An outline of independence proofs by forcing

(A still quite incomplete) lecture note of a course on forcing
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The Sections 1 and 3 of the following note grew out of a lecture note of a course for graduate students I gave in the Spring semester 2015 at Kobe university. This text was then used in the first lecture of the intensive course on forcing for graduate students at the University of Tokyo in the Autumn semester 2015. Some other sections are still to be added and/or extended parallel to the present course to provide all details of the proofs of the assertions I mentioned in the first two sections without or only with a very sketchy proof. The last section is going to contain some further simple applications forcing.

This note in the version you are reading is still incomplete and needs further brush-up. Any comments or suggestions are appreciated.

Contents

(0) The most uptodate version of this text is downloadable as:
https://fuchino.ddo.jp/notes/forcing-outline-katowice-2017.pdf

1 Metamathematical framework

metamathematics

In this section, we describe how an independence proof with the technique of forcing works. Most of the notions from set theory used in this section without further comments will be explained in the next sections.

We denote with "ZFC" the axiom system of Zermelo-Fraenkel set theory with Axiom of Choice (AC). CH denotes the Continuum Hypothesis which asserts that the cardinality of the set of the reals $\mathbb{R}$ is $\aleph_1$.(1) Under AC, this is equivalent to the assertion that, for every subset A of $\mathbb{R}$, if A is not countable (i.e. if there is no onto mapping from $\mathbb{N}$ to A) then A has the cardinality of $\mathbb{R}$ (i.e. there is a bijection from A to $\mathbb{R}$).

Let T be an arbitrary finite fragment of ZFC. (2)

We show that we can find a sufficiently large finite fragment T^* of ZFC containing all axioms of T such that(3), for any countable transitive(4) model M of T^* (cf. Corollary 2.20), we can construct a countable transitive model $M^* \supseteq M$ of T, by means of the generic extension explained in Section 3, such that

(1.1) $\quad M^* \models \neg\mathsf{CH}.$ meta-0

This implies the non-provability of CH from ZFC: Suppose, toward a contradiction, that there were a proof P_0 with

(1.2) $\quad \mathsf{ZFC} \vdash^{P_0} \mathsf{CH}.$ (5) meta-1

(1) $\aleph_1$ is the smallest uncountable cardinality. The cardinality of the reals is denoted by $2^{\aleph_0}$. Thus the Continuum Hypothesis is the equality $2^{\aleph_0} = \aleph_1$.

(2) A finite fragment of ZFC is a (concretely given) finite set of axioms of ZFC containing finitely many instances of the Axiom of Separation and the Axiom of Replacement and all other axioms. Note that, in any of such a fragment of ZFC, we we have all the axioms needed to develop the first order logic (coded in certain sets) inside the axiom system. When we say that such a fragment is "sufficiently large", we mean such a fragment of ZFC that contains all the instances of the axiom schemes of Separation and Replacement which are needed in the following arguments (where only finitely many instances of Separation and Replacement are used in any case). Sometimes we consider, in place of a finite fragment, the full axiom system of ZFC with Power Set Axiom replaced by a weaker axiom only guaranteeing that only sets with certain cardinalities have their powerset (note that $\mathcal{H}(\chi)$ for a large enough regular χ satisfies such an axiom system).

(3) The main clause of this sentence is a statement in metamathematics while the statement following this "such that" is meant to be formulated in ZFC.

(4) A set a is said to be transitive if $y \in a$ holds for any $x \in a$ and $y \in x$. Thus a is transitive if and only if, for all $x \in a$, we have $x \subseteq a$.

Let T be a finite fragment of ZFC which contains all the axioms which appear in P_0. Then we have $T \vdash^{P_0} \mathsf{CH}$. Let T^* be as above for this T.

From here on, we are working in ZFC: Let M be a countable transitive model of T^* and $M^* \supseteq M$ a countable transitive model of $T + \neg\mathsf{CH}$. In particular, $M^* \models \neg\mathsf{CH}$, or: "$M^* \models \mathsf{CH}$ does not hold". Form this we obtain a contradiction since (1.2) and $M^* \models T$, imply $M^* \models \mathsf{CH}$.

Since this argument does not depend on the choice of T [(6)], we can distill an algorithm from the argument for constructing a proof P_0' with $\mathsf{ZFC} \vdash^{P_0'} x \not\equiv x$ from a given proof P_0 of CH from ZFC.

This means that CH is not provable from ZFC as far as ZFC itself is consistent.

Likewise, we can show that, for each finite fragment T of ZFC, there is another finite fragment T^{**} of ZFC containing T such that, for any countable transitive model M of T^{**}, we can construct, by means of the forcing method, a countable transitive model $M^{**} \supseteq M$ of T such that

(1.3) $\quad M^{**} \models \mathsf{CH}.$ meta-2

With the same argument as above, we can show that if there is a proof of CH from ZFC then we can find a proof of $x \not\equiv x$ from ZFC.

Thus, we can conclude that neither CH nor $\neg\mathsf{CH}$ is provable from ZFC as far as ZFC is consistent. This situation is also formulated as "CH is independent over ZFC".

Using the forcing method we can prove many other indepenence results. The following are examples of such assertions independent from ZFC[(7)].

(1.4) ($non(null) = 2^{\aleph_0}$) Any set $A \subseteq \mathbb{R}$ of cardinality $< 2^{\aleph_0}$ is a null set (i.e. a set of Lebesgue measure 0). meta-3

(1.5) (Suslin's Hypothesis) The following four properties characterize the linear orders $\langle R, <_R \rangle$ of order-type $\langle \mathbb{R}, < \rangle$: meta-4

- (i) $<_R$ does not have minimal nor maximal element;
- (ii) $<_R$ is dense (i.e. for all $x, y \in R$ with $x <_R y$ there is $z \in R$ with $x <_R z <_R y$);

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(5) For an axiom system T and sentence φ in some formal logic $T \vdash \varphi$ denotes the assertion that there is a (formal) proof of φ from T. A formal proof of φ from T is a (concretely given finite) sequence of formulas whose last formula is φ and each component of the sequence is either one of the logical axioms of the formal system; or one of the axioms in T; or drived from some of the previous components of the sequence by one of the deduction rules of the formal system. If P is a such proof of φ from T we write $T \vdash^T \varphi$.

(6) We see later that the algorithm for finding T^* for each given T is also uniform for finite fragments T of ZFC.

(7) The following examples are not necessarily chosen from the point of view of importance but rather because of the easiness of their formulation without introducing any advanced notions.

(iii) any bounded subset of R has its supremum and infimum;

(iv) any pairwise disjoint family of open intervals in R is countable.

(1.6) (A weak form of GCH) For any sets a and b if $|\,a\,| < |\,b\,|$ [8] then we have $|\,\mathcal{P}(a)\,| < |\,\mathcal{P}(b)\,|$.

2 Prerequisites from basic set theory

prerequisites

2.1 Axioms of ZFC, ordinals and cardinals

zfc

Let $\mathcal{L}_\varepsilon$ be the language in the first order logic which consists of the single binary relation symbol ε.

