On metrization theorems beyond Bin-Nagata-Smirnov

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(July 24, 2010 (07:30 JST) version)

神戸大学 数学科 談話会 での講演

July 7（七夕）, 2010. 17:00 ～ 18:00

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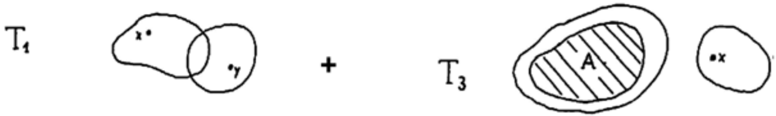
Bin-Nagata-Smirnov Metrization Theorem

Theorem 1 (Bin, Nagata, Smirnov (1950/51))

For a topological space X,
X is metrizable iff X is regular and X has a σ-locally finite base. $\square$

- A topological space $X = \langle X, \mathcal{O} \rangle$ is metrizable if there is a metric d over X s.t. the topology induced from d is identical with $\mathcal{O}$.
- A topological space X is regular if:



- A base $\mathcal{B}$ of topological space $X = \langle X, \mathcal{O} \rangle$ is σ-locally finite if $\exists\ \mathcal{B}_i \subseteq \mathcal{O},\ i \in \mathbb{N}$ s.t. $\mathcal{B} = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$ and each $\mathcal{B}_i$ is locally finite.
- $\mathcal{B} \subseteq \mathcal{O}$ is locally finite if $\forall\ x \in X\ \exists\ U \in \mathcal{O}$ s.t. $x \in U$ and $\{V \in \mathcal{B} : V \cap U \neq \emptyset\}$ is finite.

Bin-Nagata-Smirnov Metrization Theorem (2)

Theorem 1 (Bin, Nagata, Smirnov (1950/51))

For a topological space X,
X is metrizable iff X is regular and X has a σ-locally finite base. $\square$

- Bin-Nagata-Smirnov theorem is not the last word on the metrization problem!!

- In particular, the condition "X has a σ-locally finite base" is not always easily checked and sometimes impossible to check (in ZFC).

- The next theorem by Alan Dow gives an example of results on metrization problem beyond Bin-Nagata-Smirnov theorem.

Theorem 2 (Alan Dow, 1988)

For a countably compact topological space X,
if X is $\leq \aleph_1$-metrizable then X is metrizable. □

- A topological space X is said to be countably compact if every countable open cover of X has a finite subcover.
- $\aleph_1$: the first uncountable cardinal.

We have

- $\aleph_0 < \aleph_1$,
- $\aleph_1 \leq \mathfrak{c}$ where $\mathfrak{c}$ is the cardinality of $\mathbb{R}$ (the continuum).

- X is $\leq \kappa$-metrizable for a cardinal κ if every subspace Y of X of size $\leq \kappa$ is metrizable.
- "$\Leftarrow$" for the statement of Dow's theorem is trivial. So we actually have here "iff" !

Significant features of Dow's Theorem

Theorem 2 (Alan Dow, 1988)

For a countably compact topological space X, if X is $\leq \aleph_1$-metrizable then X is metrizable. □

- Not "$< \mathfrak{c}$" nor "$\leq \mathfrak{c}$" but "$\leq \aleph_1$" appears in the theorem.
  - ▷ "$\leq \aleph_1$" cannot be replaced by "$\leq \aleph_0$":

ω_1 (the first uncountable ordinal ($= \aleph_1$ as a set)) with the canonical order topology is countably compact, first countable, and $\leq \aleph_0$-metrizable but not metrizable.

- The poof of the theorem uses the method of elementary submodels in a very essential way. No other natural proof without this method is known! ▷ This shows that the theorem is not merely an easy corollary of Bin-Nagata-Smirnov theorem.
  - ▷ The method of elementary submodels is a quite common tool in set theory but it is apparently outside the technical scope of the majority of the non-logicians.

The order topology of ω_1 (1/3)

Claim 2.1

ω_1 with the canonical order topology is countably compact.

Proof. Suppose that O_k, $k \in \mathbb{N}$ are open subsets of ω_1 s.t.

(1) $\omega_1 = \bigcup_{k\in\mathbb{N}} O_k$.

We show first that $\omega_1 \setminus O_k$ is bounded for some $k \in \mathbb{N}$. Suppose otherwise. Then $\omega_1 \setminus O_k$, $k \in \mathbb{N}$ are all closed and unbounded. It follows that $\omega_1 \setminus \bigcup_{k\in\mathbb{N}} O_k = \bigcap_{k\in\mathbb{N}}(\omega_1 \setminus O_k)$ is also closed and unbounded; hence non empty in particular. This is a contradiction to (1).

We may assume that $\omega_1 \setminus O_0$ is bounded. Assume now, toward a contradiction, that $\bigcup_{k<i} O_k \neq \omega_1$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$ let $\alpha_i < \omega_1$ be s.t.

$$\alpha_i \in \omega_1 \setminus \bigcup_{k\leq i} O_k \quad \text{but} \quad (\alpha_i, \omega_1) \subseteq \bigcup_{k\leq i} O_k.$$

Then $\langle \alpha_i \,:\, i \in \mathbb{N}\rangle$ is decreasing and it is strictly decreasing at infinitely many places. A contradiction. □

The order topology of ω_1 (2/3)

Claim 2.2

ω_1 with the canonical order topology is first countable.

