Rado's Conjecture implies the Fodor-type Reflection Principle

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The main results in this note will be later integrated into the joint paper: "Rado's Conjecture and the Fodor-type Reflection Principle".

1 Introduction

introduction

We give a proof of the implication in the title. This proof will be later included in the paper [8] in preparation.

In [16], Stevo Todorvcević showed that Rado's Conjecture (RC) implies Chang's Conjecture (CC) by proving the chain of the implications:

(1.1) $\mathsf{RC} \Rightarrow$ *II* has a winning strategy in $G_\omega(\omega_2, \omega_1) \Rightarrow \mathsf{CC}^* \Rightarrow \mathsf{CC}$. RC-CC

Our proof of the Fodor-type Reflection Principle (FRP) from RC is in analogy to this proof and it is done by showing the chain of the implications:

(1.2) $\mathsf{RC} \Rightarrow$ *II* has a winning strategy in $G^{\downarrow}_\omega(\kappa) \Leftrightarrow \mathsf{CC}^{\downarrow}(\kappa) \Rightarrow \mathsf{FRP}(\kappa)$ RC-FRP

for all regular $\kappa > \aleph_1$.

Let us review first the definition of and basic facts about the principles RC and FRP, and then introduce two other principles mentioned in (1.2).

We shall call in this note a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the intial segment $\{u \in T : u \leq_T t\}$ in T below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

For a tree T and $t \in T$, we define the hight of t by $ht(t) = otp(\{s \in T : s <_T t\})$ and the αth level of T by $T_\alpha = \{t \in T : ht(t) = \alpha\}$. We write $ht_T(t)$ in place of $ht(t)$ if it is necessary to specify in which tree the hight of t should be considered.

For $t \in T$ and $\alpha \leq ht(t)$, $t \restriction \alpha$ = the αth element of $\{s \in T : s \leq_T t\}$. In spite of this convention, if T consists of functions on ordinals then we denote with $t \restriction \alpha$ for $t \in T$ rather the usual restriction of the function t on the set α.

A tree T is *special* if there are $T_i \subseteq T$, $i \in \omega$ such that elements of T_i are pairwise incomparable for each $i \in \omega$ and $T = \bigcup_{i \in \omega} T_i$.

Rado's Conjecture (RC) is the assertion:

(RC): Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

RC is known to be consistent (modulo a large large cardinal). E.g., Todorčević showed that, if κ is strongly compact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$, then we have

$$\Vdash_{\mathbb{P}} \text{“Rado's Conjecture”}.$$

For a cardinal κ and a regular cardinal $\delta < \kappa$, we denote $E^\kappa_\delta = \{\alpha < \kappa : cf(\alpha) = \delta\}$. For a regular cardinal $\kappa > \omega$ we define the Fodor-type Reflection Principle for κ by

FRP(κ): For any stationary $E \subseteq E^\kappa_\omega$ and any ladder system(1) $g : E \to [\kappa]^{\aleph_0}$, there exists $\alpha^* \in E^\kappa_{\omega_1}$ such that

$$\{x \in [\alpha^*]^{\aleph_0} : sup(x) \in E,\ g(sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

The Fodor-type Reflection Principle (FRP) is then the assertion:

(FRP): FRP(κ) holds for all regular $\kappa > \aleph_1$.

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [2], [3], [4], [5], [6], see also [7]).

(1.3) Any locally countably compact topological space X is metrizable if and only if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable

is one of such “mathematical” statements equivalent to FRP over ZFC (see [3] and [4]).

The reflection principle RP implies FRP and FRP implies Shelah's Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [6]).

For a regular κ, we denote with ${}^{\kappa\downarrow}\kappa$ the set $\{f \in {}^\kappa\kappa : f \text{ is regressive}\}$.

The game $G^\downarrow_\omega(\kappa)$ for Players I and II is defined as follows: A match in $G^\downarrow_\omega(\kappa)$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{\kappa\downarrow}\kappa & & f_1 \in {}^{\kappa\downarrow}\kappa & & \cdots & f_n \in {}^{\kappa\downarrow}\kappa & & \cdots \\ \hline II & & \delta_0 \in \kappa & & \delta_1 \in \kappa & \cdots & & \delta_n \in \kappa & \cdots \end{array} \quad (n < \omega)$$

II wins in a match of $G^\downarrow_\omega(\kappa)$ as above if

(1.4) $\{\alpha \in E^{\kappa}_{\omega_1} : f_n(\alpha) < \sup\{\delta_i : i \in \omega\}$ for all $n \in \omega\}$ is unbounded.

Let θ be a sufficiently large regular cardinal and

(1.5) $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubseteq \rangle$ where $\sqsubseteq$ is a well-ordering on $\mathcal{H}(\theta)$. calM

($\mathsf{CC}^{\downarrow}(\kappa)$): For any countable $M \prec \mathcal{M}$ with $\kappa \in M$ and $\alpha \in \kappa$, there exists a countable $M^* \prec \mathcal{M}$ such that $M \prec M^*$, $\alpha^* \geq \alpha$ and $cf(\alpha^*) = \omega_1$ for $\alpha^* = \inf(\kappa \cap M^* \setminus \sup(\kappa \cap M))$.

For a game $\mathcal{G}$ for players I and II, we denote with $\mathsf{WS}_{II}(\mathcal{G})$ the assertion that Player II has a winning strategy in the game $\mathcal{G}$.

