Reflection of some properties of uncountable structures

Sakaé Fuchino (渕野 昌)

Dept. of Computer Sciences
Kobe University
(神戸大学大学院 システム情報学研究科)

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

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The type of reflection statements we consider

- For a class $\mathcal{C}$ of structures of a same type and a property Φ, consider the following statement:

$(*)$ For any uncountable $A \in \mathcal{C}$, if many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfy Φ then A satisfies Φ as well.

- "many" above can mean "all", "club many", "stationarily may", etc.

- $(*)$ can be also formulated in contraposition as:

$(\substack{*\\*})$ For any uncountable $A \in \mathcal{C}$, if A does not satisfy Φ then there are not many $B \subseteq A$ with $B \in \mathcal{C}$ and $|B| = \aleph_1$ satisfying Φ.

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Two instances of the reflection statement

- The following two assertions are consistent over ZFC (plus some large cardinal axiom like a strongly compact cardinal).
  - Moreover, these assertions are equivalent to each other over ZFC (we do not need large cardinal here).

(†) *An uncountable boolean algebra B is openly generated if club many subalgebras of B of size $\aleph_1$ are openly generated.*

([S.F. and A. Rinot, 2011])

(‡) *An uncountable (undirected) graph $G = \langle G, E \rangle$ has countable coloring number if all subgraphs of G of size $\aleph_1$ have countable coloring number.*

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(1) $\mathrm{cf}(I) = \omega_1$;

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Fodor-type Reflection Principle (FRP) (2/2)

- Martin's Maximum implies FRP.
- FRP holds if strongly compact cardinal is Levy collapsed to $\aleph_2$.
- FRP implies the failure of the square principles.

Under Martin's Maximum we have $2^{\aleph_0} = \aleph_2$ and in the model with Levy collapse we have CH. However:

- FRP is preserved by c.c.c. generic extension. Hence
  - FRP is consistent with "arbitrary" size of the continuum.

On the other hand, FRP has influence on cardinal arithmetic:

- FRP implies Shelah's Strong Hypothesis (SSH). In particular,
  - FRP implies Singular Cardinal Hypothesis.
- SSH under FRP is used in the induction proof of "FRP $\Rightarrow$ (†)" for the case of successor of singular cardinal of countable cofinality.

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(‡) $\Rightarrow$ FRP: Suppose that FRP does not hold and let λ be minimal regular uncountable cardinal s.t. there are a stationary $E \subseteq E^\lambda_\omega$ and a mapping $g : E \to [\lambda]^{\leq\aleph_0}$ which are counterexamples of FRP.

- From E and g as above, we can construct an almost essentially disjoint ladder system $g^* : E^* \to [\lambda]^{\leq\aleph_0}$ s.t. $E^* \subseteq E^\lambda_\omega$ is stationary and $g(\alpha) \subseteq \lambda \setminus \mathrm{Lim}(\lambda)$ for all $\alpha \in E^*$.
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(‡) *An uncountable (undirected) graph $G = \langle G, E\rangle$ has countable coloring number if all subgraphs of G of size $\aleph_1$ have countable coloring number.*

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- Suppose toward a contradiction that $Col(G) > \omega$. Without loss of generality, (the underlying set of) $G = \lambda$.
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There are openly generated boolean algebras B of size $\geq \aleph_2$ which are not projective. There are club many openly generated subalgebras of B of size $\aleph_1$. By definition, all these subalgebras are projective.

There is a graph G with uncountable chromatic number s.t. all subgraphs of G of cardinality $< 2^{\aleph_0}$ are of countable chromatic number. We can even replace "$< 2^{\aleph_0}$" with "$< \beth_n$" for any $n \in \omega$.

([P. Erdős and A. Hajnal, 1961])

For a cardinal $\kappa \geq (2^{\aleph_0})^+$, let $G = \langle [\kappa]^2, E \rangle$ be defined by

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for $\{\alpha, \alpha'\}$, $\{\beta, \beta'\} \in [\kappa]^2$ with $\alpha < \alpha'$ and $\beta < \beta'$.

