On Reflection Theorems of Paracompactness

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A reflection theorem on meta-Lindelöfness

- Unless mentioned otherwise, we assume below all (topological) spaces are Hausdorff.
- The Main results are slight improvements of theorems in:

[S.F.2010]

Sakaé Fuchino, Fodor-type Reflection Principle and Balogh's reflection theorems, 京都大学数理解析研究所講究録 (RIMS Kôkyûroku) No.1686, (April, 2010), 41–58.

- **The following theorem is a prototype of the assertions I want to consider in this talk:**

Theorem 1. (S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba (2010))
Assume the Fodor-type Reflection Principle (FRP).
For a locally separable countably tight space X if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X is meta-Lindelöf.

A reflection theorem on meta-Lindelöfness

Theorem 1. (S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba (2010))

Assume the Fodor-type Reflection Principle (FRP).

For a locally separable countably tight space X if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X is meta-Lindelöf.

FRP: For any regular $\lambda \geq \aleph_2$, stationary $E \subseteq E^{\lambda}_{\omega}$ and for any $g : E \to [\lambda]^{\aleph_0}$ s.t. $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$ (i.e. g is a ladder system), there is $\eta < \lambda$ of cofinality ω_1 s.t. $\{x \in [\eta]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[\eta]^{\aleph_0}$.

Theorem 2. (S.F., H. Sakai, L. Soukup and T. Usuba (201?))

The assertion of Theorem 1. is equivalent to FRP over ZFC.

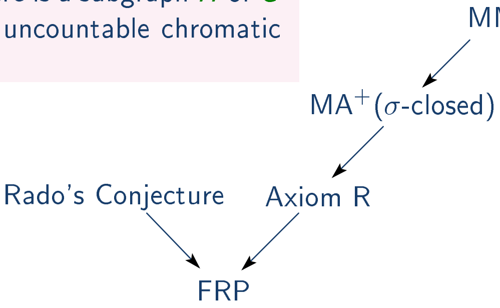
Axiom R, Rado's Conjecture and FRP

- FRP is shown to be equivalent to many mathematical reflection theorems over ZFC (S.F., H. Sakai, L. Soukup and T. Usuba (201?)).
- Some of the mathematical reflection theorems are previously known to be consequences of Axiom R.
- Axiom R implies FRP.
- Axiom R implies $2^{\aleph_0} \leq \aleph_2$.
- FRP imposes practically no restriction on the size of the continuum. More exactly, FRP is preserved by c.c.c. generic extension. In particular the reflection statements of Theorem 1 and many other theorems are consistent, say with $2^{\aleph_0} = \aleph_{2012}$.
- Rado's Conjecture implies FRP. Thus the reflection statements of Theorem 1 and many other theorems are also consequences of Rado's Conjecture. Rado's Conjecture also imply $2^{\aleph_0} \leq \aleph_2$ but Axiom R and Rado's Conjecture are independent.

FRP, Axiom R, Rado's Conjecture etc.

Axiom R: For any cardinal $\lambda \geq \aleph_2$, for any stationary $S \subseteq [\lambda]^{\aleph_0}$ and for any ω_1-club $C \subseteq [\lambda]^{\aleph_1}$, there is $I \in C$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

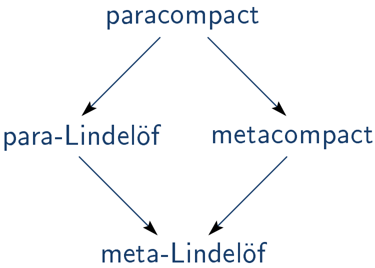
Rado's Conjecture: For any linear ordering I and any intersection graph G consisting of some intervals of I, if G has uncountable chromatic number then there is a subgraph H of G of cardinality $\aleph_1$ with uncountable chromatic number.



Paracompactness versus meta-Lindelöfness

A space $(X, \mathcal{O})$ is **meta-Lindelöf** if any open covering of X has an open refinement which is point countable.

A space $(X, \mathcal{O})$ is **paracompact** if any open covering of X has an open refinement which is locally finite.



A simple example

Example 3. Let $X = \mathbb{R}$ and $Z = \{\frac{1}{z} : z \in \mathbb{Z}\}$. For $x \in \mathbb{R}$, $x \neq 0$ let $\mathcal{B}(x) = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n \in \omega\}$ and $\mathcal{B}(0) = \{(-\frac{1}{n}, \frac{1}{n}) \setminus Z : n \in \omega\}$. $\mathcal{B}(x)$, $x \in X$ build a nbhd bases of a topology $\mathcal{O}$.
Then we have

(0) $\langle X, \mathcal{O} \rangle$ is separable.
(1) $\langle X, \mathcal{O} \rangle$ is not regular but Hausdorff.
(2) $\langle X, \mathcal{O} \rangle$ is meta-Lindelöf (or even metacompact and para-Lindelöf).
(3) $\langle X, \mathcal{O} \rangle$ is not paracompact.

