On Reflection of List Chromatic Number

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Reflection theorems

▶ The following assertions are consistent (relative to some fairly large cardinal) and they are equivalent to each other over ZFC.

[S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, 2010], [S.F. and A. Rinot, 2011], [S.F., H. Sakai, L. Soukup and T. Usuba, preprint].

For any locally countably compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

An uncountable boolean algebra B is openly generated if club many subalgebras of B of size $\aleph_1$ are openly generated.

An uncountable graph G has countable coloring number if all subgraphs of G of size $\aleph_1$ have countable coloring number.

▷ [Z. Balogh, 2002], [S.F., 1994], [W. Fleissner, 1986] proved that the assertions above are theorems under Fleissner's Axiom R and independent from ZFC.

Set-theoretic core of the reflection theorems

- The reflection assertions of the last slide are all equivalent to the following set-theoretic principle which we called **Fodor-type Reflection Principle** (see [S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, 2010] and [S.F., H. Sakai, L. Soukup and T. Usuba, preprint]):

(FRP):

For any regular uncountable λ and any stationary $S \subseteq E^{\lambda}_{\omega} = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ and any mapping $g : S \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ (a reflection point of g) s.t.

(1) $\mathrm{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;

(3) *for any regressive* $f : S \cap I \to \lambda$ *s.t.* $f(\alpha) \in g(\alpha)$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \lambda$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

Non reflection theorems

- Slight modifications of reflection assertions above are often simply false in ZFC:

There are non metrizable spaces X s.t. all subspaces of X of cardinality $< |X|$ are metrizable. (A. Hajnal and I. Juhász, 1976)

There are non projective boolean algebras B s.t. club many subalgebras of B of cardinality $\aleph_1$ are projective.

For $1 < n < \omega$ there is graph G of uncountable chromatic number s.t. all subgraphs of G of cardinality $< \beth_n$ are of countable chromatic number (P. Erdős and A. Hajnal, 1961).

Reflection to a large cardinal

- The second and third examples of the non-reflection can be repaired if the “reflection cardinal” is set to be a very large cardinal:

Proposition. Suppose that κ is a supercompact cardinal. For any boolean algebra B if club many subalgebras of B of cardinality $< \kappa$ are projective then B is projective.

Reflection to a large cardinal

Proposition 1. Suppose that κ is a supercompact cardinal. For any boolean algebra B, if club many subalgebras of B of cardinality $< \kappa$ are projective, then B is projective.

Proof.

- Suppose that B is a boolean algebra with $\lambda = |\,B\,|$ s.t. club many subalgebras of B of size $< \kappa$ are projective. Without loss of generality the underlining set of B is λ.
- Let $f : [B]^{<\omega} \to B$ be s.t., for any $C \in [\lambda]^{<\kappa}$ with $|\,C\,| \subseteq C$, if C is closed w.r.t. f then $C = B \restriction C$ is projective.
- Let $j : V \to M$ be an elementary embedding to a transitive class M s.t. $crit(j) = \kappa$, $j(\kappa) > \lambda = |\,B\,|$ and ${}^{\lambda}M \subseteq M$.
- By elementarity, we have $M \models$"$j(f)$ witnesses that club many subalgebras of $j(B)$ of size $< j(\kappa)$ are projective".
- Since $j''B \in M$, $M \models |\,j''B\,| < j(\kappa)$ and $j''B$ is closed w.r.t. $j(f)$, it follows that $M \models$"$j''B$ is projective". Hence $B \cong f''B$ is projective. $\square$

Reflection to a large cardinal

Proposition 1. Suppose that κ is a supercompact cardinal. For any boolean algebra B, if club many subalgebras of B of cardinality $< \kappa$ are projective, then B is projective.

- Corresponding proposition on countability of chromatic number of graphs also holds with exactly the same proof.

Proposition 2. Suppose that κ is a supercompact cardinal. For any graph G, if all subgraphs of G of cardinality $< \kappa$ are of countable chromatic number then G is of countable chromatic number.

Reflection to a large cardinal

Proposition 1. Suppose that κ is a supercompact cardinal. For any boolean algebra algebra B, if club many subalgebras of B of cardinality $< \kappa$ are projective, then B is projective.

Proposition 2. Suppose that κ is a supercompact cardinal. For any graph G, if all subgraphs of G of cardinality $< \kappa$ are of countable chromatic number then G is of countable chromatic number.

