Linear Algebra I Exercises I

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This list of exercises (and its possible further update) is downloadable as:

`http://kurt.scitec.kobe-u.ac.jp/~fuchino/kobe/lin-alg1-ss14-ex1.pdf`

Some other materials connected to to the lecuture might be found at:

`http://kurt.scitec.kobe-u.ac.jp/~fuchino/kobe/index.html`

A lecture note of the course will be also linked to this page in the course of the semester.

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1. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

Calculate (1) AB, (2) $B + C$, (3) $7A - 3B$, (4) $AB + AC$
(Hint for (4): use The Distributive Law (see **4.**) to simplify the calculation),

2. Let

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Calculate AB, BA, A^2, B^2, A^3, B^3.

3. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Calculate A^2, A^3, A^4.

4. For $l \times m$ matrix $A = [a_{i,j}]$, and $m \times n$ matrices $B = [b_{j,k}]$, $C = [c_{j,k}]$, show that the following equation (The Distributive Law) always holds: $A(B + C) = AB + AC$. Show that the corresponding calculation rule $(A + B)C = AC + BC$ also holds (note that the size of matrices should be declared differently for this equation).

5. For $a_1, \ldots, a_n \in \mathbb{R}$, let $diag(a_1, \ldots, a_n)$ be the $n \times n$ matrix $D = [d_{i,j}]$ (the diagonal matrix with diagonal entries $a_1, \ldots, a_n$) defined by:

$$d_{i,j} = \begin{cases} a_i, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}$$

(1) What is $diag(2, 3, -1, 4)$?

(2) Show the following equation:
$$diag(a_1, a_2, \ldots, a_n)diag(b_1, b_2, \ldots, b_n) = diag(a_1b_1, a_2b_2, \ldots, a_nb_n).$$

(3) Show that, for an $m \times n$ matrix $A = [\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_n]$,
$$A\ diag(a_1, a_2, \ldots, a_n) = [a_1\mathbf{a}_1\ a_2\mathbf{a}_2\ \cdots\ a_n\mathbf{a}_n]$$
holds.

$\boxed{\textbf{6.}}$ Find the matices M_{φ_1}, $M_{\varphi 2}$ corresponding to the following linear mappings φ_1, φ_2:

$$\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^2; \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \text{(projection)}$$

$$\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^5; \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(canonical embedding)}$$

$\boxed{\textbf{7.}}$ Show the following:
(1) If $a_1, ..., a_n \in \mathbb{R}$ are all $\neq 0$ then $diag(a_1, ..., a_n)$ is invertible and $(diag(a_1, a_2, ..., a_n))^{-1} = diag(\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_n})$.
(2) If at least one of $a_1, ..., a_n \in \mathbb{R}$ is equal to 0 then $diag(a_1, ..., a_n)$ is not invertible.

$\boxed{\textbf{8}}$ (1) Suppose that both of $n \times n$-matrices A and B are invertible. Show that then the matrices AB and BA are invertible as well.

(2) Auppose that A, B are $n \times n$-matrices and A is invertible. Show that AB is invertible if and only if B is invertible[1].

[1] Actually we can even show: For $n \times n$-matrix A and B, AB (or BA) is invertible if and only if both A and B are invertible. But for the proof of this theorem, we need some deep results we will learn later. In contrast, the assertion of $\boxed{\textbf{8}}$, (2) can be proved quite easily.