# On Reflection of List Chromatic Number

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## Reflection theorems



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*For any locally countably compact Hausdorff space $X$ if all subspaces of $X$ of cardinality $\leq \aleph_1$ are metrizable then $X$ is also metrizable.*

*An uncountable boolean algebra $B$ is openly generated if club many subalgebras of $B$ of size $\aleph_1$ are openly generated.*

*An uncountable graph $G$ has countable coloring number if all subgraphs of $G$ of size $\aleph_1$ have countable coloring number.*

▷ [Z. Balogh, 2002], [S.F., 1994], [W. Fleissner, 1986] proved that the assertions above are theorems under Fleissner's Axiom R and independent from ZFC.

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# Set-theoretic core of the reflection theorems

- The reflection assertions of the last slide are all equivalent to the following set-theoretic principle which we called **Fodor-type Reflection Principle** (see [S.F., I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, 2010] and [S.F., H. Sakai, L. Soukup and T. Usuba, preprint]):

**(FRP):**

*For any regular uncountable $\lambda$ and any stationary $S \subseteq E^\lambda_\omega = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \omega\}$ and any mapping $g : S \rightarrow [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ (a reflection point of $g$) s.t.*

(1) $\mathrm{cf}(I) = \omega_1$;

(2) $g(\alpha) \subseteq I$ *for all* $\alpha \in I \cap S$;

(3) *for any regressive* $f : S \cap I \rightarrow \lambda$ *s.t.* $f(\alpha) \in g(\alpha)$ *for all* $\alpha \in S \cap I$, *there is* $\xi^* < \lambda$ *s.t.* $f^{-1}{}''\{\xi^*\}$ *is stationary in* $sup(I)$.

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# Non reflection theorems



- Slight modifications of reflection assertions above are often simply false in ZFC:

There are non metrizable spaces $X$ s.t. all subspaces of $X$ of cardinality $< |X|$ are metrizable. (A. Hajnal and I. Juhász, 1976)

There are non projective boolean algebras $B$ s.t. club many subalgebras of $B$ of cardinality $\aleph_1$ are projective.

For $1 < n < \omega$ there is graph $G$ of uncountable chromatic number s.t. all subgraphs of $G$ of cardinality $< \beth_n$ are of countable chromatic number (P. Erdős and A. Hajnal, 1961).

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# Reflection to a large cardinal



- The second and third examples of the non-reflection can be repaired if the "reflection cardinal" is set to be a very large cardinal:

**Proposition.** Suppose that $\kappa$ is a supercompact cardinal. For any boolean algebra $B$ if club many subalgebras of $B$ of cardinality $< \kappa$ are projective then $B$ is projective.

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Proof.

- Suppose that $B$ is a boolean algebra with $\lambda = |B|$ s.t. club many subalgebras of $B$ of size $< \kappa$ are projective. Without loss of generality the underlining set of $B$ is $\lambda$.
- Let $f : [B]^{<\omega} \to B$ be s.t., for any $C \in [\lambda]^{<\kappa}$ with $|C| \subseteq C$, if $C$ is closed w.r.t. $f$ then $C = B \restriction C$ is projective.
- Let $j : V \to M$ be an elementary embedding to a transitive class $M$ s.t. $crit(j) = \kappa$, $j(\kappa) > \lambda = |B|$ and ${}^{\lambda}M \subseteq M$.
- By elementarity, we have $M \models$"$j(f)$ witnesses that club many subalgebras of $j(B)$ of size $< j(\kappa)$ are projective".
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**Proposition 1.** Suppose that $\kappa$ is a supercompact cardinal. For any boolean algebra $B$, if club many subalgebras of $B$ of cardinality $< \kappa$ are projective, then $B$ is projective.

Proof.
- Suppose that $B$ is a boolean algebra with $\lambda = |B|$ s.t. club many subalgebras of $B$ of size $< \kappa$ are projective. Without loss of generality the underlining set of $B$ is $\lambda$.
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# Reflection/non reflection of list chromatic number

**Theorem (P. Komjáth, [preprint]).**

(1) (MA(Cohen)) *For any graph $G$ of cardinality $\aleph_1$, we have $\aleph_0 < Chr(G) \Leftrightarrow \aleph_0 < List(G)$.*

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for any $\sigma$-closed p.o. forcing the cardinality of $A$ to be $\leq \aleph_1$.
  - Projective boolean algebras are openly generated. There are openly generated boolean algebras which are not projective.