Locally separable and Lindelöf spaces

Proposition 4. For a locally (separable & Lindelöf) space X, the following are equivalent:

(a) X has an open partition into Lindelöf spaces;
(b) X is paracompact;
(c) X is meta-Lindelöf.

Corollary 5. (to Theorem 1.) (FRP) For a locally (separable & Lindelöf) countably tight space X if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X is paracompact.

- The assertion of Corollary 5 is still equivalent to FRP over ZFC.

Regular locally compact spaces under FRP

Theorem 6. (G. Gruenhage and P. Koszmider, 1996)
Assume $\mathrm{MA}_{\aleph_1}$. For any normal locally compact space X, X is paracompact if $\Leftrightarrow$ X is meta-Lindelöf.

Theorem 7. (S. Watson, 1982)
Assume $\mathrm{MA}_{\aleph_1}(\sigma\text{-centered})$ and that a Suslin tree exists. Then there is a normal locally compact space X which is meta-Lindelöf but not paracompact.

- Thus the following assertion is independent from ZFC:

(*) For any normal locally compact space X,
X is paracompact $\Leftrightarrow$ X is meta-Lindelöf

Corollary 8. (*) above is independent from ZFC $+$ FRP.

Proof. FRP is presereved by c.c.c. generic extension. □

Countably tight locally Lindelöf spaces

- The following theorem was proved first under Axiom R by Balogh (2002):

> **Theorem 9.**([S.F.2010]) Assume FRP. Suppose that X is locally Lindelöf and countably tight. If every open subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact then X itself is paracompact.

- The assertion of Theorem 9 is also equivalent to FRP.
- The following theorem was also proved first under Axiom R by Balogh (2002):

> **Theorem 10.** Assume FRP^R. Suppose that X is a countably tight locally Lindelöf space such that (1) for all open subspaces Y of X with $L(Y) \leq \aleph_1$ have $L(\overline{Y}) \leq \aleph_1$ and, (2) every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact. Then X itself is paracompact.

Countably tight locally Lindelöf spaces

Theorem 10. Assume FRP^R. Suppose that X is a countably tight locally Lindelöf space such that (1) for all open subspaces Y of X with $L(Y) \leq \aleph_1$ have $L(\overline{Y}) \leq \aleph_1$ and, (2) every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact. Then X itself is paracompact.

FRP^R: For any regular $\lambda \geq \aleph_2$, ω_1-club $\mathcal{I} \subseteq [\lambda]^{\aleph_1}$, stationary $E \subseteq E^\lambda_\omega$ and for any ladder system $g : E \to [\lambda]^{\aleph_0}$, there is $I \in \mathcal{I}$ s.t. $\{x \in [I]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[I]^{\aleph_0}$.

- Axiom R $\Rightarrow$ FRP^R $\Rightarrow$ FRP
- FRP^R is still preserved by c.c.c. generic extension.

Sketch of the proof of Theorem 10.

Theorem 10. Assume FRP^R. Suppose that X is a countably tight locally Lindelöf space such that (1) for all open subspaces Y of X with $L(Y) \leq \aleph_1$ have $L(\overline{Y}) \leq \aleph_1$ and, (2) every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact. Then X itself is paracompact.

- By induction on $\lambda \geq \aleph_1$ we prove the following:

$(*)_\lambda$: For any countably tight locally Lindelöf space X with $L(X) = \lambda$, if (1) for all open subspaces Y of X with $L(Y) \leq \aleph_1$ have $L(\overline{Y}) \leq \aleph_1$ and, (2) every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact, then X itself is paracompact.

- For $\lambda = \aleph_1$ this is trivial.

Sketch of the proof of Theorem 10. (2/5)

$(*)_\lambda$: For any countably tight locally Lindelöf space X with $L(X) = \lambda$, if (1) for all open subspaces Y of X with $L(Y) \leq \aleph_1$ have $L(\overline{Y}) \leq \aleph_1$ and, (2) every clopen subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact, then X itself is paracompact.

- For $\lambda = \aleph_1$ this is trivial. Assume $\lambda > \aleph_1$.
- Assume that $(*)_{\lambda'}$ holds for all uncountable $\lambda' < \lambda$. We prove that $(*)_\lambda$ then also holds.
- Let X be as in $(*)_\lambda$ and let $\{L_\alpha : \alpha < \lambda\}$ be an open covering of X consisting of Lindelöf subspaces.
- By (1) and since X is countably tight
  $\mathcal{I} = \{I \in [\lambda]^{\aleph_1} : \bigcup_{\alpha \in I} L_\alpha$ is clopen in $X\}$ is an ω_1-club.
  By (2), $\bigcup_{\alpha \in I} L_\alpha$ for each $I \in \mathcal{I}$ is paracompact.
- ▷ **Case I.:** λ is regular.
- Let $X_\beta = \bigcup_{\alpha < \beta} L_\alpha$ for $\beta < \lambda$ and $E = \{\beta < \lambda : X_\beta \neq \overline{X_\beta}\}$.

