Reflection Theorems in Topology

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http://fuchino.ddo.jp/slides/kobe-top-12-12-13pf.pdf

Reflection and non-reflection

Theorem 1. (A. Dow, 1988)

For any countably compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

Theorem 2. (A. Hajnal and I. Juhász, 1976)

For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.

Sketch of the Proof

Theorem 3. (S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba 2011, S.F., Soukup, H.Sakai and Usuba, 201?) *The following assertion is equivalent to the Fodor-type reflection principle (FRP) over ZFC:*

For any locally countably compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

Reflection Theorems equivalent to FRP

- The following assertions among many others are also kown to be equivalent to FRP over ZFC:

(S.F., 2008, S.F., L.Soukup, H.Sakai and T.Usuba, 201?) *For every T_1-space X with point countable base, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated.*

(S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba 2011, S.F., Soukup, H.Sakai and Usuba, 201?) *For every locally separable countably tight topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf, then X itself is also meta-Lindelöf.*

(S.F., Soukup, H.Sakai and Usuba, 201?) *For every countably tight topological space X of local density $\leq \aleph_1$, if all subspace of cardinality $\leq \aleph_1$ are collectionwise Hausdorff, then X is collectionwise Hausdorff.*

Proof of the equivalences

- The "$\Rightarrow$" direction of the equivalence proofs are quite involved (we skip it here).
- Most of the implications "$\neg$FRP $\Rightarrow \cdots$" can be proved by the following construction of topological spaces:

> **Fact.** *If $\neg$FRP holds then there is a regular cardinal λ with $ADS^{-}(\lambda)$: there are stationary $E^* \subseteq E^{\lambda}_{\omega}$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \restriction \alpha$ is essentially disjoint for all $\alpha < \lambda$.*

Proof of the equivalences

Fact. *If $\neg$FRP holds then there is a regular cardinal λ with ADS$^{-}(\lambda)$: there are stationary $E^* \subseteq E^\lambda_\omega$ and a ladder system $g^* : E^* \to [\lambda]^{\aleph_0}$ s.t. $g^* \restriction \alpha$ is essentially disjoint for all $\alpha < \lambda$.*

- Let λ, E^*, g^* be as above. We may assume that $g^*(\alpha) \cap E^* = \emptyset$ for all $\alpha \in E^*$.
- Let $X = E^* \cup \bigcup_{\alpha \in E^*} g^*(\alpha)$ and $\mathcal{O}$ be the topology on X generated from

$$\mathcal{B} = \{\{\alpha\} : \alpha \in \textstyle\bigcup_{\alpha \in E^*} g^*(\alpha)\} \cup \{g^*(\alpha) \cup \sup\{\alpha\} \setminus x : \alpha \in E^*, x \in [g^*(\alpha)]^{<\aleph_0}\}$$

- Any subspace Y of X of cardinality $< \lambda$ is metrizable:
  Since $g^*(\alpha)$, $\alpha \in E^* \cap Y$ are essentially disjoint, Y can be partitioned into disjoint metrizable open subspaces.
- X itself is not metrizable since it is not meta-Lindelöf:
  Consider the open covering $\mathcal{B}$ of X. Fodor's Lemma imples that there is no point countable open refinement.

Another reflection statement on metrizability

- Recall the reflection theorems on metrizability on the first slide.

- What is about the metrizability of spaces of countable character?

- Restriction to the spaces of countable tightness already makes the reflection number of metrizability consistently $< \infty$:

Theorem 4. Suppose that κ is a/the ω_1-strongly compact cardinal. For any countably tight X, if all subspaces of X of cardinality $< \kappa$ are metrizable, then X itself is also metrizable.

Another reflection statement on metrizability

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Proof.

- For X as above, let $A = X \dot\cup [X]^{\aleph_0} \dot\cup \mathbb{R}$.
- Let $E^{\mathfrak{A}}, C^{\mathfrak{A}} \subseteq A^2$ and $Q^{\mathfrak{A}}$ be defined by:
  $x\ E^{\mathfrak{A}}\ y :\Leftrightarrow x \in X,\ y \in [X]^{\aleph_0}$ and $x \in y$;
  $x\ R^{\mathfrak{A}}\ y :\Leftrightarrow x \in X,\ y \in [X]^{\aleph_0}$ and $x \in \overline{y}$; $Q^{\mathfrak{A}} = \mathbb{Q}$.
  Note that, by countable tightness of X, C decides the topology of X. Let $\leq^{\mathfrak{A}}$ be the canonical less than or equal to relation on $\mathbb{R}$.
- Let T be the $\mathcal{L}_{\omega_1,\omega}$ theory in $L = \{E, C, ..., d(\cdot,\cdot), \leq, c_a\}_{a\in A}$ consisting of quantifier free diagram of $\mathfrak{A} = \langle A, E^{\mathfrak{A}}, C^{\mathfrak{A}}, ...\rangle$ and the assertion "d generates the topology introduced by C".
- Then T is $< \kappa$-satisfiable. Since κ is ω_1-strongly compact, T is satisfiable. A metric of X can be constructed from a model of T. □