- For a class $\mathcal{C}$ of structures of the same type and a predicate $P(x)$ on elements of $\mathcal{C}$, the **reflection cardinal of P** is defined as

$$\mathit{refl}(\mathcal{C}, P) = \min\{\kappa : \text{for all } B \in \mathcal{C}, \text{ if club many } C \subseteq B,\ C \in \mathcal{C} \text{ with } |C| < \kappa \text{ has the porperty } P \text{ then } B \text{ also have } P\}$$

- J. Bagaria proved that Vopenka principle is equivalent to the statement: "For every $\mathcal{C}$ and 'algebraic' P as above, $\mathit{refl}(\mathcal{C}, P)$ is defined.".

Reflection/non reflection of list chromatic number

Theorem (P. Komjáth, [preprint]).

(1) (MA(Cohen)) *For any graph G of cardinality $\aleph_1$, we have* $\aleph_0 < Chr(G) \Leftrightarrow \aleph_0 < List(G)$.

(2) *For any graph G of cardinality $\aleph_1$, if $\aleph_0 < Col(G)$ then, for* $\mathbb{P} = \mathrm{Fn}(\omega_2, 2, < \aleph_1)$, *we have* $\Vdash_{\mathbb{P}}$ "$\aleph_0 < List(G)$".

- Let $(\dagger)$ be the assertion:

Any uncountable graph G has countable list chromatic number if all subgraphs of G of size $\aleph_1$ have countable list chromatic number.

Corollary (S.F. and H. Sakai [in preparation]).

(1) "ZFC+FRP+$\neg(\dagger)$" *is consistent.*

(2) "ZFC+FRP+$(\dagger)$" *is consistent.*

Reflection/non reflection of list chromatic number

- Let (†) be the assertion:

Any uncountable graph G has countable list chromatic number if all subgraphs of G of size $\aleph_1$ have countable list chromatic number.

Corollary (S.F. and H. Sakai [in preparation])

(1) "ZFC+FRP+¬(†)" *is consistent.*

(2) "ZFC+FRP+(†)" *is consistent.*

Proof. (1): Let $M \models$ ZFC + FRP. Let $M[G]$ be a c.c.c.-generic extension where MA(Cohen) also holds. FRP still holds in $M[G]$. By (1) of Komjáth's theorem countability of list chromatic number behaves like countability of chromatic number. Hence by Erdős-Hajnal's example, we have the negation of (†).

(2): Levy collapse of a strongly compact cardinal to $\aleph_2$ forces FRP. By (2) of Komjáth's theorem, countability of list chromatic number behaves like countability of coloring number in this model. Hence by [S.F.,H.Sakai,...], (†) for list chromatic number holds. □

Problems

- Is there a reflection statement on groups which is similar to (or, even equivalent to) the FRP ?
- Is "ZFC + FRP + (†) consistent with ¬CH ?
- Find more examples of natural reflexion statements which are independent from FRP.

Tēnā koutou !

感谢您的关注！

Thank you for your attention !

Vielen Dank für Ihre Aufmerksamkeit !

%̂$#%(*#+@} !

Openly generated boolean algebras

- A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F.
  - Note that subalgebra of a boolean algebra is not necessarily free.
  - Free boolean algebras are projective. There are projective boolean algebras which are not free.
- A boolean algebra A is **openly generated** if

$$\Vdash_{\mathbb{P}} \text{“} A \text{ is projective”}$$

  for any σ-closed p.o. forcing the cardinality of A to be $\leq \aleph_1$.
  - Projective boolean algebras are openly generated. There are openly generated boolean algebras which are not projective.

back

Openly generated boolean algebras

- A boolean algebra A is **projective** if $A \oplus F$ is free for a sufficiently large free boolean algebra F.
  - Note that subalgebra of a boolean algebra is not necessarily free.
- A boolean algebra A is **openly generated** if $\Vdash_{\mathbb{P}}$ "A is projective" for any σ-closed p.o. forcing the cardinality of A to be $\leq \aleph_1$.

Proposition (L. Heindorf, 198?). *A Boolean algebra A is openly generated if and only if there is a mapping $f : A \to [A]^{<\aleph_0}$ (Freese-Nation mapping) s.t., for all $a, b \in A$ with $a \leq_A b$, there is $c \in f(a) \cap f(b)$ s.t. $a \leq_A c \leq_A b$.*

back

Club (stationarily) many subsets of size $\aleph_1$

- For a set X, let $[X]^{\aleph_1} = \{A \subseteq X : |A| = \aleph_1\}$.
- $[X]^{\kappa}$, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$ etc. for any cardinal κ are defined similarly.