Then $Chr(G) \geq \aleph_0$ but $Chr(G \restriction [X]^2) = \aleph_0$ for all $X \in [\kappa]^{\leq 2^{\aleph_0}}$.

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Corollary (S.F. and H. Sakai [in preparation])

(1) "ZFC+FRP+$\neg$(‡) for list chromatic number" *is consistent.*

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Proof. (1): Let $M \models$ ZFC + FRP. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. Hence, by (1) of Komjáth's theorem and Erdős-Hajnal's example we have the (strong) negation of (‡) for list chromatic number.

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(2) "ZFC+FRP+(‡) for list chromatic number" *is consistent.*

Proof. (1): Let $M \models$ ZFC + FRP. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. Hence, by (1) of Komjáth's theorem and Erdős-Hajnal's example we have the (strong) negation of (‡) for list chromatic number.

(2): Levy collapse of a strongly compact cardinal makes FRP true. So by [S.F.,H.Sakai,. . .] and (2) of Komjáth's theorem, (‡) for list chromatic number holds. □

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Openly generated boolean algebras

- A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F.
  - Note that subalgebra of a boolean algebra is not necessarily free.
- A boolean algebra A is **openly generated** if $\Vdash_{\mathbb{P}}$ "A is projective" for any σ-closed p.o. forcing the cardinality of A to be $\leq \aleph_1$.

Proposition (L. Heindorf, 198?). *A Boolean algebra A is openly generated if and only if there is a mapping $f : A \to [A]^{<\aleph_0}$ (Freese-Nation mapping) s.t., for all $a, b \in A$ with $a \leq_A b$, there is $c \in f(a) \cap f(b)$ s.t. $a \leq_A c \leq_A b$.*

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Club (stationarily) many subsets of size $\aleph_1$

- For a set X, let $[X]^{\aleph_1} = \{A \subseteq X : |A| = \aleph_1\}$.
  - $[X]^{\kappa}$, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ etc. for any cardinal κ are defined similarly.

- $\mathcal{F} \subseteq [X]^{\aleph_1}$ is **club** (closed unbounded) if
  - For any upward directed $\mathcal{F}' \subseteq \mathcal{F}$ (w.r.t. $\subseteq$) of cardinality $\leq \aleph_1$, we have $\bigcup \mathcal{F}' \in \mathcal{F}$; and,
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Graphs with countable coloring number

- A structure $\mathcal{G} = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v\ E\ w$ (or $w\ E\ v$) if the vertices $v, w \in G$ are connected in the graph.
- $\mathcal{G}$ has **coloring number** $\leq \kappa$ if there is a well-ordering $\sqsubset$ of G s.t., for all $v \in G$, we have $\left|\{w \in G : w \sqsubset v, w\ E\ v\}\right| < \kappa$
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- A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v\,E\,w$ (or $w\,E\,v$) if the vertices v, $w \in G$ are connected in the graph.
- G has **coloring number** $\leq \kappa$ if there is a well-ordering $\sqsubset$ of G s.t., for all $v \in G$, we have $\left|\{w \in G : w \sqsubset v,\ w\,E\,v\}\right| < \kappa$
- G has **countable coloring number** if it has coloring number $\leq \aleph_0$.
- The coloring number of G (notation: $Col(G)$) is defined by
$$Col(G) = \min\{\kappa : G \text{ has coloring number} \leq \kappa\}.$$

Theorem (P. Erdős and A. Hajnal, 1961). *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \to [G]^{<\kappa}$ s.t., for all v, $w \in G$, if $v\,E\,w$ then $v \in f(w)$ or $w \in f(v)$.*

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Chromatic and list chromatic number of graphs

- For a graph $G = \langle G, E \rangle$, a mapping $f : G \to \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all $v, w \in G$ with $v \mathrel{E} w$. The **chromatic number** $Chr(G)$ of a graph G is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \to \kappa\}.$$

- It is easy to see that $Chr(G) \leq Col(G)$ holds for all graph G.
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