The axiom system of Zermelo-Fraenkel set theory with Axiom of Choice (ZFC) is a collection of the first order sentences in the language $\mathcal{L}_\varepsilon$ which correspond to the following assertions:[9]

Axiom of Extensionality: For any a and b, $a = b$ holds if and only if, for any c, we have ($c \in a$ holds if and only if $c \in b$ holds).

Axiom of Empty Set: There is a which contains no element.

By the Axiom of Extensionality the set a in Axiom of Empty Set is determined uniquely. We denote this a by $\emptyset$ and call it the *emptyset*.

Pairing Axiom: For any a and b there is c whose elements are exactly a and b.

The set c is also unique for given a and b by the Axiom of Extensionality. We denote this c with $\{a, b\}$. If $a = b$ we denote $\{a, b\}$ also with $\{a\}$ (or $\{b\}$) and call it singleton a (or singleton b).

For a, b we define the ordered pair of a and b to be $\{\{a\}, \{a, b\}\}$.[10]

Ordered pairs satisfy the property:

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[8] For a set a we denote with $|\,a\,|$ the cardinality of a.

[9] The intended "interpretation" of ε is the element relation: read $x\ \varepsilon\ y$ as x is an element of y. We formulate the axioms of ZFC with the intention that all mathematical objects we are talking about are sets. Thus we do not need the predicate "set". Although we might say "for a set a ...", "for all set a ..." etc. to make the narration to sound more natural. Also later when we are going to talk about classes which are metamathemaical sort of generalization of the notion of sets, we often say "there exists a set ..." and "for all sets ..." to make clear that we are not talking metamathematically about classes.

When we are using mathematical everyday language to describe what is going on in the formal system of ZFC more intuitively, the binary relation simbol ε in $\mathcal{L}_\varepsilon$ and the symbol for the equality $\equiv$ in the formal language are often replaced with the element relation $\in$ and the equality $=$.

[10] This definition of the ordered pair was invented by Kuratowski in [Kuratowski 1921] much later than the Zerlmelo's [Zermelo 1908] where the axiom system similar to the subsystem Z of our ZFC (see below) was introduced.

(2.1) for any a, b, a', b', $\langle a, b\rangle = \langle a', b'\rangle$ if and only if $a = a'$ and $b = b'$ hold. axioms-0

Axiom of Union: For any a there is b which consists of all elements of elements of a.

b as in the Axiom of Union is also unique for given a. We denote this b by $\bigcup a$. If $a = \{c, d\}$, $\bigcup a$ is also denoted by $c \cup d$. It is $\bigcup\{a\} = a$.

The following axiom is actually a scheme of axioms where we formulate an axiom for each $\mathcal{L}_\varepsilon$ formula $\varphi = \varphi(x, x_1, ..., x_n)$:

Axiom of Separation$_\varphi$: For any a_1,..., a_n and a there is b which consists of all elements c of a satisfying $\varphi(c, a_1, ..., a_n)$.

The set b is unique for the concretely given formula φ and given a_0,...a_n and a. This b is denoted by

(2.2) $\{x \in a : \varphi(x, a_1, ..., a_n)\}$.

An application is the existence of difference set: for a, b we denote with $a \setminus b$ the set

(2.3) $\{x \in a : x \notin b\}$

whose existence is garanteed by the instance of the Axiom of Separation for φ being $\neg(x_0 \; \varepsilon \; x_1)$.

[definition of Cartesian products, functions (domain, range, image, 1-1,onto, etc.), binary relations... should be put here.]

l-axioms-0

Lemma 2.1 *For any formula $\varphi(x, x_1, ..., x_n)$ and for any $a_1, ..., a_n$, suppose that there is a_0 such that $\varphi(a_0, a_1, ..., a_n)$. Then there is c such that, for any d, ($d \in c$ if and only if $d \in a$ holds for all a such that $\varphi(a, a_1, ..., a_n)$).*

Proof. Let a_0 be as above. Then $c = \{d \in a_0 : \forall x\,(\varphi(x, a_1, ..., a_n) \rightarrow d \in x)\}$ is as desired. This set exists by the Separation Axiom. ❑

The set c as in Lemma 2.1 is also denoted by $\bigcap\{x : \varphi(x, a_1, ..., a_n)\}$. If $d = \{x : \varphi(x, a_1, ..., a_n)\}$ for some set c then we also denote this set by $\bigcap d$.

Axiom of Infinity: There is a such that $\emptyset \in a$ and for any $b \in a$ we have $b \cup \{b\} \in a$.

Unlike other existence axioms we introduced sofar, Axiom of Infinity does not specify a single set. However by this axiom and Lemma 2.1 we obtain the minimal set with the property in the axiom:

(2.4) $\bigcap\{x : \emptyset \in x,$ for any $y\,(y \in x \rightarrow y \cup \{y\} \in x)\}$. axioms-1

This set is denoted by ω. Intuitively ω consists of $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$,.... Since the narural numbers 1, 2, 3,... are later defined to be the sets $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$,... respectively, ω is considered to be the set of all natural numbers.

A set a is a subset of b if (for every c, if $c \in a$ then $c \in b$) holds. If a is a subset of b we write $a \subseteq b$.

Power Set Axiom: For any a there is b such that b consists of all subsets of a.

b in the Power Set Axiom is called the power set of a and denoted by $\mathcal{P}(a)$.

The following axiom is also a scheme of axioms for each $\mathcal{L}_\varepsilon$-froamula ψ:

Axiom of Replacement$_\psi$: For any set A and c_0,..., c_{n-1}, if, for each $a \in A$ there is unique b such that$\psi(a, b, A, c_0, ..., c_{n-1})$ holds then there is a set B which consists of all b such that there is an $a \in A$ with $\psi(a, b, A, c_0, ..., c_{n-1})$.

The set B is the Axiom of Replacement is also denoted by

$$(2.5) \qquad \{b\,:\,\psi(a, b, A, c_0, ..., c_{n-1}),\ \text{for some } a \in A\}.$$

By letting $((\varphi(x, y, ...) \land x \equiv y)\ \lor\ (\neg\varphi(x, y, ...) \land y \equiv \emptyset))$ to be the formula ψ in the Axiom of Replacement, the set $\{b\,:\,\psi(a, b, A, c_0, ..., c_{n-1}),\ \text{for some } a \in A\}$ or this set set minus $\{\emptyset\}$ is $\{b \in A\,:\,\varphi(b, c_0, ..., c_{n-1})\}$. Thus the Axiom of Replacement (together with the Axiom of Extensionality, the Axiom of Emptyset and the Axiom of Pairs) implies Axiom of Separation.

One of the reasons why we included the Axiom of Separation in spite of the redundancy is that we can define in this way the subtheories Z and ZC of ZF and ZFC by simply declaring that Z (ZC resp.) be ZF (ZFC resp.) $-$ the Axiom of Replacement.

Z and ZC correspond to the modernized versions of Zermelo's Axiom System and Zermelo's Axiom System with the Axiom of Choice respectively.