Using these notations, the main part of the implications we are going to establish in Section 3 and Section 4 can be put together as

$$\mathsf{RC} \;\Rightarrow\; \mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa)) \;\Leftrightarrow\; \mathsf{CC}^{\downarrow}(\kappa)$$

for all $\kappa > \aleph_1$.

2 Two Lemmas on Trees

lemmas-on-trees

Let us begin with a review of two well-known lemmas on trees.

A subset D of a tree T is dense open if D is an end-segment of T [2] and for any $t \in T$ there is $t' \in D$ such that $t \leq_T t'$. A tree T is a Baire tree if, for any dense open $D_n \subseteq T$, $n \in \omega$, $\bigcap_{n\in\omega} D_n$ is dense open.

special-tree-0

Lemma 2.1 (1) *Suppose that T is a tree without maximal elements. If T is a Baire tree then T is not special.*

(2) *If T is a tree of hight $< \omega_1$ then T is special.*

Proof. (1): We prove the contraposition: Suppose that T is a special tree without maximal elements. Let $T = \bigcup_{n\in\omega} T_n$ where T_n is pairwise incomparable for all $n \in \omega$. Then $D_n = \{t \in T : t$ is incomparable with all $t' \in T_n$ or $t' \lneqq_T t$ for some $t' \in T\}$ for each $n \in \omega$ is an dense open subset of T but $\bigcap_{n\in\omega} D_n = \emptyset$.

(2): Let $\eta = ht(T)$ then $T = \bigcup_{\xi<\eta} T_\xi$ with $T_\xi = \{t \in T : ht(t) = \xi\}$ for $\xi \leq \eta$ is a partition of T into countably may pairwise incomparable subsets. ❏ (Lemma 2.1)

PDL

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(1) For $E \subseteq E^{\kappa}_{\omega}$, a *ladder system* on E is a mapping $g : E \to [\kappa]^{\aleph_0}$ which maps each element $\alpha \in E$ to a countable cofinal subset $g(\alpha)$ of α.

(2) I.e., for any $t \in D$ and $t' \in T$ with $t \leq_T t'$ we have $t' \in D$.

Lemma 2.2 (Pressing Down Lemma for trees, Todorčević [13]) *If a tree T is not special then, for any regressive mapping*[3] $f : T \to T$, *there is a $t \in T$ such that $f^{-1}{}''\{t\}$ is a non special tree (as a subtree of T)*[4] *.*

Proof. Suppose that $f : T \to T$ is a regressive function such that $f^{-1}{}''\{t\}$ is special for all $t \in T$. We show that T is then also special.

For each $t \in T$ let $g_t : f^{-1}{}''\{t\} \to \omega$ be such that

(2.1) $\quad g_t^{-1}{}''\{n\}$ is pariwise incomparable for all $n \in \omega$. rado-0

For $t \in T$, let $\overline{u}_t = \langle u_0^t, u_1^t, ..., u_{n_t}^t\rangle$ be the descending sequence of elements of T defined by $u_0^t = t$, $u_{n+1}^t = f(u_n^t)$ for all $n < n_t$ and $u_{n_t}^t$ is a minimal element in T. Note that we have $u_k^t \in f^{-1}{}''\{u_{k+1}^t\}$ for all $k < n_t$.

Let $l : T \to {}^{\omega>}\omega$ be defined by

$$l(t) = \begin{cases} \emptyset, & \text{if } t \text{ is a minimal element of } T; \\ \langle g_{u_1^t}(u_0^t), g_{u_2^t}(u_1^t), ..., g_{u_{n_t}^t}(u_{n_t-1}^t)\rangle, & \text{otherwise.} \end{cases}$$

Claim 2.2.1 *l witnesses that T is special.*

$\vdash$ Note that $|{}^{\omega>}\omega| = \aleph_0$. Suppose $t, t' \in T$, $t \neq t'$ but $l(t) = l(t')$. It is enough to show that t and t' are then incomparable.

If $l(t) = l(t') = \emptyset$ then t and t' are two different minimal elements of T and hence are incomparable. Otherwise both t and t' are non minimal in T. If $u_{n_t}^t \neq u_{n_{t'}}^{t'}$ then $u_{n_t}^t$ and $u_{n_{t'}}^{t'}$ are incomparable as two different minimal elements and hence $t >_T u_{n_t}^t$ and $t' >_T u_{n_{t'}}^{t'}$ are also incomparable. Thus we may assume that $u_{n_t}^t = u_{n_{t'}}^{t'}$. Let $k = min\{n < n_t : u_{n+1}^t = u_{n+1}^{t'}\}$. Then we have $g_{u_{k+1}^t}(u_k^t) = g_{u_{k+1}^{t'}}(u_k^{t'}) = g_{u_{k+1}^t}(u_k^{t'})$. By (2.1), u_k^t and $u_k^{t'}$ are incomparable. Since $u_k^t \leq_T t$ and $u_k^{t'} \leq_T t'$. It follows that t and t' are incomparable as well. $\dashv$ (Claim 2.2.1)

□ (Lemma 2.2)

We shall use the following slight strengthening of the Pressing Down Lemma. For a tree T we denote $\lim(T) = \{t \in T : ht_T(t) \text{ is a limit ordinal}\}$.

xPDL

Corollary 2.3 *For a tree T, suppose that $f : \lim(T) \to T$ is regressive. If $f^{-1}{}''\{t\}$ is special for all $t \in T$, then T is special.*

Proof. Let $\bar{f} : T \to T$ be the extension of f as above defined by

(2.2) $\quad \bar{f}(t) = f(t \restriction \alpha)$ where α is the largest limit ordinal below $ht_T(t) + 1$ or 0 if $ht_T(t) < \omega$. RC-game-0

[3] A mapping $f : T \to T$ (or partial mapping $f : T' \to T$ for some $T' \subseteq T$) is said to be *regressive* if we have $f(t) <_T t$ if $t \in T$ is not a minimal element in T, and $f(t) = t$ otherwise.

