Linear Algebra II – final exam on Jan. 30

(2014/01/30)

2013/14 Fall Semester, Sakaé Fuchino (TA: Diego Mejía)

---

① Calculate the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -5 \\ -2 & 3 & -1 & 4 \\ 7 & -1 & 3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 9 & 6 & 7 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & 4 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

② Find the inverse of the following matrix: $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$

③ Diagonalize the matrix: $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

④ Which of the following are linear mappings. Explain why. Determine the matrices M_φ corresponding to the linear mappings φ among the following.

(a) $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$ (b) $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

(c) $\varphi_3 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x\sin\theta \\ y\cos\theta \end{bmatrix}$ (d) $\varphi_4 : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

(e) $\varphi_5 : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ 0 \\ x \end{bmatrix}$ (f) $\varphi_6 : \mathbb{R}^2 \to \mathbb{R}^4; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(g) $\varphi_7 : \mathbb{R} \to \mathbb{R}^2; x \mapsto \begin{bmatrix} x \\ x^2 \end{bmatrix}$ (h) $\varphi_8 : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ x+2y \end{bmatrix}$

⑤ Let $\varphi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation counter-clockwise through the angle θ ($0 \leq \theta < 2\pi$) around the origin.

(a) Show that φ_θ is a linear mapping.

(b) Determine the matrix M_{φ_θ} corresponding to the linear mapping φ_θ.

(c) Find a geometric explanation for the fact that M_{φ_θ} does not have any eigenvector if $\theta \neq 0$.

(d) For which θ can M_{φ_θ} be diagonalized?

⑥ Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that $\varphi(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\varphi(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$. (a) Find the matrix M_φ representing the linear mapping φ. (b) Decide $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}(\varphi)$.

⑦ Show that any linear mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ satisfies the following: (a) $\varphi(\mathbf{0}) = \mathbf{0}$. (b) If $\varphi(\mathbf{a}) \neq \mathbf{0}$ for some $\mathbf{a} \in \mathbb{R}^m$ then $\varphi(\mathbf{a}+\mathbf{b}) \neq \varphi(\mathbf{b})$ for all $\mathbf{b} \in \mathbb{R}^m$.

⑧ Suppose that ε is an eigenvalue of an $n \times n$ matrix A. Show that

$$\{\mathbf{u} \in \mathbb{R}^n \,:\, \mathbf{u} \text{ is an eigenvector of } A \text{ with eigen value } \varepsilon\} \cup \{\mathbf{0}\}$$

is a linear subspace of $\mathbb{R}^n$.

Example of Possible Answers

① (a):

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -5 \\ -2 & 3 & -1 & 4 \\ 7 & -1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -5 \\ 0 & 7 & -3 & 10 \\ 0 & -15 & 10 & -23 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 & -5 \\ 7 & -3 & 10 \\ -5 & 10 & -23 \end{vmatrix} = -134$$

(b):

$$\begin{vmatrix} 1 & 9 & 6 & 7 \\ 0 & 3 & 9 & 6 \\ 0 & 2 & 6 & 4 \\ 1 & 4 & 4 & 0 \end{vmatrix} = 0,$$

since the third row of the matrix is $\frac{2}{3}$ times the second row.

②:

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \end{array}\right]$$

Hence $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$.

③: Solving the characteristic polynomial, we obtain the eingenvalues of our matrix A as:

$$\begin{vmatrix} 3-\varepsilon & -1 \\ -1 & 2-\varepsilon \end{vmatrix} = 0 \Leftrightarrow \varepsilon^2 - 5\varepsilon + 5 = 0 \Leftrightarrow \varepsilon = \frac{5 \pm \sqrt{5}}{2}.$$

For the eigenvalue $\varepsilon = \dfrac{5+\sqrt{5}}{2}$ of A, one of the non-trivial solution of the equation $A\mathbf{x} = \varepsilon\mathbf{x}$ is $\begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix}$.
For $\varepsilon = \dfrac{5-\sqrt{5}}{2}$, $\begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix}$. Thus, letting $U = \begin{bmatrix} 2 & 2 \\ 1-\sqrt{5} & 1+\sqrt{5} \end{bmatrix}$, we have $U^{-2}AU = \begin{bmatrix} \frac{5+\sqrt{5}}{2} & 0 \\ 0 & \frac{5-\sqrt{5}}{2} \end{bmatrix}$.

④ (a): φ_1 is not a linear mapping since $\varphi_1(\mathbf{0}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \mathbf{0}$.

(b): φ_2 is a linear mapping since φ_2 is represented as $\varphi_2(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

In particular $M_{\varphi_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

(c): φ_3 is a linear mapping since φ_3 is represented as $\varphi_3(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

In particular $M_{\varphi_3} = \begin{bmatrix} \sin\theta & 0 \\ 0 & \cos\theta \end{bmatrix}$.