The significance of these axiom systems lies in the fact that most of the conventional modern mathematics can be done in the framework of either Z or ZC while ZC is consistencywise strictly weaker than ZFC (which can be proved to be equiconsistent with Z): we shall show later that *consis*$(\ulcorner\ulcorner\mathsf{ZC}\urcorner\urcorner)$ is provable in ZFC.

Axiom of Foundation: For any non-empty set x there is $y \in x$ such that for any $z \in x$ we have $z \notin y$.

y in the Axiom of Foundation is called an $\in$-minimal element of x.

Axiom of Choice: For any set x with $\emptyset \notin x$. There is a mapping $f : x \to \bigcup x$ such that $f(y) \in y$ holds for all $y \in x$.

[the rest will be written later]

2.2 Induction and recursive definition on well-founded sets and classes ind-rec

The most general form of the principle of induction and recursive definition we rely on is formulated in the following Theorem 2.5.

We begin with some definitions. Note that the following is a generalization of the corresponding assertions on sets since a set a can be always seen as the class $\{x : x \in a\}$.

For a class $\mathbf{X}$ and a binary class relation $\mathbf{R}$ on $\mathbf{X}$, the class relation $\mathbf{R}$ is said to be *set-like* if, for all $x \in \mathbf{X}$, the class

$$I_{\mathbf{R}}(x) = \{y \in \mathbf{X} : y \mathrel{\mathbf{R}} x\} \tag{2.6}$$ ind-0

is a set.

$\mathbf{R}$ is *well-founded* if, for all non-empty set $a \subseteq \mathbf{X}$, there is an $\mathbf{R}$ minimal element of a (i.e. such $x \in a$ that $y \not\mathrel{\mathbf{R}} x$ for all $y \in a$). Note that in general a minimal element of a is not unique.

For a set-like $\mathbf{R}$ on $\mathbf{X}$ and $x \in \mathbf{X}$, we define

$$I^0_{\mathbf{R}}(x) = \{x\}; \tag{2.7}$$ ind-1

$$I^{n+1}_{\mathbf{R}}(x) = \bigcup\{I_{\mathbf{R}}(y) : y \in I^n_{\mathbf{R}}(x)\}. \tag{2.8}$$ ind-2

Note that we have in particular $I^1_{\mathbf{R}}(x) = I_{\mathbf{R}}(x)$.

The *transitive closure* and the *weak transitive closure* of x with respect to $\mathbf{R}$ is defined by

$$\mathit{trcl}_{\mathbf{R}}(x) = \textstyle\bigcup_{n\in\omega} I^n_{\mathbf{R}}(x) \text{ and} \tag{2.9}$$ ind-3

$$\mathit{trcl}^-_{\mathbf{R}}(x) = \textstyle\bigcup_{n\in\omega\setminus 1} I^n_{\mathbf{R}}(x). \tag{2.10}$$ ind-4

If $\mathbf{R}$ is well-founded, we have $\mathit{trcl}^-_{\mathbf{R}}(x) = \mathit{trcl}_{\mathbf{R}}(x) \setminus \{x\}$.

If $\mathbf{R}$ is set-like then $\mathit{trcl}_{\mathbf{R}}(x)$ and $\mathit{trcl}^-_{\mathbf{R}}(x)$ are sets.

Note that the class binary relation $\in = \{\langle x, y\rangle : x \in y\}$ on V is set-like and well-founded (under the Axiom of Foundation).

For the set-like well-founded class relation $\in$, we often abbreviate $\mathit{trcl}_{\in}(x)$ and $\mathit{trcl}^-_{\in}(x)$ as $\mathit{trcl}(x)$ and $\mathit{trcl}^-(x)$ respectively.

For a class relation $\mathbf{R}$ on a class $\mathbf{X}$, $\mathbf{A} \subseteq \mathbf{X}$ is said to be transitive with respect to $\mathbf{R}$ (or $\mathbf{R}$-transitive for short) if, for any $x \in \mathbf{A}$ and $y \in \mathbf{X}$ with $y \mathrel{\mathbf{R}} x$, we alway have $y \in \mathbf{A}$. $\mathbf{A}$ is transitive with respect to $\mathbf{R}$ if and only if $\mathit{trcl}_{\mathbf{R}}(x) \subseteq \mathbf{A}$ for all $x \in \mathbf{A}$.

The next lemma follows immediately from the definition of $\mathit{trcl}_{\mathbf{R}}(\cdot)$:

L-ind-a

Lemma 2.2 *Suppose that $\mathbf{R}$ is a set-like binary relation on a class $\mathbf{X}$.*

(1) *For any $x \in \mathbf{X}$, $\mathit{trcl}_{\mathbf{R}}(x)$ is a set transitive with respect to $\mathbf{R}$.*

(2) *For $x \in \mathbf{X}$, $\mathit{trcl}_{\mathbf{R}}(x)$ is the $\subseteq$-minimal $\mathbf{R}$-transitive set y with $x \in y$; If $\mathbf{R}$ is well-founded then $\mathit{trcl}^-_{\mathbf{R}}(x)$ is the $\subseteq$-minimal $\mathbf{R}$-transitive set y with $I_{\mathbf{R}}(x) \subseteq y$.*

(3) *For all* x, $y \in \mathbf{X}$, $y \in trcl_{\mathbf{R}}(x) \;\Leftrightarrow\;$ *there is a sequence* s *of elements of* $\mathbf{X}$ *of finite length* $n \in \omega \setminus 1$ *such that* $s(k+1)\ \mathbf{R}\ s(k)$ *for all* $k \in \omega$ *with* $k+1 < n$, $s(0) = x$ *and* $s(n-1) = y$.

Proof. (1): That $trcl_{\mathbf{R}}(x)$ is a set follows from the set-likeness of $\mathbf{R}$ and the Axiom of Replacement (By induction we can prove that all $I^n_{\mathbf{R}}(x)$, $n \in \omega$ are sets. Then the Axiom of Replacement and the Axiom of Union imply that $\bigcup_{n\in\omega} I^n_{\mathbf{R}}(x)$ is a set).

Suppose that $y \in trcl_{\mathbf{R}}(x)$ and $z\ \mathbf{R}\ y$. Let $n \in \omega$ be such that $y \in I^n_{\mathbf{R}}(x)$ then $z \in I^{n+1}_{\mathbf{R}}(x)$ and hence $z \in trcl_{\mathbf{R}}(x)$.

(2): Suppose that $y \subseteq \mathbf{X}$ is $\mathbf{R}$-transitive and $x \in y$. Then by induction we can show that $I^n_{\mathbf{R}}(x) \subseteq y$ for all $n \in \omega$. Thus $trcl_{\mathbf{R}}(x) = \bigcup_{n\in\omega} I^n_{\mathbf{R}}(x) \subseteq y$.