[4] In contraposition, this means that, if there is a regressive function $f : T \to T$ such that $f^{-1}{}''\{t\}$ is special for all $t \in T$, then T is special.

$\bar{f}$ is still regressive and $\bar{f}^{-1\,\prime\prime}\{t\}$ is special for all $t \in T$. By Pressing Down Lemma (Lemma 2.2) it follows that T is special. □ (Lemma 2.3)

3 RC implies Player *II* has a winning strategy in $G^{\downarrow}_{\omega}(\kappa)$.

RCimpliesWS

For a cardinal $\kappa > \omega_1$, let θ be a sufficiently large regular cardinal and let $\mathcal{M}$ be as in (1.5). For $M \in [\mathcal{M}]^{\aleph_0}$, let

(3.1) $$D_M = \{\alpha \in E^{\kappa}_{\omega_1} \,:\, f(\alpha) < \sup(\kappa \cap M) \text{ for all } f \in {}^{\kappa\downarrow}\kappa \cap M\}.$$ RC-game-1

Clearly we have $D_M \supseteq \sup(\kappa \cap M) \cap E^{\kappa}_{\omega_1}$.

Let

(3.2) $$\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)} = \{M \in [\mathcal{M}]^{\aleph_0} \,:\, M \prec \mathcal{M},\, D_M \text{ is bounded}\}.$$ RC-game-2

Note that

(3.3) D_M is bounded if and only if there is an $\alpha^* < \kappa$ such that, for any $\alpha \in E^{\kappa}_{\omega_1}$ with $\alpha^* < \alpha$, there is some $f \in {}^{\kappa\downarrow}\kappa \cap M$ such that $f(\alpha) \geq \sup(\kappa \cap M)$. RC-game-2-0

The following is trivial:

L-RC-game-a

Lemma 3.1 *Suppose that κ is an uncountable cardinal with $cf(\kappa) = \omega$. Then we have*

(1) *II has a winning strategy in $G^{\downarrow}_{\omega}(\kappa)$ and*

(2) *$\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$ is non stationary in $[\kappa]^{\aleph_0}$.*

L-RC-game-0

Lemma 3.2 *For any infinite cardinal κ, $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ if and only if $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$ is non-stationary in $[\mathcal{M}]^{\aleph_0}$.*

Proof. By Lemma 3.1, we may assume that $cf(\kappa) > \omega$.

Suppose that *II* has a winning strategy in $G^{\downarrow}_{\omega}(\kappa)$. Let

$$C = \{M \in [\mathcal{M}]^{\aleph_0} \,:\, M \prec \mathcal{M} \text{ and } \kappa \in M\}.$$

Then C is a club in $[\mathcal{M}]^{\aleph_0}$. We show that C is disjoint from $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$. Suppose $M \in C$. By the assumtion above and elementarity, there is a winning strategy $\sigma \in M$ for player *II* in $G^{\downarrow}_{\omega}(\kappa)$. Let f_0, δ_0, f_1, δ_1, ... be a match in $G^{\downarrow}_{\omega}(\kappa)$ such that

(3.4) f_n, $\delta_n \in M$ for all $n \in \omega$; RC-game-3

(3.5) *II* plays according to the winning strategy σ in this match; and RC-game-4

(3.6) $\{f_n \,:\, n \in \omega\} = {}^{\kappa\downarrow}\kappa \cap M$. RC-game-5

Note that

(3.7) $$\sup\{\delta_n \,:\, n \in \omega\} \leq \sup(\kappa \cap M).$$ RC-game-6

Since $I\!I$ wins in this match by (3.5), $\{\alpha \in E^{\kappa}_{\omega_1} : f_n(\alpha) < \sup\{\delta_i : i \in \omega\}$ for all $n \in \omega\}$ is unbounded. Hence, by (3.7) and (3.6), $D_M = \{\alpha \in E^{\kappa}_{\omega_1} : f(\alpha) < \sup(\kappa \cap M)$ for all $f \in {}^{\kappa\downarrow}\kappa \cap M\}$ is unbounded in κ. Thus $M \notin \mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$.

Suppose now that $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$ is non-stationary. Then there is a club $\mathcal{C} \subseteq [\mathcal{M}]^{\aleph_0}$ disjoint from $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$. Player $I\!I$ can then choose her moves in such way that, if f_0, δ_0, f_1, $\delta_1, \dots$ are the moves of a match, she chooses an increasing sequence $M_n \in \mathcal{C}$, $n \in \omega$ along with her moves δ_n, $n \in \omega$ such that

(3.8) $\quad f_0, \dots, f_n \in M_n,$ RC-game-7

(3.9) $\quad \delta_n = \sup(\kappa \cap M_n)$, $\delta_{n+1} \in M_n$ for all $n \in \omega$; and RC-game-9

Since $\mathcal{C}$ is a club, $M = \bigcup_{n\in\omega} M_n \in \mathcal{C}$ and hence $M \notin \mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$. By (3.9), $\sup(M \cap \kappa) = \sup_{n\in\omega} \delta_n$.