Proof. For $\alpha \in \omega_1$, if α is a successor ordinal then α is an isolated point. Otherwise α has the countable neighborhood base:

$$\{(\beta, \alpha + 1) : \beta < \alpha\}.$$

□

Claim 2.3

ω_1 with the canonical order topology is $\leq \aleph_0$-metrizable.

Proof. For any countable $Y \subseteq \omega_1$, there is $\alpha < \omega_1$ s.t. $Y \subseteq \alpha$. But since α (with its canonical order) is an order preserving embedding of α into $\mathbb{R}$, α is metrizable and hence also Y. □

The order topology of ω_1 (3/3)

Claim 2.4

ω_1 with the canonical order topology is not metrizable.

Proof. Suppose that there is a metric d which induces the order topology of ω_1.
For all $\alpha \in Lim(\omega_1)$, let $n_\alpha \in \mathbb{N} \setminus \{0\}$ be s.t. $B_d(\alpha, \frac{2}{n_\alpha}) \subseteq \alpha + 1 = (-1, \alpha + 1)$ and $\beta_\alpha < \alpha$ be s.t. $\beta_\alpha \in B_d(\alpha, \frac{1}{n_\alpha})$.
By Fodor's lemma there is $n^* \in \mathbb{N}$ and $\beta^* < \omega_1$ s.t.

$$S = \{\alpha \in Lim(\omega_1) \; : \; n_\alpha = n^* \text{ and } \beta_\alpha = \beta^*\}$$

is stationary and hence, in particular, infinite.
Let $\alpha_0, \alpha_1 \in S$ be s.t. $\alpha_0 < \alpha_1$. Then we have

$$d(\alpha_0, \alpha_1) \leq d(\alpha_0, \beta^*) + d(\beta^*, \alpha_1) \leq \frac{1}{n_{\alpha_0}} + \frac{1}{n_{\alpha_1}} = \frac{2}{n^*}.$$

Thus, $\alpha_1 \in B_d(\alpha_0, \frac{2}{n^*}) = B_d(\alpha, \frac{2}{n_{\alpha_0}}) \subseteq \alpha_0 + 1$. This is a contradiction. □

Balogh's Theorem

The following would be a natural generalization of Dow's Theorem:

> For a locally countably compact topological space X,
>
> if X is $\leq \aleph_1$-metrizable then X is metrizable.

We cannot prove the assertion above in the "conventional" framework of mathematics:

- If we assume $V = L$ (the axiom asserting that the set-theoretic universe consists of constructible sets in the sense of Gödel) then the assertion above is false.

- Zoltan Balogh (posth. 2002) showed that Axiom R (this is e.g. a consequence of Martin's Maximum) implies the assertion above (Balogh's metrization theorem).

Set-theoretic reformulation of Balogh's Theorem

The assertion (Balogh's metrization theorem):

> For a locally countably compact topological space X,
>
> if X is $\leq \aleph_1$-metrizable then X is metrizable.

is equivalent to the following set-theoretic principle called Fodor-type Reflection Principle (FRP):

> For any regular cardinal $\kappa > \aleph_1$ and any stationary $S \subseteq E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ and $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that
>
> - $\mathrm{cf}(I) = \omega_1$;
> - $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
> - for any regressive $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{''}\{\xi^*\}$ is stationary in $sup(I)$.

Assertions equivalent to FRP (1/3)

Theorem 3 (S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba (2010); S.F., H. Sakai, L. Soukup, T. Usuba (201?))

Over ZFC, FRP *is equivalent to the following assertion:*

> *For a* *locally* *countably compact topological space* X,
>
> *if* X *is* $\leq \aleph_1$*-metrizable then* X *is metrizable.*

- Via FRP a lot of "mathematical statements" can be proved to be equivalent to the Balogh's metrization theorem.

Assertions equivalent to FRP (2/3)

- For an infinite graph $G = \langle G, \mathcal{E} \rangle$, the coloring number of G ($col(G)$) is defined as

$$col(G) = \min\{\mu : \text{there is a well-ordering } \prec \text{ of } G \text{ s.t. } |\{y \in G : y \prec x \text{ and } \{x, y\} \in \mathcal{E}\}| < \mu \text{ for all } x \in G\}.$$

Theorem 4 (S.F., H. Sakai, L. Soukup, T. Usuba (201?))

Over ZFC, FRP *is equivalent to the following assertion:*

For any infinite graph $G = \langle G, \mathcal{E} \rangle$, *if*

$$col(G \restriction I) \leq \aleph_0 \text{ holds for all } I \in [G]^{\leq \aleph_1},$$

then $col(G) \leq \aleph_0$.

Assertions equivalent to FRP (3/3)

- A Boolean algebra B is openly generated if there is a mapping $f : B \to [B]^{<\aleph_0}$ s.t., for any b, $c \in B$ with $b \leq c$, there is $d \in f(b) \cap f(c)$ s.t. $b \leq d \leq c$.

Theorem 5 (S.F., A. Rinot (201?))

Over ZFC, FRP *is equivalent to the following assertion:*

> *For any Boolean algebra* B*,*
> *if there are closed-unboundedly many openly generated subalgebras* C *of* B *of cardinality* $\leq \aleph_1$ *then* B *is openly generated.*

- The proof of this theorem uses the fact that FRP implies Shelah's Strong Hypothesis (SSH)

Thank you for your attention!