The transitivity of $trcl^-(x)$ is shown similarly to the proof of (1). If $\mathbf{R}$ is also well-founded then we can prove that $x \not\mathbf{R}\ x$ for all x (otherwise $\{x\}$ would be a counter-example to the well-foundedness of $\mathbf{R}$) and, for any $z \in \mathbf{X}$ with $zI^n_{\mathbf{R}}(x)$ we have $z \neq x$ (otherwise we can contruct a counter-example to the well-foundedness of $\mathbf{R}$ using the characterization of $trcl_{\mathbf{R}}(x)$ in (3) below). From this fact it follows that $trcl^-_{\mathbf{R}}(x) = \bigcup_{n\in\omega\setminus 1} I^n_{\mathbf{R}}(x)$.

Now suppose that $y \subseteq \mathbf{X}$ is $\mathbf{R}$-transitive and $I_{\mathbf{R}}(x) \subseteq y$. Then by induction we can show that $I^n_{\mathbf{R}}(x) \subseteq y$ for all $n \in \omega \setminus 1$. Thus $trcl^-_{\mathbf{R}}(x) = \bigcup_{n\in\omega\setminus 1} I^n_{\mathbf{R}}(x) \subseteq y$.

(3): Let

(2.11) $\quad C = \{y \in \mathbf{X} :$ there is a sequence s of elements of $\mathbf{X}$ of finite length $n \in \omega \setminus 1$ such that $s(k+1)\ \mathbf{R}\ s(k)$ for all $k \in \omega$ with $k+1 < n$, $s(0) = x$ and $s(n-1) = y\}$.

We have to show that $C = trcl_{\mathbf{R}}(x)$. $x \in C$: the sequence $\{\langle \emptyset, x\rangle\}$ witnesses this. C is transitive: if $y \in C$ and $z\ \mathbf{R}\ y$, then letting s a sequence witnessing the fact that $y \in C$, $\tilde{s} = s \cup \{\langle \ell(s), z\rangle\}$ witnesses $z \in C$. By (2) it follows that $trcl_{\mathbf{R}}(x) \subseteq C$.

On the other hand, we can prove by induction that

(2.12) $\quad$ If $y \in C$ and a sequence s of length n witnesses this, then $y \in I^n_{\mathbf{R}}(x)$

holds for all $n \in \omega \setminus 1$. This shows $trcl_{\mathbf{R}}(s) \supseteq C$. ❑ (Lemma 2.2)

L-ind-0

Lemma 2.3 *Suppose that* $\mathbf{R}$ *is set-like well-founded class relation on the class* $\mathbf{X}$*. Then any non-empty class* $\mathbf{A} \subseteq \mathbf{X}$ *has the* $\mathbf{R}$*-minimal element.*

Proof. Let $a \in \mathbf{A}$ be arbitrary. Then $trcl_{\mathbf{R}}(a) \cap \mathbf{A}$ is a non-empty subset of $\mathbf{A}$. An $\mathbf{R}$-minimal element a_0 of this set is an $\mathcal{R}$-minimal element of $\mathcal{A}$. ❑ (Lemma 2.3)

For a set-like class binary relation $\mathbf{R}$ on a class $\mathbf{X}$, let $\mathbf{R}^{trcl} \subseteq \mathbf{X}^2$ be defined by

(2.13) $\quad x\ \mathbf{R}^{trcl}\ y \;\Leftrightarrow\; x \in trcl^-_{\mathbf{R}}(y)$. ind-4-0

Note that this relation is eqivalent to

(2.14) there is a sequence $\langle x_k : k \leq n\rangle$ for some $n \in \omega \setminus 2$ such that $x_0 = y$, $x_n = x$ and $x_{k+1}\ \mathbf{R}\ x_k$ for all $k < n$

by Lemma 2.2, (3).

Note also that, for all $x \in \mathbf{X}$,

$$trcl^-_{\mathbf{R}^{trcl}}(x) = \{y \in \mathbf{X} : y\ \mathbf{R}^{trcl}\ x\} = trcl^-_{\mathbf{R}}(x) \tag{2.15}$$ ind-6-a-0

by the following Lemma 2.4, (2) and by (2.13).

L-ind-1

Lemma 2.4 *Suppose that* $\mathbf{R}$ *is a set-like class binary relation on a class* $\mathbf{X}$*. Then*

(1) $\mathbf{R}^{trcl}$ *is set-like.*

(2) $\mathbf{R}^{trcl}$ *is a transitive relation.*

(3) *If* $\mathbf{R}$ *is well-founded then* $\mathbf{R}^{trcl}$ *is also well-founded.*

Proof. (1): By (2.15). The right side of the equation is a set since $\mathbf{R}$ is set-like.

(2): Suppose that a, b, $c \in \mathbf{X}$, $a\ \mathbf{R}^{trcl}\ b$ and $b\ \mathbf{R}^{trcl}\ c$, that is, $c \in trcl^-_{\mathbf{R}}(b)$ and $b \in trcl^-_{\mathbf{R}}(c)$. By Lemma 2.2, (2), it follows $a \in trcl^-_{\mathbf{R}}(b) \subseteq trcl^-_{\mathbf{R}}(c)$.

(3): Suppose, toward a contradiction, that $A \subseteq \mathbf{X}$ would not have any minimal element with respect to $\mathbf{R}^{trcl}$. Let

$$A^{trcl} = \{x \in \mathbf{X} : \text{there is a sequence } \langle x_k : k \leq n\rangle \text{ in } \mathbf{X} \text{ for some } n \in \omega \text{ such that } x_0, x_n \in A,\ x_{k+1}\ \mathbf{R}\ x_k \text{ for all } k < n \text{ and } x = x_i \text{ for some } i \leq n\} \tag{2.16}$$ ind-6-0

Then $A^{trcl} \subseteq \mathbf{X}$ does not have any minimal element with respect to $\mathbf{R}$. This is a contradiction to the assumption that $\mathbf{R}$ is well-founded. ❑ (Lemma 2.4)

The following theorem is proved in ZF$^-$ (ZFC without AC and Axiom of Foundation):

Theorem 2.5 (Induction and Recursion Theorem) *Suppose that* $\mathbf{R}$ *is a set-like and well-founded class binary relation on a class* $\mathbf{X}$*.* ind-recurs-thm

(1) *For any class* $\mathbf{A} \subseteq \mathbf{X}$*, if the property*

$$\text{for any } x \in \mathbf{X}, \text{ if } trcl^-_{\mathbf{R}}(x) \subseteq \mathbf{A} \text{ then } x \in \mathbf{A} \tag{2.17}$$ ind-5

holds then $\mathbf{A} = \mathbf{X}$*.*

(2) *Let*

$$\mathfrak{F} = \{f : f \text{ is a mapping on a set } D \subseteq \mathbf{X} \text{ closed with respect to } \mathbf{R}\}. \tag{2.18}$$ ind-6

is a mapping on a set of the form $trcl^-_{\mathbf{R}}(x)$?

