Reflection of some properties of uncountable structures

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The type of reflection statements we consider

- For a class $\mathcal{C}$ of structures of a same type and a property Φ, consider the following statement:

$(*)$ For any uncountable $A \in \mathcal{C}$, if many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfy Φ then A satisfies Φ as well.

- "many" above can mean "all", "club many", "stationarily may", etc.

- $(*)$ can be also formulated in contraposition as:

$\binom{*}{*}$ For any uncountable $A \in \mathcal{C}$, if A does not satisfy Φ then there are not many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfying Φ.

Two instances of the reflection statement

- The following two assertions are consistent over ZFC (plus some large cardinal axiom like a strongly compact cardinal).
  - Moreover, these assertions are equivalent to each other over ZFC (we do not need large cardinal here).

(†) *An uncountable boolean algebra B is openly generated if club many subalgebras of B of size $\aleph_1$ are openly generated.*

([S.F. and A. Rinot, 2011])

(‡) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size $\aleph_1$ have countable coloring number.*

([S.F., H. Sakai, L. Soukup and T. Usuba, preprint])

- There are **many** other "mathematical" reflection theorems known to be equivalent to the assertions above over ZFC.

Many equivalent reflection statements

- There are **many** other "mathematical" reflection theorems known to be equivalent to the assertions on the last slide over ZFC.

- The following reflection statement is also equivalent to $(\dagger)$ and $(\ddagger)$:

For any locally compact Hausdorff space X if all subspace of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

Fodor-type Reflection Principle (FRP)

- The reflection 'theorems' (†) and (‡) are both equivalent to the following set-theoretic assertion we called **Fodor-type Reflection Principle (FRP)** in [S.F.,I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, 2010].

(FRP):

For any regular uncountable λ and any stationary $S \subseteq E^\lambda_\omega = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ and mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ such that

(1) $\mathrm{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;

(3) *for any regressive* $f : S \cap I \to \lambda$ *s.t.* $f(\alpha) \in g(\alpha)$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \lambda$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

Fodor-type Reflection Principle (FRP) (2/2)

(FRP):

For any regular uncountable λ *and any stationary* $S \subseteq E^\lambda_\omega = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ *and mapping* $g : S \to [\lambda]^{\leq\aleph_0}$ *there is* $I \in [\lambda]^{\aleph_1}$ *such that*

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Under (1) and (2), (3) is equivalent to:

(3') *for a/any filtration* $\langle I_\alpha : \alpha < \omega_1\rangle$ *of* I, *the set* $\{\alpha < \omega_1 : \sup(I_\alpha) \in E \cap I, g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha\}$ *is sataionary in* ω_1.

Fodor-type Reflection Principle (FRP) (2/2)

- Martin's Maximum implies FRP.
- FRP holds if strongly compact cardinal is Levy collapsed to $\aleph_2$.
- FRP implies the failure of the square principles.

Under Martin's Maximum we have $2^{\aleph_0} = \aleph_2$ and in the model with Levy collapse we have CH. However:

- FRP is preserved by c.c.c. generic extension. Hence
- ▷ FRP is consistent with "arbitrary" size of the continuum.

On the other hand, FRP has influence on cardinal arithmetic:

- FRP implies Shelah's Strong Hypothesis (SSH). In particular,
- ▷ FRP implies Singular Cardinal Hypothesis.
- SSH under FRP is used in the induction proof of "FRP $\Rightarrow$ (†)" for the case of successor of singular cardinal of countable cofinality.

Sketch of the proof of FRP $\Leftrightarrow$ (‡)

(‡) $\Rightarrow$ FRP: Suppose that FRP does not hold and let λ be minimal regular uncountable cardinal s.t. there are a stationary $E \subseteq E^{\lambda}_{\omega}$ and a mapping $g : E \to [\lambda]^{\leq \aleph_0}$ which are counterexamples of FRP.

- From E and g as above, we can construct an almost essentially disjoint ladder system $g^* : E^* \to [\lambda]^{\leq \aleph_0}$ s.t. $E^* \subseteq E^{\lambda}_{\omega}$ is stationary and $g(\alpha) \subseteq \lambda \setminus \mathrm{Lim}(\lambda)$ for all $\alpha \in E^*$.
- Let $G = \langle \lambda, E \rangle$ be defined by

$$\alpha \mathrel{E} \beta \quad \Leftrightarrow \quad \begin{array}{l} \alpha < \beta \text{ and } \alpha \in g^*(\beta), \text{ or} \\ \beta < \alpha \text{ and } \beta \in g^*(\alpha). \end{array}$$

Then G is a counterexample of (‡).