Sketch of the proof of Theorem 10. (3/5)

- Let $X_\beta = \bigcup_{\alpha<\beta} L_\alpha$ for $\beta < \lambda$ and $E = \{\beta < \lambda : X_\beta \neq \overline{X_\beta}\}$.

Claim 1. $E \cap E^\lambda_\omega$ is non stationary in λ.

- Suppose otherwise. For $\beta \in E \cap E^\lambda_\omega$, let $p_\beta \in \overline{X_\beta} \setminus X_\beta$ and let $h(\beta) \in \lambda$ be s.t. $p_\beta \in L_{h(\beta)}$. Let $g : E \to [\lambda]^{\aleph_0}$ be a ladder system s.t. $\overline{\bigcup_{\gamma\in g(\beta)}} \ni p_\beta$ (possible since X is countably tight).
- By FRP^R, there is an $I \in \mathcal{I}$ s.t. $cf(I) = \omega_1$, I is closed w.r.t. h and I is as in the definition of FRP^R.
- By $I \in \mathcal{I}$ and (2), $\bigcup_{\alpha\in I} L_\alpha$ is paracompact. This leads to a contradiction. q.e.d. (Claim 1)

- By countable tightness of X it follows that E is non stationary in λ. Let $C = \{\xi_\beta : \beta < \lambda\}$ be a club in λ disjoint from E. Then $\mathcal{P} = \{\bigcup_{\alpha\in\xi_{\beta+1}\setminus\xi_\beta} L_\alpha : \beta \in \lambda\}$ is an open partition and, by the induction hypothesis, elements of $\mathcal{P}$ are paracompact. It follows that X is also paracompact.

Sketch of the proof of Theorem 10. (4/5)

- **Case II.:** λ is singular.
- We use the following Fact:

Theorem 11. (S.F. and A. Rinot (2011)) FRP implies SSH.

SSH: $\mathrm{cf}([\kappa]^{\mu}, \subseteq) = \kappa$ for all cardinals μ, κ with $\mu < \mathrm{cf}(\kappa)$.

Claim 2. For sufficiently large regular θ. Suppose that $M \prec \mathcal{H}(\theta)$ s.t. $\omega_1 \subseteq M$, X, $\langle L_\alpha : \alpha \in \lambda \rangle \in M$ and $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$. Then $\bigcup\{L_\alpha : \alpha \in \lambda \cap M\}$ is a clopen subspace of M.

- Let $\langle M_i : i < \mathrm{cf}(\lambda) \rangle$ e an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ s.t. $\omega_1 \subseteq M_i$, X, $\langle L_\alpha : \alpha \in \lambda \rangle \in M_i$, $[M_i]^{\aleph_0} \cap M_i$ is cofinal in $[M_i]^{\aleph_0}$ and $\lambda \subseteq \bigcup_{i<\mathrm{cf}(\lambda)} M_i$. We can find such M_i's by SSH.
- Let $X_i = \bigcup\{L_\alpha : \alpha \in \lambda \cap M_i\}$.

Sketch of the proof of Theorem 10. (5/5)

- Let $\langle M_i : i < \mathrm{cf}(\lambda)\rangle$ e an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ s.t. $\omega_1 \subseteq M_i$, X, $\langle L_\alpha : \alpha \in \lambda\rangle \in M_i$, $[M_i]^{\aleph_0} \cap M_i$ is cofinal in $[M_i]^{\aleph_0}$ and $\lambda \subseteq \bigcup_{i<\mathrm{cf}(\lambda)} M_i$.
  We can find such M_i's by SSH.
- Let $X_i = \bigcup\{L_\alpha : \alpha \in \lambda \cap M_i\}$.
- By Claim 2, X_i, $i < \mathrm{cf}(\lambda)$ are clopen subspace of X and $L(X_i) < \kappa$. By the induction hypothesis, it follows that X_i, $i < \mathrm{cf}(\lambda)$ are all paracompact. For each $i < \mathrm{cf}(\lambda)$, let $\mathcal{C}_i$ be a locally finite open refinement of the covering $\{L_\alpha : \alpha \in \lambda \cap M_i\}$ of X_i.
- Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\lambda)} \mathcal{C}_i$. For A, $B \in \mathcal{C}$, let $A \sim B :\Leftrightarrow A \cap B \neq \emptyset$ and let $\approx$ be the transitive closure of $\sim$.
- It is easy to see that each equivalence class $e \in \mathcal{C}/\approx$ is of cardinality $\leq \mathrm{cf}(\lambda) < \lambda$. Since $\bigcup e$ is a clopen subspace of X it is paracompact again by the induction hypothesis.
- Thus X is partitioned into open paracompact subspaces and hence X is paracompact as well. q.e.d. (Theorem 10.)

Some open problems

(1) Does the assertion of Theorem 10 imply FRP^R ?
(It does imply FRP)

(2) Is FRP^R really necessary for Theorem 10 ?

(3) Is FRP^R strictly stronger than FRP ?

(4) Does Rado's Conjecture imply FRP^R ?

(5) Can paracompactness in Theorems 10 replaced by meta-Lindelöfness ? If not are there ZFC counterexamples ?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