If $\mathbf{G}$ *is a class function on* $\mathfrak{F} \times \mathbf{X}$ *then there is a class function* $\mathbf{F}$ *on* $\mathbf{X}$ *such that*

$$\mathbf{F}(x) = \mathbf{G}(\mathbf{F} \restriction trcl^-_{\mathbf{R}}(x), x) \text{ holds for all } x \in \mathbf{X} \tag{2.19}$$ ind-6-a

and $\mathbf{F}$ *is the unique class function with this property.*[(11)]

Proof. Let $\mathbf{R}^{trcl}$ be the relation defined by (2.13). By Lemma 2.4 $\mathbf{R}^{trcl}$ is a well-founded set-like class relation.

(1): Suppose, toward a contradiction, that $\mathbf{A} \subsetneqq \mathbf{X}$ but $\mathbf{A} \models$ (2.17). Let $x^* \in \mathbf{X} \setminus \mathbf{A}$ be minimal with respect to $\mathbf{R}^{trcl}$. By definition of $\mathbf{R}^{trcl}$ we have that $trcl^-_{\mathbf{R}}(x^*) \subseteq \mathbf{A}$. By $\mathbf{A} \models$ (2.17) it follows that $x^* \in \mathbf{A}$. A contradiction to the choice of x^*.

(2): We first show the uniqueness of $\mathbf{F}$. Suppose that $\mathbf{F}$ and $\mathbf{F}'$ are class functions on $\mathbf{X}$ such that

$$\mathbf{F}(x) = \mathbf{G}(\mathbf{F} \restriction trcl^-_{\mathbf{R}}(x), x) \text{ for all } x \in \mathbf{X}, \text{ and} \tag{2.20}$$ ind-6-1

$$\mathbf{F}'(x) = \mathbf{G}(\mathbf{F}' \restriction trcl^-_{\mathbf{R}}(x), x) \text{ for all } x \in \mathbf{X}. \tag{2.21}$$ ind-6-2

Suppose $\mathbf{F} \neq \mathbf{F}'$. Let $x^* \in \mathbf{X}$ be $\mathbf{R}^{trcl}$-minimal with $\mathbf{F}(x^*) \neq \mathbf{F}'(x^*)$. Then we have $\mathbf{F} \restriction trcl^-_{\mathbf{R}}(x^*) = \mathbf{F}' \restriction trcl^-_{\mathbf{R}}(x^*)$. Thus by (2.20) and (2.21)

$$\mathbf{F}(x^*) = \mathbf{G}(\mathbf{F} \restriction trcl^-_{\mathbf{R}}(x^*), x^*) = \mathbf{G}(\mathbf{F}' \restriction trcl^-_{\mathbf{R}}(x^*), x^*) = \mathbf{F}'(x^*). \tag{2.22}$$

This is a contradiction to the choice of x^*.

For the proof of the existence of $\mathbf{F}$ with (2.19), let

$$\begin{aligned} \mathfrak{F}' = \{f \in \mathfrak{F} : & \operatorname{dom}(f) \text{ is transitive subset of } \mathbf{X} \text{ with respect to } \mathbf{R} \text{ and} \\ & (2.23\text{a})\text{: } \mathbf{G}(f \restriction trcl^-_{\mathbf{R}}(x), x) = f(x) \text{ holds for all } x \in \operatorname{dom}(f)\} \end{aligned} \tag{2.23}$$ ind-6-3

C-ind-0

Claim 2.5.1 $\mathfrak{F}' \neq \emptyset$.

$\vdash$ $\emptyset \in \mathfrak{F}'$ $\dashv$ (Claim 2.5.1)

C-ind-1

Claim 2.5.2 *For any* $f \in \mathfrak{F}'$ *and* $x \in \operatorname{dom}(f)$*, we have* $f \restriction trcl_{\mathbf{R}}(x) \in \mathfrak{F}'$.

$\vdash$ By definition of $\mathfrak{F}'$. $\dashv$ (Claim 2.5.2)

C-ind-2

Claim 2.5.3 *Any* f, $g \in \mathfrak{F}'$ *are compatible (as functions).*

$\vdash$ Toward a contradiction, suppose that there are f, $g \in \mathfrak{F}'$ which are not compatible to each other. Let $x^* \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$ be such that $f(x^*) \neq g(x^*)$. We may assume that x^* is minimal with respect to $\mathbf{R}^{trcl}$ with this property. Then we have $f \restriction trcl^-_{\mathbf{R}}(x^*) = g \restriction trcl^-_{\mathbf{R}}(x^*)$. Note that $trcl^-_{\mathbf{R}}(x^*) \subseteq \operatorname{dom}(f)$, $\operatorname{dom}(g)$ by Lemma 2.2, (2).

Thus, by f, $g \in \mathfrak{F}'$, it follows that $f(x^*) = g(x^*)$. This is a contradiction to the choice of x^*. $\dashv$ (Claim 2.5.3)

C-ind-3

Claim 2.5.4 *For any set* $F \subseteq \mathfrak{F}'$ *we have* $\bigcup F \in \mathfrak{F}'$.

---

(11) The definitons of $\mathbf{X}$, $\mathbf{R}$, $\mathbf{G}$ may also contain some parameters so that the class function $\mathbf{F}$ contains these parameters also as variables.

$\vdash$ $\bigcup F$ is a function by Claim 2.5.3. $\mathrm{dom}(\bigcup F)$ is transitive with respect to $\mathbf{R}$ as union of sets transitive with respect to $\mathbf{R}$. Since $f \models$ (2.23a) for all $f \in F$, it follows that $\bigcup F \models$ (2.23a). $\dashv$ (Claim 2.5.4)

By Claim 2.5.3, $\mathbf{F} = \bigcup \mathfrak{F}'$ is a class function. By the definition (2.5.1) of $\mathfrak{F}'$, we have $\mathbf{F} \models$ (2.19). $\mathrm{dom}(\mathbf{F}) = \mathbf{X}$: if this is not the case, let $x^* \in \mathbf{X} \setminus \mathrm{dom}(\mathbf{F})$ be minimal with respect to $\mathbf{R}^{trcl}$. By the minimality of x^*, we have $trcl^-_{\mathbf{R}}(x^*) \subseteq \mathrm{dom}(\mathbf{F})$. It follows that $f = \mathbf{F} \restriction trcl^-_{\mathbf{R}}(x^*) \in \mathfrak{F}'$. Let $f' = f \cup \{\langle x^*, \mathbf{G}(f, x^*)\rangle\}$. Then $f' \in \mathfrak{F}'$. Thus $x^* \in \mathrm{dom}(f') \subseteq \mathrm{dom}(\mathbf{F})$. This is a contradiction to the choice of x^*. ❏ (Theorem 2.5)

We shall call the arguments based on (1) or (2) of the theorem above as $\mathbf{R}$-induction or $\mathbf{R}$-recursion (on $\mathbf{X}$) respectively.