Thus, by the definition (3.2) of $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$, Player $I\!I$ wins in all such matches as above.

❑ (Lemma 3.2)

Let

(3.10) $\quad T_{G^{\downarrow}_{\omega}(\kappa)} = \{\langle M_\alpha : \alpha \leq \xi\rangle : \xi < \omega_1$, $\langle M_\alpha : \alpha \leq \xi\rangle$ is a continuously increasing sequence in $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$; RC-game-10

(∗) $\quad M_\alpha \in M_{\alpha+1}$ for all $\alpha < \xi$ }

We can regard $T_{G^{\downarrow}_{\omega}(\kappa)}$ as a tree with the ordering:

(3.11) $\quad$ for t, $t' \in T_{G^{\downarrow}_{\omega}(\kappa)}$, $t \leq_{T_{G^{\downarrow}_{\omega}(\kappa)}} t'$ if and only if t is an initial segment of t'. RC-game-11

For $t \in T_{G^{\downarrow}_{\omega}(\kappa)}$ with $\langle M_\alpha : \alpha \leq \xi\rangle$, we write $\ell(t) = \xi$, $M_t = M_\xi$ and $M_{t,\alpha} = M_\alpha$ for all $\alpha < \xi = \ell(t)$.

L-RC-game-1

Lemma 3.3 *For any cardinal $\kappa > \omega$, all $T \in [T_{G^{\downarrow}_{\omega}(\kappa)}]^{\aleph_1}$ are special trees.*

Proof. For $t \in T_{G^{\downarrow}_{\omega}(\kappa)}$ with $t = \langle M_\alpha : \alpha \leq \xi\rangle$, let

(3.12) $\quad d(t) = \kappa \cap M_\xi$ RC-game-11-0

and, for $T \subseteq T_{G^{\downarrow}_{\omega}(\kappa)}$, let

(3.13) $\quad d(T) = \bigcup\{d(t) : t \in T\}.$ RC-game-12

Note that $d(T) \in [\kappa]^{\leq\aleph_1}$ for $T \in [T_{G^{\downarrow}_{\omega}(\kappa)}]^{\aleph_1}$. We show by induction for $\xi < \omega_2$ that

$(3.14)_\xi$ $\quad$ If $T \in [T_{G^{\downarrow}_{\omega}(\kappa)}]^{\aleph_1}$ is such that $otp(d(T)) \leq \xi$, then T is a special tree. L-RC-game-2

Clearly this implies the present lemma.

Case I: $\xi < \omega_1$.

In this case, all $T \subseteq [T_{G^{\downarrow}_{\omega}(\kappa)}]^{\aleph_1}$ with $otp(d(T)) = \xi$ have hight $< \omega_1$ by (∗) in (3.10). Hence T is special by Lemma 2.1, (2). Thus the assertion $(3.14)_\xi$ holds.

Case II: $\xi < \omega_2$ is a successor, say $\xi = \eta + 1$, and $(3.14)_\nu$ holds for all $\nu \le \eta$.

Suppose that $otp(d(T)) \le \xi$. Since $\sup d(t)$ for each $t \in T$ is a limit ordinal, $\sup(d(T))$ is also a limit ordinal. It follows that $otp(d(T)) < \eta$. By the induction hypothesis, it follows that T is special.

Case III: $\xi < \omega_2$ is a limit ordinal of cofinality ω and $(3.14)_\nu$ holds for all $\nu < \xi$.

Suppose that $otp(d(T)) = \xi$ and let $\langle \nu_n : n \in \omega \rangle$ is an increasing sequence of ordinals cofinal in $\nu = \sup d(T)$. Then we have $T = \bigcup_{n\in\omega} T_n \cup T_\infty$ where

(3.15) $T_n = \{t \in T : d(t) \subseteq \nu_n\}$ for all $n \in \omega$; and RC-game-12-0

(3.16) $T_\infty = \{t \in T : \sup d(t) = \nu\}$. RC-game-12-1

all T_n, $n \in \omega$ are special by the induction hypothesis and T_∞ is special since elements of T_∞ are maximal elements of T by (*) in (3.10). Hence T is special as well.

Case IV: $\xi < \omega_2$ is a limit of cofinality ω_1 and $(3.14)_\nu$ holds for all $\nu < \xi$.

Suppose $T \in [T_{G^\downarrow_\omega(\kappa)}]^{\aleph_1}$ is such that $otp(d(T)) = \xi$.

Let

(3.17) $T_0 = T \setminus \{t \in T : t \text{ is maximal in } T\}$. RC-game-12-1-0

Clearly it is enough to show that T_0 is special.

If $\sup(d(T_0)) < \sup(d(T))$ then by induction hypothesis T_0 is special. Hence we may assume that $\sup(d(T_0)) = \sup(d(T))$.

Let $\nu = \sup(d(T_0))$ and let $\langle \nu_\beta : \beta < \omega_1 \rangle$ be a strictly increasing sequence of ordinals cofinal in ν.

Note that $\nu > \sup(D_{M_t})$ for all $t \in lim(T_0)$ by $(*)$ in (3.10). Hence, for any $t \in \lim(T_0)$, there is $f_t \in {}^{\kappa\downarrow}\kappa \cap M_t$ such that $f_t(\nu) \ge \sup(\kappa \cap M_t)$.