Sketch of the proof of FRP $\Leftrightarrow$ (‡) (2/3)

(‡) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size $\aleph_1$ have countable coloring number.*

FRP $\Rightarrow$ (‡): By induction on $|G|$. We assume that all subgraphs of G of cardinality $\aleph_1$ have countable coloring number.

Case I.: $|G| = \aleph_1$. G is trivially of countable coloring number.

Case II.: $|G| = \lambda$ is singular. By induction hypothesis all subgraphs of G of cardinality $< \lambda$ are of countable coloring number.

- It follows from (the most general version of) Shelah Singular Compactness Theorem that G is then also of countable coloring number.

Sketch of the proof of FRP $\Leftrightarrow$ (‡) (3/3)

Case III.: $|G| = \lambda$ is regular. Let $\langle G_\alpha : \alpha < \lambda \rangle$ be a filtration. By induction hypothesis G_α, $\alpha < \lambda$ are of countable coloring number.

- Suppose toward a contradiction that $Col(G) > \omega$. Without loss of generality, (the underlying set of) $G = \lambda$.
- By the Corollary of Erdős-Hajnal theorem,

$$E_0 = \{\alpha < \lambda : \text{there is a } c_\alpha \in [G_\alpha]^{\aleph_0} \text{ and } \eta_\alpha \in G \setminus G_\alpha \text{ s.t. } \xi \mathrel{E} \eta_\alpha \text{ for all } \xi \in c_\alpha\} \text{ is stationary.}$$

- We can show $E = E_0 \cap E^\lambda_\omega \cap \{\alpha < \lambda : G_\alpha = \alpha\}$ is stationary.
- For $\alpha \in E$ let $g(\alpha) = c_\alpha \cup \{\eta_\alpha\}$ and apply FRP to get $I \in [\lambda]^{\aleph_1}$ s.t.

(1) $\mathrm{cf}(I) = \omega_1$; (2) $g(\alpha) = c_\alpha \cup \{\eta_\alpha\} \subseteq I$ for all $\alpha \in I \cap S$;

(3') for a/any filtration $\langle I_\alpha : \alpha < \omega_1 \rangle$ of I, the set $\{\alpha < \omega_1 : \sup(I_\alpha) \in E \cap I,\ g(\sup(I_\alpha)) \cap \sup(I_\alpha) = c_{g(\sup(I_\alpha))} \subseteq I_\alpha\}$ is sataionary in ω_1. (see the characterization of FRP)

Then $Col(G \restriction I) > \omega$. But this is a contradiction to the choice of G. $\square$

Non reflection

Slight modifications of (†) and (‡) can be inconsistent with ZFC:

There are openly generated boolean algebras B of size $\geq \aleph_2$ which are not projective. There are club many openly generated subalgebras of B of size $\aleph_1$. By definition, all these subalgebras are projective.

There is a graph G with uncountable chromatic number s.t. all subgraphs of G of cardinality $< 2^{\aleph_0}$ are of countable chromatic number. We can even replace "$< 2^{\aleph_0}$" with "$< \beth_n$" for any $n \in \omega$.
([P. Erdős and A. Hajnal, 1961])

For a cardinal $\kappa \geq (2^{\aleph_0})^+$, let $G = \langle [\kappa]^2, E \rangle$ be defined by

$$\{\alpha, \alpha'\} \; E \; \{\beta, \beta'\} \;\; \Leftrightarrow \;\; \alpha' = \beta \text{ or } \beta' = \alpha$$

for $\{\alpha, \alpha'\}$, $\{\beta, \beta'\} \in [\kappa]^2$ with $\alpha < \alpha'$ and $\beta < \beta'$.

Then $Chr(G) \geq \aleph_0$ but $Chr(G \restriction [X]^2) = \aleph_0$ for all $X \in [\kappa]^{\leq 2^{\aleph_0}}$.

Non reflection (2/2)

For a cardinal $\kappa \geq (2^{\aleph_0})^+$, let $G = \langle [\kappa]^2, E \rangle$ be defined by

$$\{\alpha, \alpha'\} \; E \; \{\beta, \beta'\} \;\Leftrightarrow\; \alpha' = \beta \text{ or } \beta' = \alpha$$

for $\{\alpha, \alpha'\}$, $\{\beta, \beta'\} \in [\kappa]^2$ with $\alpha < \alpha'$ and $\beta < \beta'$.