A special form of Theorem 2.5, (2) is the $\mathbf{R}$-recursive definition of a (class) relation on $\mathbf{X}$: to define a (class) relation $\mathfrak{R} \subseteq \mathbf{X}^n$ for some given $n \in \mathbb{N}$, $n \geq 1$, we start from a class function $\mathbf{G} : \mathfrak{F} \times \mathbf{X}^n \to 2$ and take $\mathbf{F}$ as in Theorem 2.5, (2).[(12)] The desired class relation $\mathfrak{R}$ is then defined as $\mathfrak{R} = \{\langle a_0, ..., a_{n-1}\rangle : \mathbf{F}(a_0, ..., a_{n-1}) = 1\}$. [(13)] The recursive definition (3.9) of $\mathbb{P}$-names in Section 3 is one of such examples of the application of Theorem 2.5, (2).

Since the class binary relation $\in$ ($= \{\langle a, b\rangle : a \in b\}$) on any class $\mathbf{X}$ in particular on the class On of ordinals is set-like and well-founded, we obtain the following Induction and Recursion Theorem for ordinals from Theorem 2.5:

Theorem 2.6 (Induction and Recursion Theorem for Ordinals) ind-recurs-ord-thm

(1) *If* $\mathbf{A}$ *is a class of ordinals such that*

(2.24) *for any* $\alpha \in \mathsf{On}$, *if* $\alpha \subseteq \mathbf{A}$ *then* $\alpha \in \mathbf{A}$ ind-7

holds then $\mathbf{A} = \mathbf{X}$.

(2) *Let*

(2.25) $\mathfrak{F} = \{f : f$ *is a mapping on an ordinal*$\}$.

If $\mathbf{G}$ *is a class function on* $\mathfrak{F}$ *then there is a class function* $\mathbf{F}$ *on* On *such that* $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \restriction \alpha)$ *holds for all* $\alpha \in \mathsf{On}$ [(14)] *and* $\mathbf{F}$ *is the unique class function with this property.*[(15)]

❏

(12) Note that $2 = \{0, 1\}$.

(13) Of course, similarly to other applications of Theorem 2.5, (2), the real chronological order of what we would do is that we chose first the class function $\mathbf{G}$ so that the class relation $\mathfrak{R}$ difined with the $\mathbf{G}$ should be the relation as desired.

(14) Note that α can be reconstructed from $\mathbf{F} \restriction \alpha$.

(15) Also here, the definiton of $\mathbf{G}$ may also contain some parameters.

The arguments by (1) or (2) of Theorem 2.6 are called induction on $\alpha \in \mathsf{On}$ or recursion on $\alpha \in \mathsf{On}$ respectively.

Likewise, the induction or recursion arguments for $\mathbf{X} = \gamma$ and $\mathbf{R} = (\in \cap \gamma^2)$ for some ordinal γ are called induction on $\alpha \in \gamma$ or resursion on $\alpha \in \gamma$ respectively.

By recursion on $\alpha \in \mathsf{On}$, we can define von Neumann's cumulative hierarchy of sets: Let

$$(2.26)\quad V_\alpha = \begin{cases} \emptyset, & \text{if } \alpha = 0; \qquad \ldots \text{ (a)} \\ \mathcal{P}(V_\beta), & \text{if } \alpha = \beta + 1 \text{ for some } \beta \in \mathsf{On}; \qquad \ldots \text{ (b)} \\ \bigcup_{\beta < \alpha} V_\beta, & \text{if } \alpha \text{ is a limit ordinal.} \qquad \ldots \text{ (c)} \end{cases}$$ ind-8

The actual construction of the class function $\mathbf{V}(\cdot) : \mathsf{On} \to \mathsf{V};\ \alpha \mapsto V_\alpha$ follows from the application of Theorem 2.6, (2) on the class funcion $\mathbf{G}$ on $\mathcal{F}$ as in Theorem 2.6, (2) defined by

$$(2.27)\quad \mathbf{G}(f) = \begin{cases} \emptyset, & \text{if } f = \emptyset; \\ \mathcal{P}(f(\beta)), & \text{if } \mathrm{dom}(f) = \beta + 1 \text{ for some ordinal } \beta; \\ \bigcup f''\gamma, & \text{if } \mathrm{dom}(f) \text{ is a limit ordinal } \gamma. \end{cases}$$ ind-8-0

Note that the definition of $\mathbf{G}$ does not contain any recursion.

Note that the cumulative hierarchy can be introduced in ZF without the Axiom of Foundation.

Valpha

Lemma 2.7 (1) *$V_\alpha \subseteq V_\beta$ for all α, $\beta \in \mathsf{On}$ with $\alpha \leq \beta$.*

(2) *V_α is transitive for all $\alpha \in \mathsf{On}$.*

(3) *For all $\alpha \in \mathsf{On}$, we have $\mathsf{On} \cap V_\alpha = \alpha$. In particular $\alpha \notin V_\alpha$.*

(4) *V_α is closed under trcl$^-$, union of pairs and subset*[16] *for all $\alpha \in \mathsf{On}$.*

(5) *For $\alpha \in \mathsf{On}$, $a \in V_\alpha$ and $b \in a$, we have trcl$(b) \in V_\alpha$.*

(6) (in ZFC (with Axiom of Foundation)) $\mathsf{V} = \bigcup_{\alpha \in \mathsf{On}} V_\alpha$. *That is, for any set x there is $\alpha \in \mathsf{On}$ such that $x \in V_\alpha$.*

Proof. (1): We prove by induction on $\alpha \in \mathsf{On}$ that

$$(2.28)_\alpha \quad V_\beta \subseteq V_{\beta'} \text{ for all } \beta < \beta' \leq \alpha$$ ind-8-1

holds. For $\alpha = 0$ or 1 this is trivial. If α is a limit ordinal and $(2.28)_{\alpha^-}$ holds for all $\alpha^- < \alpha$ then $(2.28)_\alpha$ also holds by (2.26), (c).

Assume that $\alpha \geq 2$, $\alpha = \alpha^- + 1$ and $(2.28)_{\alpha^-}$ holds and suppose that $a \in V_{\alpha^-}$. We have to show that $a \in V_\alpha$.

If α^- is a successor ordinal, say $\alpha^- = \alpha^{--} + 1$. Then $a \subseteq V_{\alpha^{--}} \subseteq V_{\alpha^-}$ by (2.26), (b) and induction hypothesis. Thus $a \in V_{\alpha^- + 1} = V_\alpha$ by (2.26), (b).

[16] I.e., for any a, $b \in V_\alpha$, trcl$^-(a)$, $a \cup b \in V_\alpha$ and $c \in V_\alpha$ for any c with $c \subseteq a$.

If α^- is a limit $a \in V_\beta$ for some $\beta < \alpha$. We may assume that β is a successor ordinal and $\beta = \beta^- + 1$. Then $a \subseteq V_{\beta^-} \subseteq V_{\alpha^-}$ by induction hypothesis. Hence $a \in V_{\alpha^-+1} = V_\alpha$.

(2): We prove the statement by induction on $\alpha \in \mathsf{On}$. For $\alpha = 0$, $V_\alpha = \emptyset$ is apparently transitive. Suppose that $\alpha > 0$ and V_β is transitive for all $\beta < \alpha$. If α is a limit then $V_\alpha = \bigcup_{\beta<\alpha} V_\alpha$ by (2.26), (c) and it is transitive.