Noting that $\ell(t)$ is a limit ordinal and hence we have $M_t = \bigcup_{\alpha<\ell(t)} M_{t,\alpha}$, let

(3.18) $h(t) = t \restriction (\xi_0 + 1)$ where $\xi_0 = \min\{\alpha < \xi : f_t \in M_{t,\alpha}\}$. RC-game-12-2

Then we have $h : \lim(T_0) \to T_0$ and h is regressive.

Claim 3.3.1 *$h^{-1}{}''\{u\}$ is special for all $u \in T_0$.*

$\vdash$ Suppose $u \in T_0$. Since M_u is countable it is enough to show that $T_f = \{t \in h^{-1}{}''\{u\} : f_t = f\}$ is special for each $f \in {}^{\kappa\downarrow}\kappa \cap M_u$. Since $f(\nu) < \nu$, there is $\beta^* < \omega_1$ such that $f(\nu) \le \nu_{\beta^*}$. For any $t \in T_f$, we have $\sup(d(t)) \le f_t(\nu) = f(\nu) \le \nu_{\beta^*}$. Thus $T_f \subseteq \{t \in T_0 : d(t) \subseteq \nu_{\beta^*}\}$. The subtree of $T_{G^\downarrow_\omega(\kappa)}$ on the right side of the inclusion is special by the induction hypothesis. Hence T_f is also special.

$\dashv$ (Claim 3.3.1)

By Corollary 2.3, it follows that T_0 is special and hence, also in this case, T is special and $(3.14)_\xi$ holds. □ (Lemma 3.3)

T-RC-WSII

Theorem 3.4 *Assume that* RC *holds. Then* $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ *holds.*

Proof. By RC and Lemma 3.3, $T_{G^{\downarrow}_{\omega}(\kappa)}$ is special. Assume, toward a contradiction, that II does not have any winning strategy. By Lemma 3.2, this means that $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$ is stationary.

We shall prove that $T_{G^{\downarrow}_{\omega}(\kappa)}$ is then a Baire tree. By Lemma 2.1, (1), this is a contradiction to the speciality of $T_{G^{\downarrow}_{\omega}(\kappa)}$.

Suppose that D_n, $n \in \omega$ are dense open subsets of $T_{G^{\downarrow}_{\omega}(\kappa)}$ and $t \in T_{G^{\downarrow}_{\omega}(\kappa)}$. We have to show that there is $t' \in T_{G^{\downarrow}_{\omega}(\kappa)}$ such that $t \leq_{T_{G^{\downarrow}_{\omega}(\kappa)}} t'$ and $t' \in \bigcap_{n\in\omega} D_n$. Let $\tilde{\mathcal{M}}$ be the expansion of $\mathcal{M}$ obtained by adding D_0, D_1,... and $\mathcal{S}_{G^{\downarrow}_{\omega}(\kappa)}$ as unary relations. Let $M \prec \tilde{\mathcal{M}}$ be countable such that

(3.19) $t \in M$ and RC-game-12-3

(3.20) $M \in \mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$. RC-game-12-4

Let x_n, $n \in \omega$ be an enumeration of M.

Let $\langle t_n : n \in \omega\rangle$ be an increasing sequence in $T_{G^{\downarrow}_{\omega}(\kappa)}$ such that

(3.21) $t_n \in M$ for all $n \in \omega$; RC-game-12-5

(3.22) $t_0 = t$; RC-game-13

(3.23) $t_{n+1} \in D_n \cap M$ for all $n \in \omega$ (this is possible by (3.19) and the elementarity of M); and RC-game-14

(3.24) x_n is an element of the last component of t_{n+1} for all $n \in \omega$. RC-game-15

Let

(3.25) $t' = \bigcup\{t_n : n \in \omega\}^\frown\langle M\rangle$.

Then $t' \in T_{G^{\downarrow}_{\omega}(\kappa)}$ by (3.24) and (3.20). $t \leq_{T_{G^{\downarrow}_{\omega}(\kappa)}} t'$ by (3.22) and (3.20). $t \in D_n$ for all $n \in \omega$ by (3.23). □ (Theorem 3.4)

4 Other implications

otherimpl

WSIICC

Proposition 4.1 $\mathsf{CC}^{\downarrow}(\kappa)$ *is equivalent to* $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$.

Proof. First, assume that Player II has a winning stratety in $G^{\downarrow}_{\omega}(\kappa)$. For any countable $M \prec \mathcal{M}$ with $\kappa \in M$, there is a winning strategy $\sigma \in M$ for Player II in $G^{\downarrow}_{\omega}(\kappa)$.

Let $\langle f_n, \delta_n : n \in \omega\rangle$ be a match in $G^{\downarrow}_{\omega}(\kappa)$ such that

(4.1) f_n, $\delta_n \in M$ for all $n \in \omega$; game-CC-0

(4.2) Player II has played according to σ in this match; game-CC-1

(4.3) $\{f_n : n \in \omega\} = {}^{\kappa\downarrow}\kappa \cap M$. game-CC-2

Since Player *II* wins in this match, there is an $\alpha^* \in E^\kappa_{\omega_1}$ such that $\sup(\kappa\cap M) < \alpha^*$ and $f(\alpha^*) < \sup(\kappa\cap M)$ for all $f \in {}^{\kappa\downarrow}\kappa \cap M$.