Then $Chr(G) \geq \aleph_0$ but $Chr(G \restriction [X]^2) = \aleph_0$ for all $X \in [\kappa]^{\leq 2^{\aleph_0}}$.

Proof.

- $Chr(G \restriction [X]^2) = \aleph_0$ for any $X \in [\kappa]^{\leq 2^{\aleph_0}}$:

Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a family of pairwise incomparable subsets of ω with $|\mathcal{F}| = 2^{\aleph_0}$ and $f : X \to \mathcal{F}$ be an injection. For $\{\alpha, \alpha'\} \in [X]^2$ with $\alpha < \alpha'$, let $c(\{\alpha, \alpha'\}) = n_{\alpha,\alpha'}$ for some $n_{\alpha,\alpha'} \in f(\alpha) \setminus f(\alpha')$. If $c(\{\alpha, \alpha'\}) = c(\{\beta, \beta'\})$, then $\{\alpha, \alpha'\} \not\!E \{\beta, \beta'\}$. I.e., c is a good coloring of $G \restriction [X]^2$.

- $Chr(G \restriction [X]^2) > \aleph_0$ for any $X \in [\kappa]^{>2^{\aleph_0}}$:

Suppose that $X \in [\kappa]^{>2^{\aleph_0}}$ and $g : [X]^2 \to \omega$. By Erdős-Rado Theorem, there is $Y \in [X]^{\aleph_1}$ s.t. Y is homogeneous w.r.t. g. Let α, β, $\gamma \in Y$ with $\alpha < \beta < \gamma$. Then we have $g(\{\alpha, \beta\}) = g(\{\beta, \gamma\})$ and $\{\alpha, \beta\} \; E \; \{\beta, \gamma\}$. $\square$

Theorem (P. Komjáth, [preprint])

(1) (MA(Cohen)) *For any graph G of cardinality $\aleph_1$, we have* $\aleph_0 < Chr(G) \Leftrightarrow \aleph_0 < List(G)$.

(2) *For any graph G of cardinality $\aleph_1$, if $\aleph_0 < Col(G)$ then, for* $\mathbb{P} = \mathrm{Fn}(\omega_2, 2, < \aleph_1)$, *we have* $\Vdash_{\mathbb{P}}$ "$\aleph_0 < List(G)$".

Corollary (S.F. and H. Sakai [in preparation])

(1) "ZFC+FRP+¬(‡) for list chromatic number" *is consistent.*

(2) "ZFC+FRP+(‡) for list chromatic number" *is consistent.*

Proof. (1): Let $M \models$ ZFC + FRP. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. Hence, by (1) of Komjáth's theorem and Erdős-Hajnal's example we have the (strong) negation of (‡) for list chromatic number.

(2): Levy collapse of a strongly compact cardinal makes FRP true. So by [S.F.,H.Sakai,...] and (2) of Komjáth's theorem, (‡) for list chromatic number holds. □

Problems

- Is there a reflection statement on groups which is similar to (or, e.g. equivalent to) $(\ddagger)$?

- Is "ZFC + FRP + $(\ddagger)$ for list chromatic number" consistent with $\neg$CH ?

- Find more examples of natural reflexion statements which are independent from FRP.

$$\aleph_{\omega+1}$$

Openly generated boolean algebras

- A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F.

- Note that subalgebra of a boolean algebra is not necessarily free.

- A boolean algebra A is **openly generated** if $\Vdash_{\mathbb{P}}$ "A is projective" for any σ-closed p.o. forcing the cardinality of A to be $\leq \aleph_1$.

Proposition (L. Heindorf, 198?). *A Boolean algebra A is openly generated if and only if there is a mapping $f : A \to [A]^{<\aleph_0}$ (Freese-Nation mapping) s.t., for all a, $b \in A$ with $a \leq_A b$, there is $c \in f(a) \cap f(b)$ s.t. $a \leq_A c \leq_A b$.*

back

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Club (stationarily) many subsets of size $\aleph_1$

- For a set X, let $[X]^{\aleph_1} = \{A \subseteq X \; : \; |A| = \aleph_1\}$.
- $[X]^{\kappa}$, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ etc. for any cardinal κ are defined similarly.