- $\mathcal{F} \subseteq [X]^{\aleph_1}$ is **club** (closed unbounded) if
  - For any upward directed $\mathcal{F}' \subseteq \mathcal{F}$ (w.r.t. $\subseteq$) of cardinality $\leq \aleph_1$, we have $\bigcup \mathcal{F}' \in \mathcal{F}$; and,
  - for any $A \in [X]^{\aleph_1}$ there is $B \in \mathcal{F}$ with $A \subseteq B$.
- $\mathcal{S} \subseteq [X]^{\aleph_1}$ is **stationary** if $\mathcal{S} \cap \mathcal{F} \neq \emptyset$ for all club $\mathcal{F} \subseteq [X]^{\aleph_1}$.
- "There are club many subsets of X of size $\aleph_1$ with ..." means "$\{A \in [X]^{\aleph_1} : ...\}$ contains a club subset of $[X]^{\aleph_1}$". Similarly for "stationarily many".

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Reflection and non-reflection of metrizability

- The following is a theorem under ZFC:

Theorem (A. Dow, 1988).

For any countably compact Hausdorff space X if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable then X is also metrizable.

- In contrast, the reflection does not hold if we drop the compactness of the space totally:

Theorem (A. Hajnal and I. Juhász, 1976)

For any uncountable cardinal κ there is a non-metrizable space X of cardinality κ s.t. all subspaces Y of X of cardinality $< \kappa$ are metrizable.

Proof. For every regular uncountable κ, $(\kappa + 1, \mathcal{O})$ with $\mathcal{O} = \mathcal{P}(\kappa) \cup \{(\kappa \setminus x) \cup \{\kappa\} : x \in [\kappa]^{<\kappa}\}$ is such a space. □

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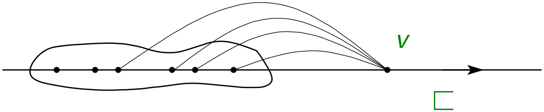
Graphs with countable coloring number

- A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation E is non-reflective and symmetric. G is the set of vertices and $v\ E\ w$ (or $w\ E\ v$) if the vertices v, $w \in G$ are connected in the graph.
- G has **coloring number** $\leq \kappa$ if there is a well-ordering $\sqsubset$ of G s.t., for all $v \in G$, we have $|\{w \in G : w \sqsubset v,\ w\ E\ v\}| < \kappa$



$\{w \in G : w \sqsubset v,\ w\ E\ v\}$

- The **coloring number** of G (notation: $Col(G)$) is defined by
$$Col(G) = \min\{\kappa : G \text{ has coloring number} \leq \kappa\}.$$
- G has **countable coloring number** if $Col(G) \leq \aleph_0$.

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Graphs with countable coloring number (2/2)

Theorem (P. Erdős and A. Hajnal, 1961). *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \to [G]^{<\kappa}$ s.t., for all $v, w \in G$, if $v \mathrel{E} w$ then $v \in f(w)$ or $w \in f(v)$.*

Corollary. *A graph $G = \langle G, E \rangle$ has coloring number $\leq \kappa$ if and only if G has a filtration $\langle G_\alpha : \alpha < \lambda \rangle$ s.t.*

- *all G_α, $\alpha < \lambda$ have coloring number $\leq \kappa$; and*
- *for all $\alpha < \lambda$ and $v \in G \setminus G_\alpha$, $|\{w \in G_\alpha : w \mathrel{E} v\}| < \kappa$.*

Corollary. *A graph $G = \langle G, E \rangle$ has coloring number $> \kappa$ if for any filtration $\langle G_\alpha : \alpha < \lambda \rangle$ of G either*

- *for stationarily many $\alpha < \lambda$, we have $col(G_\alpha) > \kappa$; or*
- *for stationarily many $\alpha < \lambda$, $|\{w \in G_\alpha : w \mathrel{E} v_\alpha\}| \geq \kappa$ for some $v_\alpha \in G \setminus G_\alpha$.*

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Chromatic and list chromatic number of graphs

- For a graph $G = \langle G, E \rangle$, a mapping $f : G \to \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all v, $w \in G$ with $v \, E \, w$. The **chromatic number** $Chr(G)$ of a graph G is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \to \kappa\}.$$

- It is easy to see that $Chr(G) \leq Col(G)$ holds for all graph G.
- The **list chromatic number** $List(G)$ of a graph G is defined by

$$\begin{aligned} List(G) = \min\{\kappa : \; & \text{for any } \lambda \text{ and } g : G \to [\lambda]^{\kappa}, \\ & \text{there is a choice function } f \text{ for } g \\ & \text{which is a good coloring of } G\} \end{aligned}$$

- We have $Chr(G) \leq List(G) \leq Col(G)$.

Chromatic and list chromatic number of graphs

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- We have $Chr(G) \leq List(G) \leq Col(G)$.

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