$M^* = sk(M \cup \{\alpha^*\})$ is then as in $\mathsf{CC}^\downarrow(\kappa)$: Suppose that α is an ordinal $< \alpha^*$ in this M^*. Then there is a Skolem function h such that $\alpha = h(\alpha^*, x_1, ..., x_n)$ for some $x_1, ..., x_n \in M$. But then there is some regressive $f \in {}^{\kappa\downarrow}\kappa \cap M$ such that $h(\alpha^*, x_1, ..., x_n) = f(\alpha^*)$. It follows that $\alpha < \sup(\kappa \cap M)$.

For the other implication, assume that $\mathsf{CC}^\downarrow(\kappa)$ holds. In a match f_0, δ_0, f_1, δ_1, ... of $G^\downarrow_\omega(\kappa)$, Player *II* can choose an increasing sequence M_n of countable elementary submodels of $\mathcal{M}$ along with her moves δ_0, δ_1,... such that $\kappa \in M_0$; $f_n \in M_n$, $\delta_n = \sup(\kappa \cap M_n)$ and $\delta_n < \delta_{n+1}$ for all $n \in \omega$. Player *II* always wins in such a match.

❑ (Proposition 4.1)

Proposition 4.2 *Suppose that κ is a regular cardinal $> \aleph_1$. Then $\mathsf{CC}^\downarrow(\kappa)$ implies $\mathsf{FRP}(\kappa)$.*

Proof. Suppose that $E \subseteq E^\kappa_\omega$ is stationary and $g : E \to [\kappa]^{\aleph_0}$ is a ladder system.

Let $\bar{M} \prec \mathcal{M}$ be such that $g \in \bar{M}$ and let

$$\alpha_0 = \kappa \cap \bar{M} \in E. \tag{4.4}$$ CC-FRP-0

Note that we need the regularity of κ to guarantee that there is such $\bar{M}$ as above.

Let $M \prec \bar{M}$ be countable such that $g \in M$, $\sup(\kappa \cap M) = \alpha_0$ and $g(\alpha_0) \subseteq M$.

By $\mathsf{CC}^\downarrow(\kappa)$, there is a countable $M^* \prec \mathcal{M}$ such that $M \prec M^*$ and $cf(\alpha^*) = \omega_1$ for $\alpha^* = \min(\kappa \cap M^* \setminus \sup(\kappa \cap M))$.

The following claim shows that α^* is as in the definition of $\mathsf{FRP}(\kappa)$.

Claim 4.2.1 $S = \{x \in [\alpha^*]^{\aleph_0} : g(\sup(x)) \subseteq x\}$ *is stationary.*

$\vdash$ Note that $S \in M^*$. Thus, it is enough to show that $S \cap C \neq \emptyset$ for all $C \in \mathcal{P}([\alpha^*]^{\aleph_0}) \cap M^*$ which is a club in $[\alpha^*]^{\aleph_0}$. .

Let $h \in M^*$ be such that $h : {}^{\omega>}\alpha^* \to \alpha^*$ and $C_h \subseteq C$. Let $x^* = \alpha^* \cap M^*$.

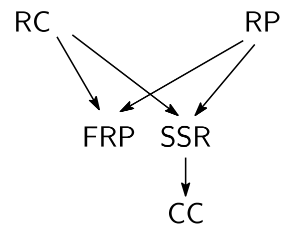
Then $\sup x^* = \alpha_0$ and $x^* \supseteq \alpha^* \cap M = \kappa \cap M \supseteq g(\alpha_0) = g(\sup(x^*))$. Thus $x^* \in S$. Since x^* is closed with respect to h by $h \in M^*$, it follows that $x^* \in S \cap C \neq \emptyset$.

$\dashv$ (Claim 4.2.1)

❑ (Proposition 4.2)

5 The diagram

Doebler [1] proved that Rado's Conjecture implies the semi-stationary reflection priciple (SSR, also known as (†)). Toshimichi Usuba (unpublished) proved that $\mathsf{FRP}$and SSR are independent (modulo a large cardinal). Hiroshi Sakai (unpublished) proved that FRP does not imply CC (modulo a supercompact cardinal).



6 À la recherche de la borne supérieure

Let us denote the reflection principle to internally club subset by $\mathsf{RP}_{\mathsf{IC}}$. Thus $\mathsf{RP}_{\mathsf{IC}}$ is the principle which states:

> For any cardinal κ, stationary $S \subseteq [\kappa]^{\aleph_0}$ and $a \in \mathcal{M}$, there is an internally club $M \prec \mathcal{M}$ of cardinality $\aleph_1$ such that, κ, S, $a \in M$ and $S \cap M$ is stationary in $[\kappa \cap M]^{\aleph_0}$.

Here, $M \prec \mathcal{M}$ is said to be internally club if $[M]^{\aleph_0} \cap M$ contains a club subset of $[M]^{\aleph_0}$. For $M \prec \mathcal{M}$ of cardinality $\aleph_1$, this is equivalent to the existence of a filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha \in \omega_1$.

L-sup-1

Proposition 6.1 $\mathsf{RP}_{\mathsf{IC}}$ *implies* $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ *for all* $\kappa > \aleph_1$.

Proof. Assume that $\mathsf{RP}_{\mathsf{IC}}$ holds. Toward a contradiction, suppose that $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ does not hold for a regular $\kappa > \aleph_1$. By Lemma 3.2, it follows that $\mathcal{B}_{G^{\downarrow}_{\omega}(\kappa)}$ is a stationary subset of $[\mathcal{M}]^{\aleph_0}$.

