Linear Algebra II – Possible set of questions in the final exam on Jan. 30

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① Calculate the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 5 & 7 \\ 1 & 3 & 9 & 27 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 9 & 6 & 7 \\ 0 & 6 & 1 & 4 \\ 0 & 9 & 1 & 0 \\ 1 & 4 & 4 & 0 \end{bmatrix}$

② Find the inverse of the following matrix: $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 0 & 2 & -2 \\ 0 & 1 & 3 & 1 \\ 2 & -7 & 3 & -1 \end{bmatrix}$

③ Diagonalize the matrix: $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

④ Which of the following are linear mappings. Explain why. Determine the matrices M_φ corresponding to the linear mappings φ among the following.

(a) $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$ (b) $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

(c) $\varphi_3 : \mathbb{R}^2 \to \mathbb{R}^2; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x\sin\theta \\ y\cos\theta \end{bmatrix}$ (d) $\varphi_4 : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

(e) $\varphi_5 : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ 0 \\ x \end{bmatrix}$ (f) $\varphi_6 : \mathbb{R}^2 \to \mathbb{R}^4; \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(g) $\varphi_7 : \mathbb{R} \to \mathbb{R}^2; x \mapsto \begin{bmatrix} x \\ x^2 \end{bmatrix}$

⑤ Let $\varphi_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation counter-clockwise through the angle θ around the origin.

(a) Show that φ_θ is a linear mapping.

(b) Determine the matrix M_{φ_θ} corresponding to the linear mapping φ_θ.

(c) Find a geometric explanation for the fact that M_{φ_θ} does not have any eigenvector if $\theta \neq 0$.

(d) Can M_{φ_θ} be diagonalized for $\theta \neq 0$?

⑥ Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear mapping such that $\varphi(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\varphi\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$. (a) Find the matrix M_φ representing the linear mapping φ. (b) Decide $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}(\varphi)$.

⑦ Show that any linear mapping $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ satisfies the following: (a) $\varphi(\mathbf{0}) = \mathbf{0}$. (b) If $\varphi(\mathbf{a}) = \mathbf{0}$ for some $\mathbf{a} \in \mathbb{R}^m$ then $\varphi(\mathbf{a}+\mathbf{b}) = \varphi(\mathbf{b})$ for all $\mathbf{b} \in \mathbb{R}^m$.