Reflection numbers

- We can consider the following cardinal numbers (they can be ∞):

$$\mathfrak{Refl} = \min\{\kappa : \text{for all } \underline{\text{locally compact}}\ X, \text{ if all supspace of } X \text{ of card.} < \kappa \text{ are metrizable, then } X \text{ is also metrizable } \};$$

$$\mathfrak{Refl}^* = \min\{\kappa : \text{for all } \underline{\text{countably tight}}\ X, \text{ if all supspace of } X \text{ of card.} < \kappa \text{ are metrizable, then } X \text{ is also metrizable } \};$$

- $\aleph_1 \leq \mathfrak{Refl} = \mathfrak{Refl}_{\text{FRP}} \leq \mathfrak{Refl}^* \leq \omega_1$-strongly compact cardinal

Open questions:

- (Hamburger's Problem) What is the possible value of $\mathfrak{Refl}^*$? Can it be $\aleph_2$? (It is known that $\mathfrak{Refl}^* < 2^{\aleph_0}$ is consistent).
- Can $\mathfrak{Refl}$ or $\mathfrak{Refl}^*$ be exactly the ω_1-strongly compact cardinal?

Similar open problems on reflection numbers

- $\aleph_1 \leq \mathfrak{Refl} \leq \mathfrak{Refl}_{Rado} \leq \mathfrak{Refl}_{Galvin}$
  $\leq \mathfrak{Refl}_{chr} \leq$ the ω_1-strongly compact cardinal

- For the definition of these cardinals see e.g.
  http://fuchino.ddo.jp/slides/wien12-06-22-pf.pdf

- $\mathfrak{Refl}_{Rado} = \aleph_1$ is Rado Conjecture — proved to be consistent (modulo some large cardinals) by S. Todorčević.

- $\beth_\omega \leq Refl_{chr}$ (Erdő and Hajnal, 1968)

- $\mathfrak{Refl}_{Galvin} = \aleph_1$ is Galvin Conjecture — status: open.

- Is there any relationship between $\mathfrak{Refl}_{Rado}$ and $\mathfrak{Refl}^*$, or between $\mathfrak{Refl}_{Galvin}$ and $\mathfrak{Refl}^*$?


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Sketch of the proof of Theorem 2

Theorem 2. (A. Hajnal and I. Juhász, 1976)

For any uncountable cardinal κ there is a non-metrizable space X s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.

Proof. For every cardinal $\kappa' \geq \kappa$ of cofinality $\geq \kappa,\ \omega_1$, $(\kappa'+1, \mathcal{O})$ with

$$\mathcal{O} = \mathcal{P}(\kappa') \cup \{(\kappa' \setminus x) \cup \{\kappa'\} : x \subseteq \kappa',\ x \text{ is bounded in } \kappa'\}$$

is such a space:

- Any subspace of size $< \kappa'$ is discrete and hence metrizable.
- $\kappa' + 1$ itself is not metrizable since the point κ' has character $\geq \mathrm{cf}(\kappa') > \aleph_0$. □

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Fodor-type reflection principle (FRP)

(FRP):

For any regular uncountable λ *and any stationary* $S \subseteq E^{\lambda}_{\omega} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ *and any mapping* $g : S \to [\lambda]^{\leq\aleph_0}$ *there is* $I \in [\lambda]^{\aleph_1}$ *(a reflection point of* g*) s.t.*

(1) $\mathrm{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;

(3) *for any regressive* $f : S \cap I \to \lambda$ *s.t.* $f(\alpha) \in g(\alpha)$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \lambda$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

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Left-separated topological spaces

A topological space X is **left-separated** if there is a well-ordering $<$ of X s.t. all initial segments of X w.r.t. $<$ are closed subsets of X.

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$$\cdots\,](\,\cdots$$

meta-Lidelöf, and collectionwise Hausdorff spaces

A topological space X is said to be **meta-Lindelöf** if every open cover of X has an open refinement which is point countable.

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T_1 space is **collectionwise Hausdorff** if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets which simultaneously separates D, that is, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.

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$\mathsf{ADS}^{-}(\lambda)$

For $X \subseteq \lambda$, $g : X \to [\lambda]^{\aleph_0}$ is a **ladder system** if $otp(g(\alpha)) = \omega$ and $g(\alpha)$ is a cofinal subset of α for all $\alpha \in X$.

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$g : X \to \mathcal{P}(Y)$ is **essentially disjoint** if there is $h : X \to [Y]^{<\aleph_0}$ s.t. $g(x) \setminus h(x)$, $x \in X$ are pairwise disjoint.

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ω_1-strongly compact cardinal

A cardinal κ is ω_1-**strongly compact** if it is the smallest κ with the property that, for any $\mathcal{L}_{\omega_1,\omega}$ theory T if all subtheories of T of cardinality $< \kappa$ are satisfiable (i.e. T is $< \kappa$-satisfiable) then T itself is satisfiable.

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Countable tightness

A topological space X is **countably tight** if, for any $x \in X$ and $Y \subseteq X$ with $x \in \overline{Y}$, there is $Y' \in [Y]^{\leq \aleph_0}$ s.t. $x \in \overline{Y'}$.

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