(d) φ_4 is not a linear mapping since $\varphi_4(\mathbf{0}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$.

(e): φ_5 is a linear mapping since φ_5 is represented as $\varphi_5(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

In particular $M_{\varphi_5} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

(f): φ_6 is a linear mapping since φ_6 is represented as $\varphi_6(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

In particular $M_{\varphi_6} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(g): φ_7 is not a linear mapping since for example $2\varphi_7(1) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \varphi_7(2)$.

(h): φ_8 is a linear mapping since φ_8 is represented as $\varphi_8(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

In particular $M_{\varphi_8} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$.

⑤ (a): φ_θ is linear since, for any vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^2$ and $a \in \mathbb{R}$ (2) the rotaion of $\mathbf{x}+\mathbf{y}$ is equal to the addition of the vectors $\mathbf{x}$ rotated and $\mathbf{y}$ rotated, (2) $a\mathbf{x}$ rotated is the same as a times $vecv$ rotated.

(b): Sicne $\varphi_\theta(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\varphi_\theta(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -sin\theta \\ \cos\theta \end{bmatrix}$, we have $M_{\varphi_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(c): If $\theta \neq 0$ then vectors are rotated by non-zero angle. In particular, for any non-zero vector $\mathbf{x} \in \mathbb{R}^2$ $\varphi_\theta(\mathbf{x})$ has different direction as $\alpha\mathbf{x}$ for any $\alpha \in \mathbb{R}$ if $\theta \neq 0$ and $\theta \neq \pi$. This explains that φ_θ has no eigenvector.

(d): By (c), φ_θ cannot diagonalized if $\theta \neq 0$ and $\theta \neq \pi$, If $\theta = 0$ then φ_θ is the identity mapping and its matrix M_{φ_θ} is the 2×2 unit matrix E. Thus M_{φ_θ} is diagonalization of itself and hence diagonalizable. If $\theta = \pi$ then each $\mathbf{x}$ is sent to $-\mathbf{x}$ by the mapping φ_θ. Thus $M_{\varphi_\theta} = -E$.

⑥ (a):

Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{3}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$, we have $\varphi(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = -\frac{1}{3}\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}\right) = -\frac{1}{3}\begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Similarly, since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3}\left(2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$, $\varphi(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \frac{1}{3}\left(2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

It follows that $M_\varphi = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -3 & 0 \end{bmatrix}$.

(b): $\mathrm{Im}(\varphi) = \{a\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} : a \in \mathbb{R}\}$. $\mathrm{Ker}(\varphi) = \{\begin{bmatrix} 0 \\ a \end{bmatrix} : a \in \mathbb{R}\}$.

⑦ (a): We have $\varphi(\mathbf{0}) = \varphi(\mathbf{0}+\mathbf{0}) = \varphi(\mathbf{0})+\varphi(\mathbf{0})$ by the additivity of the linear mapping. By subtracting $\varphi(\mathbf{0})$ from the both sides of the equation, we obtain $\mathbf{0} = \varphi(\mathbf{0})$ as desired.

(b): We have $\varphi(\mathbf{a}+\mathbf{b}) = \varphi(\mathbf{a}) + \varphi(\mathbf{b})$. Thus we would have $\mathbf{0} = \varphi(\mathbf{b})$ if $\varphi(\mathbf{a}+\mathbf{b}) = \varphi(\mathbf{a})$.

⑧ Let $X = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}$ is an eigenvector of A for the eigen value $\varepsilon\} \cup \{\mathbf{0}\}$, and suppose $\mathbf{a}$, $\mathbf{b} \in X$ and c, $d \in \mathbb{R}$. It is enough to show that $c\mathbf{a} + d\mathbf{b} \in X$. Note first that we have $A\mathbf{a} = \varepsilon\mathbf{a}$ and $A\mathbf{b} = \varepsilon\mathbf{b}$ (independently of whether a or b is $\mathbf{0}$ or not).

If $c\mathbf{a} + d\mathbf{b} = \mathbf{0}$ then this is clear. So suppose $c\mathbf{a} + d\mathbf{b} \neq \mathbf{0}$. We have $A(c\mathbf{a} + d\mathbf{b}) = cA\mathbf{a} + dA\mathbf{b} = c\varepsilon\mathbf{a} + d\varepsilon\mathbf{b} = \varepsilon(c\mathbf{a} + d\mathbf{b})$. This shows that $c\mathbf{a} + d\mathbf{b}$ is an eigenvector of A for the eigenvalue ε and hence $c\mathbf{a} + d\mathbf{b} \in X$ as desired.