Suppose that α is a successor ordinal and $\beta^* = \sup \alpha$.(17) For $a \in V_\alpha$ and $b \in a$, $a \subseteq V_{\beta^*}$ by (2.26), (b) and $b \in V_{\beta^*}$. Since we have $V_{\beta^*} \subseteq V_\alpha$ by (1), it follow that $b \in V_\alpha$. This shows that V_α is transitive.

(3): We show the statement again by induction on $\alpha \in \mathsf{On}$. For $\alpha = 0$ the statement is trivial by (2.26), (a).

Suppose that $\alpha > 0$ and $\mathsf{On} \cap V_\beta = \beta$ holds for all $\beta < \alpha$. If α is limit then $\mathsf{On} \cap V_\alpha = \bigcup_{\beta<\alpha}(\mathsf{On} \cap V_\beta) = \alpha$ by (2.26), (c).

Suppose that α is a successor ordinal and $\alpha_0 = \sup \alpha$. We have $\mathsf{On} \cap V_{\alpha_0} = \alpha_0$ by induction hypothesis. By (2.26), (b), $\xi \in \mathsf{On} \cap V_\alpha$ if and only if $\xi \in \mathsf{On}$ and $\xi \subseteq \alpha_0$ if and only if $\xi \in \alpha_0$ or $\xi = \alpha_0$. Thus $\mathsf{On} \cap V_\alpha = \alpha_0 \cup \{\alpha_0\} = \alpha_0 + 1 = \alpha$.

(4): By induction on $\alpha \in \mathsf{On}$, we prove that

$(2.29)_\alpha$ V_α is closed under trcl^-, union of pairs and subset ind-8-2

for all $\alpha \in \mathsf{On}$.

Susppose that $(2.29)_\beta$ holds for all $\beta < \alpha$. If α is a limit then $(2.29)_\alpha$ clearly holds by (2.26), (c).

Suppose that α is a successor ordinal and let $\alpha_0 = \sup \alpha$. Let a, $b \in V_\alpha$ and $c \subseteq a$. Then we have a, $b \subseteq V_{\alpha_0}$ by (2.26), (b). By (2) above, $\mathit{trcl}^-(a) \subseteq V_{\alpha_0}$. We also have $a \cup b \subseteq V_{\alpha_0}$ and $c \subseteq V_{\alpha_0}$. Thus $\mathit{trcl}^-(a)$, $a \cup b$, $c \in V_{\alpha_0+1} = V_\alpha$.

(5): Suppose that $a \in V_\alpha$ and $b \in a$. By (4) above we have $b \in a \subseteq \mathit{trcl}^-(a) \in V_\alpha$. It follows that $\mathit{trcl}(b) \subseteq \mathit{trcl}^-(a) \in V_\alpha$. Since V_α is closed under subset by (4), it follows that $\mathit{trcl}(b) \in V_\alpha$.

(6): Note that, by the Axiom of Foundation, we may apply Theorem 2.5, (1) to $\mathbf{X} = \mathsf{V}$ and $\mathbf{R} = \in$. Thus, we have to show that, if $y \in V_\alpha$ for some $\alpha \in \mathsf{On}$ for all $y \in x$ then $x \in V_\alpha$ for some $\alpha \in \mathsf{On}$. For each $y \in x$, let $\alpha_y = \min\{\alpha \in \mathsf{On} : y \in V_\alpha\}$ and $\alpha^* = \sup\{\alpha_y : y \in x\}$. Then, by (1), we have $x \subseteq V_{\alpha^*}$. By (2.26), it follows that $x \in V_{\alpha^*+1}$. ❑ (Lemma 2.7)

For a set a, let

$$(2.30) \quad \mathit{rank}(a) = \min\{\alpha \in \mathsf{On} : a \in V_{\alpha+1}\}.$$ ind-9

(17) Note that $\alpha = \alpha_0 + 1$ holds.

By Lemma 2.7, (7), $rank(a)$ is well-defined[(18)] for all sets a in ZFC (with the Axiom of Foundation).

Lemma 2.8 (ZFC without Axiom of Foundation) (1) *For any sets a and b, if $rank(a)$ is well-defined and $b \in a$ then $rank(b)$ is also well-defined and $rank(b) < rank(a)$.* L-rank

(2) *For any set a, if $rank(a)$ is well-defined, we have $rank(a) = \sup\{rank(b)+1 : b \in a\}$.*

Proof. (1): Since $a \in V_{rank(a)+1}$, we have $a \subseteq V_{rank(a)}$ by (2.26). It follows that $b \in V_{rank(a)}$. Thus $rank(b)$ is well-defined and $rank(b) < rank(a)$.

(2): Let $\alpha = \sup\{rank(b)+1 : b \in a\}$. Note that $rank(b)$ is well-defined for all $b \in a$ by (1). For all $b \in a$, we have $b \in V_{rank(b)+1} \subseteq V_\alpha$ by Lemma 2.7, (1). It follows that $a \subseteq V_\alpha$ and $a \in V_{\alpha+1}$ thus $rank(a) \leq \alpha$. On the other hand, if $\beta < \alpha$ then there is $b \in a$ such that $rank(b) \geq \beta$ Thus $a \notin V_{\beta+1}$ (If $a \in V_{\beta+1}$ we would have $a \subseteq V_\beta$ and hence $b \in V_\beta$. This is a contradiction to $rank(b) \geq \beta$). This shows that $rank(a) \geq \alpha$. Thus $rank(a) = \alpha = \sup\{rank(b)+1 : b \in a\}$. ❑ (Lemma 2.8)

By Lemma 2.7, (3), we have $rank(\alpha) = \alpha$ for all $\alpha \in \mathsf{On}$.

Over ZF^- the equality $\mathsf{V} = \bigcup_{\alpha \in \mathsf{On}} V_\alpha$ is actually equivalent to the Axiom of Foundation: we have already seen that this equation follows from the Axiom of Foundation. So, working in ZF^-, assume that the equality holds. Then $rank(a)$ is well-defined for all sets a. Suppose hat a is an arbitrary non-empty set. Let b be an element of a with minimal $rank(b)$. Then b is an $\in$-minimal element of a by Lemma 2.8, (1).

For an infinite cardinal κ, let

(2.31) $\mathcal{H}(\kappa) = \{a : |\,trcl(a)\,| < \kappa\}$. ind-10

Elements of $\mathcal{H}(\kappa)$ are said to be *hereditarily of cardinality* $< \kappa$.