By $\mathsf{RP}_{\mathsf{IC}}$, there is an $N \prec \mathcal{M}$ such that $N = \bigcup_{\alpha<\omega_1} N_\alpha$ where

(6.1) $\kappa \in N_0$; sup-a

(6.2) N_α, $\alpha < \omega_1$ are countable and $N_\alpha \prec \mathcal{M}$; sup-0

(6.3) $\langle N_\alpha : \alpha < \omega_1 \rangle$ is continuously increasing with respect to $\subseteq$; sup-1

(6.4) $N_\alpha \in N_{\alpha+1}$ for all $\alpha < \omega_1$; and sup-2

(6.5) $S_0 = \{\alpha < \omega_1 : D_{N_\alpha}$ is bounded$\}$ is stationary. sup-3

For $\alpha \in S_0$, let $\xi_\alpha < \kappa$ be the lowest upper bound of D_{N_α}. Note that, by elementarity, we have $\xi_\alpha \in N_{\alpha+1}$. Let $\xi^* = \sup_{\alpha\in S_0} \xi_\alpha$. Since S_0 is unbounded in ω_1 it follows that

(6.6) $\xi^* = \sup(\kappa \cap N)$. sup-4

For each $\alpha \in S_0$, there is $f_\alpha \in {}^{\kappa\downarrow}\kappa \cap N_\alpha$ such that

(6.7) $f_\alpha(\xi^*) \geq \sup(\kappa \cap N_\alpha)$. sup-5

By (generalized) Fodor's lemma, there is a stationary $S_1 \subseteq S_0$ and $f \in {}^{\kappa\downarrow}\kappa$ such that $f_\alpha = f$ for all $\alpha \in S_1$. By (6.6) and (6.7), we have $f(\xi^*) \geq \xi^*$. But this is a contradiction to the regresseness of f. ❑ (Proposition 6.1)

Doebler actually showed in [1] that RC implies $\mathsf{WS}_{II}(G_\omega([\kappa]^{\aleph_1},\omega_1))$ for the following game $G_\omega([\kappa]^{\aleph_1},\omega_1)$ and this implies SSR.

The game $G_\omega([\kappa]^{\aleph_1},\omega_1)$ for Players I and II is defined as follows: A match in $G_\omega([\kappa]^{\aleph_1},\omega_1)$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & f_1 \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & \cdots \\ \hline II & & \delta_0 \in \omega_1 & & \delta_1 \in \omega_1 & \cdots & & \delta_n \in \omega_1 & \cdots \end{array} \quad (n<\omega)$$

II wins in a match of $G_\omega([\kappa]^{\aleph_1},\omega_1)$ as above if

$$\{a \in [\kappa]^{\aleph_1} \,:\, f_n(a) < \sup\{\delta_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

The principle $\mathsf{WS}_{II}(G^{\downarrow}_\omega([\kappa]^{\aleph_1}))$ for the following game $G^{\downarrow}_\omega([\kappa]^{\aleph_1})$ introduced below is still a conclusion of RC as well as of $\mathsf{RP_{IC}}$ while it implies both $\mathsf{WS}_{II}(G^{\downarrow}_\omega(\kappa))$ and $\mathsf{WS}_{II}(G_\omega([\kappa]^{\aleph_1},\omega_1))$ for all cardinal $\kappa > \aleph_1$.

A function $f : [\kappa]^{\aleph_1} \to \kappa$ is said to be regressive if $f(a) \in a$ holds for all $a \in [\kappa]^{\aleph_1}$. Let

(6.8) $\quad {}^{[\kappa]^{\aleph_1}\downarrow}\kappa = \{f \in {}^{[\kappa]^{\aleph_1}}\kappa \,:\, f \text{ is regressive}\}.$ sup-6

For a cardinal $\kappa > \aleph_1$ and an ω_1-club $\mathcal{C} \subseteq [\kappa]^{\aleph_1}$, the game $G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ for Players I and II is defined as follows: A match in $G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots \\ \hline II & & d_0 \in [\kappa]^{\aleph_0} & & d_1 \in [\kappa]^{\aleph_0} & \cdots & & d_n \in [\kappa]^{\aleph_0} & \cdots \end{array} \quad (n<\omega)$$

II wins in a match in $G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ as above if

$$\{a \in \mathcal{C} \,:\, f_n(a) \in \bigcup\{d_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

For $\mathcal{C} = \{a \in [\kappa]^{\aleph_1} \,:\, cf(\sup(a)) = \omega_1\}$, we denote $G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ simply by $G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1})$.

The game $G^{\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ for Players I and II, a weakening of $G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$, is defined as follows: A match in $G^{\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots \\ \hline II & & \delta_0 \in \kappa & & \delta_1 \in \kappa & \cdots & & \delta_n \in \kappa & \cdots \end{array} \quad (n<\omega)$$

II wins in a match in $G^{\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ as above if

$$\{a \in \mathcal{C} \,:\, f_n(a) < \sup\{\delta_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

For $\mathcal{C} = \{a \in [\kappa]^{\aleph_1} \,:\, cf(\sup(a)) = \omega_1\}$, we denote $G^{\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1})$ simply by $G^{\downarrow}_\omega([\kappa]^{\aleph_1})$.

