Linear 1　expected/possible questions in the Final Exam

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This list may be yet extended until 6. June. Existing questions may also be further corrected and/or edited. Please check the file on the internet till shortly before the exam.

In the final exam on 6. June questions which are variants of some of the questions below will be asked. In the test, not the calculation skill but rather your understanding of the materials will be asked.
In addition, at the test, there will be one or two more challenging questions not included in this list.　　The most up-to-date version of this list is downloadable as:

`http://fuchino.ddo.jp/kobe/lin-alg-1-2-j-1q-pre-final.pdf`

$\boxed{\text{I.}}$ Let $A = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Answer the following questions:

(1) Compute the following:

(a) $BA + CA$,　(b) $(B + C)^2$,　(c) $(B + C)^8$,　(d)C^5,　(e) C^{57},　(f) D^2,　(g) D^{567}
(h) $(CD)^{27}$

(2) Find the 2×2-matrix F such that $(B + C)F = E$ where E denotes the 2×2 unit matrix.

(3) C is a rotation matrix. To which angle does this rotation matrix belong?

(4) Noting (3), find the 2×2-matrix G which satisfies $G^3 = C$.

(5) Show the uniqueness of the matrices F (2). Show that, in (4), there are at least two different G's with $G^3 = C$.

(6) Show that, for any $n \in \mathbb{N}$, there is a 2×2 matrix A such that $A \neq E$, $A/= E$,..., $A^n \neq E$ but $A^{n+1} = E$. (Hint: find such A among rotation matrices.)

$\boxed{\text{II.}}$

(1) Let $\mathbb{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbb{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and let θ be the angle between the plane vectors $\mathbb{a}$ and $\mathbb{b}$. Compute $\sin\theta$.

(2) For the plane vectors $\mathbb{a}$ and $\mathbb{b}$ as in (1) compute the area of the parallelogram spanned by $\mathbb{a}$ and $\mathbb{b}$.

$\boxed{\text{III.}}$ Let $\mathbb{a} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $\mathbb{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(1) Find all space vectors $\mathbb{c}$ perpendicular to the plane spanned by $\mathbb{a}$ and $\mathbb{b}$.

(2) What is the area of the parallelogram spanned by $\mathbb{a}$ and $\mathbb{b}$?

$\boxed{\text{IV.}}$ Solve the following systems of linear equations:

Possible answers, hints and explanations

$\boxed{\text{I.}}$: (3): $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$ Note that from this it follows that $C = R_{\frac{\pi}{2}}$, $C^4 = E$ and $C^5 = E$.

(4): Rotaion by angle $\frac{\pi}{2}$ is equal to three times rotations by angle $\frac{\pi}{6}$. Thus $C = R_{\frac{\pi}{2}} = (R_{\frac{\pi}{6}})^3$ and $G = R_{\frac{\pi}{6}}$. is as desired

(5): Since $\frac{5\pi}{6} \times 3 = \frac{15\pi}{6} = 2\pi + \frac{\pi}{2}$, we have $(R_{\frac{5\pi}{6}})^3 = R_{\frac{\pi}{2}}$. Thus $G = R_{\frac{\pi}{2}}$ also satisfies the equation $G^3 = C$.

(6): An argument similar to that of (4) shows that $A = R_{\frac{2\pi}{n}}$ is as desired.

$\boxed{\text{II.}}$:

(1) : Since $\det([\mathbb{a}\mathbb{b}]) = |\mathbb{a}| \cdot |\mathbb{b}| \cdot \sin\theta$ (see the proof of Theorem 5.4),

$$\sin\theta = \frac{\det([\mathbb{a}\mathbb{b}])}{|\mathbb{a}| \cdot |\mathbb{b}|}$$

$\boxed{\text{III.}}$: (1): Remembering the geometrical characterization of inner product, a space vector $\mathbb{c}$ is perpendicular to the plane spanned by $\mathbb{a}$ and $\mathbb{b}$ if and only if $\mathbb{a} \cdot \mathbb{c} = 0$, $\mathbb{b} \cdot \mathbb{c} = 0$.

(2): Let $\mathbb{c}$ a vector perpendicular to the plane spanned by $\mathbb{a}$ and $\mathbb{b}$ with $|\mathbb{c}| = 1$. By the geometrical characterization of the determinant of 3×3 matrices, $|\det([\mathbb{a}\mathbb{b}\mathbb{c}])|$ is the area of the parallelogram spanned by $\mathbb{a}$ and $\mathbb{b}$.