- $\mathcal{F} \subseteq [X]^{\aleph_1}$ is **club** (closed unbounded) if
  - For any upward directed $\mathcal{F}' \subseteq \mathcal{F}$ (w.r.t. $\subseteq$) of cardinality $\leq \aleph_1$, we have $\bigcup \mathcal{F}' \in \mathcal{F}$; and,
  - for any $A \in [X]^{\aleph_1}$ there is $B \in \mathcal{F}$ with $A \subseteq B$.
- $\mathcal{S} \subseteq [X]^{\aleph_1}$ is **stationary** if $\mathcal{S} \cap \mathcal{F} \neq \emptyset$ for all club $\mathcal{F} \subseteq [X]^{\aleph_1}$.
- "There are club many subsets of X of size $\aleph_1$ with ..." means "$\{A \in [X]^{\aleph_1} \; : \; ...\}$ contains a club subset of $[X]^{\aleph_1}$". Similarly for "stationarily many".

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Graphs with countable coloring number

- A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v\ E\ w$ (or $w\ E\ v$) if the vertices v, $w \in G$ are connected in the graph.
- G has **coloring number** $\leq \kappa$ if there is a well-ordering $\sqsubset$ of G s.t., for all $v \in G$, we have $|\{w \in G : w \sqsubset v, w\ E\ v\}| < \kappa$
- G has **countable coloring number** if it has coloring number $\leq \aleph_0$.
- The coloring number of G (notation: $Col(G)$) is defined by

$$Col(G) = \min\{\kappa : G \text{ has coloring number} \leq \kappa\}.$$

Theorem (P. Erdős and A. Hajnal, 1961). *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \to [G]^{<\kappa}$ s.t., for all v, $w \in G$, if $v\ E\ w$ then $v \in f(w)$ or $w \in f(v)$.*

continue

Graphs with countable coloring number (2/2)

Theorem (P. Erdős and A. Hajnal, 1961). *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \to [G]^{<\kappa}$ s.t., for all $v, w \in G$, if $v \, E \, w$ then $v \in f(w)$ or $w \in f(v)$.*

Corollary. *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if G has a filtration $\langle G_\alpha : \alpha < \lambda \rangle$ s.t.*

- *all G_α, $\alpha < \lambda$ have coloring number $\leq \kappa$; and*
- *for all $\alpha < \lambda$ and $v \in G \setminus G_\alpha$, $|\{w \in G_\alpha : w \, E \, v\}| < \kappa$.*

Corollary. *A graph $G = \langle G, E \rangle$ has coloring number $> \kappa$ if for any filtration $\langle G_\alpha : \alpha < \lambda \rangle$ of G either*

- *for stationarily many $\alpha < \lambda$, we have $col(G_\alpha) > \kappa$; or*
- *for stationarily many $\alpha < \lambda$, $|\{w \in G_\alpha : w \, E \, v_\alpha\}| \geq \kappa$ for some $v_\alpha \in G \setminus G_\alpha$.*

back

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Corollary. *A graph $G = \langle G, E\rangle$ has coloring number $\leq \kappa$ if and only if G has a filtration $\langle G_\alpha : \alpha < \lambda\rangle$ s.t.*

- *all G_α, $\alpha < \lambda$ have coloring number $\leq \kappa$; and*
- *for all $\alpha < \lambda$ and $v \in G \setminus G_\alpha$, $|\{w \in G_\alpha : w\ E\ v\}| < \kappa$.*

Corollary. *A graph $G = \langle G, E\rangle$ has coloring number $> \kappa$ if for any filtration $\langle G_\alpha : \alpha < \lambda\rangle$ of G either*

- *for stationarily many $\alpha < \lambda$, we have $col(G_\alpha) > \kappa$; or*
- *for stationarily many $\alpha < \lambda$, $|\{w \in G_\alpha : w\ E\ v_\alpha\}| \geq \kappa$ for some $v_\alpha \in G \setminus G_\alpha$.*

back

Chromatic and list chromatic number of graphs

- For a graph $G = \langle G, E \rangle$, a mapping $f : G \to \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all v, $w \in G$ with $v\, E\, w$. The **chromatic number** $Chr(G)$ of a graph G is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \to \kappa\}.$$

- It is easy to see that $Chr(G) \leq Col(G)$ holds for all graph G.
- The **list chromatic number** $List(G)$ of a graph G is defined by

$$\begin{aligned} List(G) = \min\{\kappa : \ & \text{for any } \lambda \text{ and } g : G \to [\lambda]^{\kappa}, \\ & \text{there is a choice function } f \\ & \text{which is a good coloring of } G\} \end{aligned}$$

- We have $Chr(X) \leq List(X) \leq Col(X)$.

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