L-sup-2

Lemma 6.2 (1) $\mathsf{WS}_{II}(G^{\downarrow}_\omega([\kappa]^{\aleph_1}))$ *implies* $\mathsf{WS}_{II}(G^{\downarrow}_\omega(\kappa))$.

(2) $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_\omega([\kappa]^{\aleph_1}))$ *implies* $\mathsf{WS}_{II}(G_\omega([\kappa]^{\aleph_1},\omega_1))$.

(3) $\mathsf{WS}_{II}(G^{\downarrow\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1}))$ *implies* $\mathsf{WS}_{II}(G^{\downarrow\mathcal{C}}_\omega([\kappa]^{\aleph_1}))$ *for any* ω_1*-club* $\mathcal{C} \subseteq [\kappa]^{\aleph_1}$.

Proof. (1): Suppose that σ is a winning strategy of Player *II* in $G^{\downarrow}_{\omega}([\kappa]^{\aleph_1})$.

In a match f_0, δ_0, f_1, δ_1,... in $G^{\downarrow}_{\omega}(\kappa)$ where $f_n \in {}^{\kappa\downarrow}\kappa$ and $\delta_n \in \kappa$ for $n \in \omega$, Player *II* thinks that she is playing the match f'_0, δ_0, f'_1, δ_1,... in $G^{\downarrow}_{\omega}([\kappa]^{\aleph_1})$ where she applies σ for the choice of her moves in this match where $f'_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa$, $n \in \omega$ are defined by

(6.9) $\quad f'_n(u) = \min(u \setminus f_n(\sup(u)))$ for all $u \in [\kappa]^{\aleph_1}$. sup-7

Since Player *II* wins in the match f'_0, δ_0, f'_1, δ_1,..., the set

(6.10) $\quad U = \{a \in [\kappa]^{\aleph_1} : cf(a) = \omega_1$ and $f'_n(a) < \sup\{\delta_i : i \in \omega\}$ for all $n \in \omega\}$ is cofinal in $[\kappa]^{\aleph_1}$. sup-8

For any $\alpha < \kappa$ let $a \in U$ be such that $\{\alpha\} \subseteq a$ and let $\alpha^* = \sup(a)$. Then we have $\alpha < \alpha^*$, $cf(\alpha^*) = \omega_1$ and $f_n(\alpha^*) \leq f'_n(a) < \sup\{\delta_i : i \in \omega\}$ for all $n \in \omega$ by (6.9) and (6.10). This shows that

(6.11) $\quad \{\alpha \in E^{\kappa}_{\omega_1} : f_n(\alpha) < \sup\{\delta_i : i < \omega\}$ for all $n \in \omega\}$ is unbounded in κ sup-9

and thus Player *II* wins in all such matches.

(2): Suppose again that σ is a winning strategy of Player *II* in $G^{\downarrow}_{\omega}([\kappa]^{\aleph_1})$.

In a match f_0, δ_0, f_1, δ_1,... in $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ where $f_n \in {}^{[\kappa]^{\aleph_1}}\omega_1$ and $\delta_n \in \omega_1$ for $n \in \omega$, Player *II* thinks that she is playing the match f'_0, d_0, f'_1, d_1,... in $G^{\downarrow}_{\omega}([\kappa]^{\aleph_1})$ and applies σ for the choice of her moves where $f'_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa$, $n \in \omega$ are defined by

(6.12) $$f'_n(u) = \begin{cases} f_n(u), & \text{if } f_n(u) \in u, \\ \min(u), & \text{otherwise.} \end{cases}$$ sup-10

She let then

(6.13) $\quad \delta_n = \sup(\omega_1 \cap d_n)$ for all $n \in \omega$. sup-9-0

Since Player *II* wins in the match f'_0, d_0, f'_1, d_1,...

(6.14) $\quad$ the set $U = \{a \in [\kappa]^{\aleph_1} : cf(a) = \omega_1$ and $f'_n(a) \in \bigcup\{d_i : i \in \omega\}$ for all $n \in \omega\}$ is cofinal in $[\kappa]^{\aleph_1}$. sup-11

For any $b \in [\kappa]^{\aleph_1}$, let $a \in U$ be such that $b \cup \omega_1 \subseteq a$. Then, for all $n \in \omega$, we have

(6.15) $\quad f_n(a) = f'_n(a) \in \bigcup\{d_i : i \in \omega\} \leq \sup(\{\delta_i : i \in \omega\})$

by (6.12), (6.13) and (6.14). This shows that Player *II* wins in all such matches.

(3): Similarly to the proof of (2). ❑ (Lemma 6.2)

The principle asserting that $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ holds for all $\kappa \geq \aleph_2$ is called $\mathtt{G}^{\downarrow\downarrow}$ in [9]. Unfortunately RC does not imply this principle. In [9], it is shown that $\mathtt{G}^{\downarrow\downarrow}$ implies $\mathsf{RP}_{\mathtt{IS}}$. It is known $\mathsf{RP}_{\mathtt{IS}}$ or even RP does not follow from RC: in [12], it is shown that a supercompact cardinal κ can be forced to be a strongly compact cardinal in such a way that, if κ is Leváy-collapsed to be ω_2 by the countably closed p.o. the reflection of stationarity of stationary subset $\subseteq [\aleph_3]^{\aleph_0}$ does not hold. On the other hand, it is known that RC always holds if a strongly compact cardinal is Leváy-collapsed to ω_2 by the countably closed p.o. see math-notes-11.

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