数理論理学特論 2017 年 first quarter レポート課題

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Solve (at least two but preferably all of) the following exercises and report on your solutions (with as much detailed explanations as possible). Deadline: June 9, 2017 (Fri.). The grading will be done according to the report.

This list of the exercises may be extended yet in the course of May. Please check possible updates of this file under the URL:

`http://fuchino.ddo.jp/kobe/logic-tokuron-ss17-report.pdf`

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0. Write down formulas in $\mathcal{L}_\in = \{\in\}$ representing axioms of ZFC. For readability, you may use "macros", that is, for example, to formulate the Axiom of Union you may declare that $sub(x, y)$ abbreviates "$\forall u\,(u \in x \;\leftrightarrow\; u \in y)$ where u is a variable symbol different from x and y, and define the formula representing the axiom as $\forall x \exists y \forall z\,(z \in y \;\leftrightarrow\; \exists w\,(sub(w, x) \land z \in w))$.

1. Show that the following rules hold for elements of ω in Z: $a + b = b + a$, $ab = ba$, $a(b + c) = ab + ac$.

2. In Z we have, (a) ${}^{\emptyset}X = \{\emptyset\}$ for all X. (b) ${}^{X}\emptyset = \emptyset$ for all X (c) $\bigcup \emptyset = \emptyset$.

3. In Z, we have that for any x, y, there is y' such that (1) there is a bijection from y to y' and (2) $x \cap y' = \emptyset$.

4. Show that the following holds as a corollary to (the proof of) the Speedup Theorem: for any concretely given consistent theory T in $\mathcal{L}^*$ which extends Z, there is a formula $\varphi = \varphi(x_0)$ such that $\mathsf{Z} \vdash \varphi(\underline{n})$ for all $n \in \mathbb{N}$ but $\mathsf{Z} \nvdash \forall x_0\,(x_0 \in \omega \rightarrow \varphi(x_0))$.

5. Prove the Second Incompleteness Theorem as a corollary of the Speedup Theorem.

6. For any concretely given consistent theory T in $\mathcal{L}^*$ which extends Z, there is a consistent extension T' of T with a formula $\varphi = \varphi(x_0)$ such that $T' \vdash \exists x_0(x_0 \in \omega \land \varphi(x_0))$ but $T' \vdash \neg\varphi(\underline{n})$ for all $n \in \omega$.