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# Club (stationarily) many subsets of size $\aleph_1$

- For a set $X$, let $[X]^{\aleph_1} = \{A \subseteq X : |A| = \aleph_1\}$.
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# Reflection and non-reflection of metrizability

- The following is a theorem under ZFC:

**Theorem (A. Dow, 1988).**

*For any countably compact Hausdorff space $X$ if all subspaces of $X$ of cardinality $\leq \aleph_1$ are metrizable then $X$ is also metrizable.*

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**Theorem (A. Hajnal and I. Juhász, 1976)**

*For any uncountable cardinal $\kappa$ there is a non-metrizable space $X$ of cardinality $\kappa$ s.t. all subspaces $Y$ of $X$ of cardinality $< \kappa$ are metrizable.*

Proof. For every regular uncountable $\kappa$, $(\kappa+1, \mathcal{O})$ with $\mathcal{O} = \mathcal{P}(\kappa) \cup \{(\kappa \setminus x) \cup \{\kappa\} : x \in [\kappa]^{<\kappa}\}$ is such a space. □

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# Graphs with countable coloring number

- A structure $G = \langle G, E \rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation $E$ is non-reflective and symmetric. $G$ is the set of vertices and $v\ E\ w$ (or $w\ E\ v$) if the vertices $v, w \in G$ are connected in the graph.

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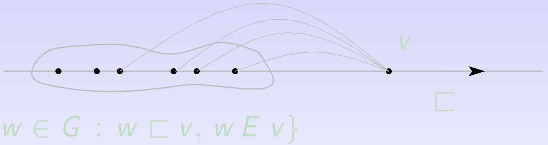


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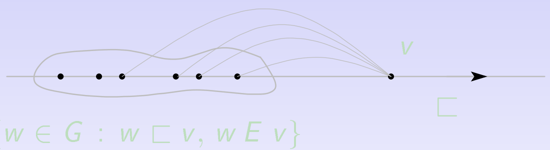


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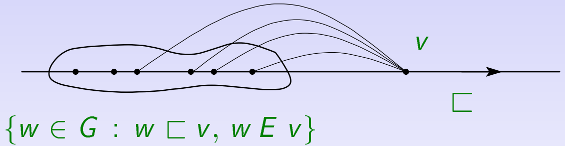


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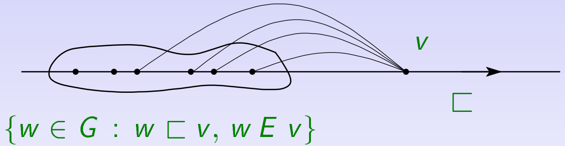


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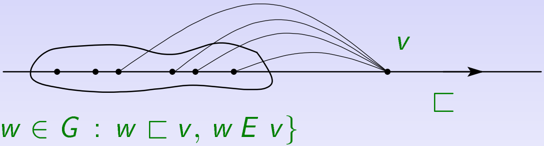


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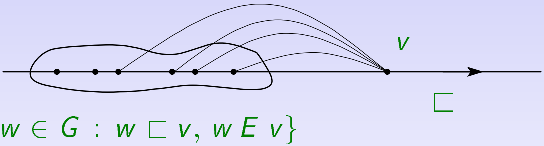


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- $G$ has **countable coloring number** if $Col(G) \leq \aleph_0$.

back

# Graphs with countable coloring number

- A structure $G = \langle G, E\rangle$ with $E \subseteq G^2$ is a **graph** if the binary relation $E$ is non-reflective and symmetric. $G$ is the set of vertices and $v\ E\ w$ (or $w\ E\ v$) if the vertices $v$, $w \in G$ are connected in the graph.

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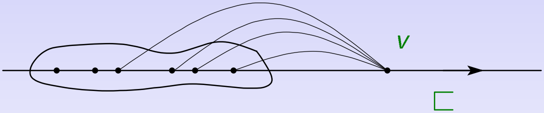


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## Graphs with countable coloring number (2/2)

**Theorem (P. Erdős and A. Hajnal, 1961).** *A graph $G = \langle G, E\rangle$ has coloring number $\leq \kappa$ if and only if there is a mapping $f : G \to [G]^{<\kappa}$ s.t., for all $v$, $w \in G$, if $v\ E\ w$ then $v \in f(w)$ or $w \in f(v)$.*

**Corollary.** *A graph $G = \langle G, E\rangle$ has coloring number $\leq \kappa$ if and only if $G$ has a filtration $\langle G_\alpha : \alpha < \lambda\rangle$ s.t.*

- *all $G_\alpha$, $\alpha < \lambda$ have coloring number $\leq \kappa$; and*
- *for all $\alpha < \lambda$ and $v \in G \setminus G_\alpha$, $|\{w \in G_\alpha : w\ E\ v\}| < \kappa$.*

**Corollary.** *A graph $G = \langle G, E\rangle$ has coloring number $> \kappa$ if for any filtration $\langle G_\alpha : \alpha < \lambda\rangle$ of $G$ either*

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# Chromatic and list chromatic number of graphs

- For a graph $G = \langle G, E \rangle$, a mapping $f : G \to \kappa$ is a **good coloring** if $f(v) \neq f(w)$ for all $v, w \in G$ with $v \, E \, w$. The **chromatic number** $Chr(G)$ of a graph $G$ is defined by

$$Chr(G) = \min\{\kappa : \text{there is a good coloring } f : G \to \kappa\}.$$

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