (1)}$$

Hence $\text{trcl}_R(x) \supseteq t$.

The second equality is proved similarly.

(3): If A is transitive and $x \in A$ then

$$\text{trcl}_R(x) \subseteq A \quad \text{by (2)}$$

Conversely if $\text{trcl}_R(x) \subseteq A$ for all $x \in A$ then for y with

$$y R x \quad y \in \text{ext}_R^1(x) \subseteq \text{trcl}_R(x) \subseteq A \quad \text{Hence } y \in A$$

Lemma 3 Suppose \mathcal{R} is a class \mathcal{R} is a well-founded and set-like class relation. Then for any class $\mathcal{Y} \subseteq \mathcal{X}$ there is a minimal element of \mathcal{Y} w.r.t \mathcal{R} .

proof Let $y_0 \in \mathcal{Y}$ and consider $S = \text{trcl}_{\mathcal{R}}(y_0)$. S is $\neq \emptyset$ since \mathcal{R} is set-like.

$y_0 \in \mathcal{Y} \cap S \Rightarrow \mathcal{Y} \cap S$ has a minimal element. y_1 is a minimal element of \mathcal{Y} . Suppose otherwise.

Then there is $y_2 \in \mathcal{Y}$ s.t. $y_2 \mathcal{R} y_1$

Since $y_1 \in D = \text{trcl}_{\mathcal{R}}(y_0)$ there is $m \in \mathbb{N}$ s.t.

$y_1 \in \text{ext}_{\mathcal{R}}^m(y_0)$ it follows that $y_2 \in \text{ext}_{\mathcal{R}}^{m+1}(y_0)$

Then $y_2 \in \text{trcl}_{\mathcal{R}}(y_0)$. A contradiction to the minimality of y_1 . \square

Theorem (Recursive definition of class function)

Suppose that \mathcal{X} is a class and \mathcal{R} is a well-founded set-like class relation on \mathcal{X} .

(1) If $\mathcal{Y} \subseteq \mathcal{X}$ satisfies that

- (*) For any $y \in \mathcal{Y}$ if
- (**) For any $x \in \mathcal{X}$ with $x \mathcal{R} y$ we have $x \in \mathcal{Y}$

then $y \in \mathcal{Y}$

then $\mathcal{Y} = \mathcal{X}$

(2) Let $\mathcal{H} = \{ f : f \text{ is a function } \text{dom}(f) \text{ is a transitive subset of } \mathcal{X} \}$

If G is a class function on $\mathcal{H} \times \mathcal{X}$

then there is a class function f s.t.

For all $x \in \mathcal{X}$ $f(x) = G(\mathcal{H} \upharpoonright \text{trcl}^-(x), x)$

f is unique with (*).

proof next time in 2 weeks.

Proof the generalization of Mostowski's Lemma:

Let $\Pi : \mathcal{X} \rightarrow V$ be defined by, for $a \in \mathcal{X}$,

$$\Pi(a) = \{ \Pi(b) \mid b \in \text{ext}_{\mathcal{R}}^-(a) \}$$

Π is taken by Thm 4 (2):

$$G : \mathcal{H} \times \mathcal{X} \rightarrow V$$

$$G(\langle f, a \rangle) = \begin{cases} \{ f(b) \mid b \mathcal{R} a \}, & \text{if } \text{dom}(f) = \text{trcl}_{\mathcal{R}}^-(a) \\ \emptyset, & \text{otherwise} \end{cases}$$

Then f is Thm 4 (2) for this G satisfies the property of Π

Let $M = \Pi''\mathcal{X}$ Then $\circlearrowleft \Pi$ is 1-1

① M is transitive

② $\Pi : \langle \mathcal{X}, \mathcal{R} \rangle \xrightarrow{\cong} \langle M, \in \rangle$ \square