Elements of $\mathcal{H}(\aleph_0)$ are said to be *hereditarily finite* and elements of $\mathcal{H}(\aleph_1)$ *hereditarily countable.*

L-hered-0

Lemma 2.9 *Let κ be an infinite cardinal.* (0) *$\mathcal{H}(\kappa)$ is transitive.*

(1) *If $\kappa < \kappa'$ then $\mathcal{H}(\kappa) \subseteq \mathcal{H}(\kappa')$. If κ is a limit cardinal then we have $\mathcal{H}(\kappa) = \bigcup_{\mu<\kappa} \mathcal{H}(\mu)$.*

(2) *$\mathcal{H}(\kappa) \subseteq V_\kappa$. In particular $\mathcal{H}(\kappa)$ is a set.*

(3) *$|\,\mathcal{H}(\kappa)\,| = 2^{<\kappa}$.*

(4) *If λ is a regular cardinal with $|\,V_\delta\,| < \lambda$ for an ordinal δ, then $V_\delta \in \mathcal{H}(\lambda)$.*

Proof. (0): If $x \in \mathcal{H}(\kappa)$ and $y \in x$ then $trcl(y) \subseteq trcl(x)$ by Lemma 2.2, (2). Thus $|\,trcl(y)\,| \leq |\,trcl(x)\,| < \kappa$ and $y \in \mathcal{H}(\kappa)$.

(18) In particular $rank(a) \neq 0$ for all non-empty a.

(1): If $a \in \bigcup_{\mu<\lambda} \mathcal{H}(\kappa)$, then $|\,trcl(a)\,| < \kappa \leq \kappa'$) Hence $a \in \mathcal{H}(\kappa')$.

Suppose that κ is a limit cardinal. By the proof above, we have $\bigcup_{\mu<\kappa} \mathcal{H}(\mu) \subseteq \mathcal{H}(\kappa)$. If $a \in \mathcal{H}(\kappa)$, there is $\mu_0 < \kappa$ such that $|\,trcl(a)\,| = \mu_0$. Since $a \in \mathcal{H}((\mu_0)^+)$ and $(\mu_0)^+ < \kappa$, it follows $a \in \bigcup_{\mu<\lambda} \mathcal{H}(\mu)$.

(2): By (1), it is enough to show the subset relation holds for a regular κ.

Suppose κ is regular. We show that $rank(x) < \kappa$ holds for all $x \in \mathcal{H}(\kappa)$. This can be shown by $\in$-induction: Suppose that $rank(y) < \kappa$ for all $y \in x$. We have to show that $rank(x) < \kappa$. By Lemma 2.8, (2) we have

$$(2.32)\quad rank(x) = \max\{rank(y)+1 \,:\, y \in x\}$$

Since $|\,x\,| \leq |\,trcl(x)\,| < \kappa$ and κ is regular, the right side of the equation (2.32) is $< \kappa$.

(3): For each $x \in \mathcal{H}(\kappa)$ we can find $\alpha < \kappa$ and well founded $e \subseteq (\alpha)^2$ such that there is a isomprphism i from $\langle trcl(x), \in\rangle$ to $\langle \alpha, e\rangle$ with $i(x) = 0$. x can be reconstructed from $\langle \alpha, e\rangle$ (this follows from the next theorem) and there are at most $2^{<\kappa}$ many pairs of the form $\langle \alpha, e\rangle$ as avove. On the other hand $|\,\mathcal{H}(\kappa)\,| \geq 2^{<\kappa}$ since $[\kappa]^{<\kappa} \subseteq \mathcal{H}(\kappa)$.

(4) : By induction on $\alpha \leq \delta$, we show that $V_\alpha \in \mathcal{H}(\lambda)$. If we have proved that $V_\alpha \in \mathcal{H}(\lambda)$ for some $\alpha < \delta$ then $V_{\alpha+1} = \mathcal{P}(V_\alpha) \subseteq \mathcal{H}(\lambda)$. Since $|\,V_{\alpha+1}\,| \leq |\,V_\delta\,| < \lambda$. It follows that $V_{\alpha+1} \in \mathcal{H}(\lambda)$. If $\gamma \leq \delta$ is a limit ordinal and we know that $V_\alpha \in \mathcal{H}(\lambda)$ for all $\alpha < \gamma$. Then $V_\alpha \subseteq \mathcal{H}(\lambda)$ for all $\alpha < \gamma$ and thus $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha \subseteq \mathcal{H}(\lambda)$. Similarly to the previous case, by $|\,V_\gamma\,| \leq |\,V_\delta\,| < \lambda$, it follows that $V_\gamma \in \mathcal{H}(\lambda)$. ❑ (Lemma 2.9)

A class binary relation $\mathbf{R}$ on a class $\mathbf{X}$ is said to be *extensional* if, for any $a, b \in \mathbf{X}$, $\{c \in \mathbf{X} \,:\, c\,\mathbf{R}\,a\} = \{c \in \mathbf{X} \,:\, c\,\mathbf{R}\,b\}$ is equivalent to $a = b$.

Theorem 2.10 (Mostowski's Collapsing Lemma) *Suppose that* $\mathbf{R}$ *is a class binary relation on a class* $\mathbf{X}$ *which is set-like, extensional and well-founded. Then there is a transitive classs* $\mathbf{M}$ *with a class isomorphism* $\Pi : (\mathbf{X}, \mathbf{R}) \overset{\cong}{\to} (\mathbf{M}, \in)$*. For* $(\mathbf{X}, \mathbf{R})$ *as avove, these* $\mathbf{M}$ *and* Π *are uniquely determined.* T-class-mostowski

Proof. We define the class function $\Pi : \mathbf{X} \to \mathsf{V}$ by letting

$$(2.33)\quad \Pi(a) = \{\Pi(b) \,:\, b\,\mathbf{R}\,a\}$$ wf-9

for all $a \in \mathbf{X}$.

This definition of Π is actually the following application of Theorem 2.5, (2): let $\mathfrak{F}$ be as in Theorem 2.5, (2) for our $\mathbf{X}$ and define the class function $\mathbf{G}$ on $\mathfrak{F} \times \mathbf{X}$ by

$$(2.34)\quad \mathbf{G}(f,a) = \begin{cases} \{f(b) \,:\, b\,\mathbf{R}\,a\}, & \text{if } \mathrm{dom}(f) = trcl^-_{\mathbf{R}}(a); \\ \emptyset, & \text{otherwise} \end{cases}$$ wf-10

for each $\langle f, a\rangle \in \mathfrak{F} \times \mathbf{X}$. $\mathbf{G}$ is well-defined since $\mathbf{R}$ is set-like. Π = the unique $\mathbf{F}$ in Theorem 2.5, (2) for this $\mathbf{G}$ then satisfies (2.33).

The uniqueness of Π follows from Theorem 2.5, (2). Π is injective by the extensionality of $\mathbf{R}$. Let $\mathbf{M} = \Pi''\mathbf{X}$. Then, Π is 1-1 by extensionality of $\mathbf{R}$ and it satisfies (2.33). It follows that $\mathbf{M}$ is transitive and $\Pi : \langle \mathbf{X}, \mathbf{R} \rangle \overset{\cong}{\to} \langle \mathbf{M}, \in \rangle$. ❑ (Theorem 2.10)

2.3 Skolem hull and elementary submodels

skolem

The following arguments are carried out not in meta-mathematics but rather inside the set theory. For example, $\mathcal{L}$-formulas in the following are finite sequences as set theoretic objects (in particular not necessarily sets corresponding to concretely given finite sequences in meta-mathematics).

Of the following definitions and assertions, the syntactical notions on the first order logic are a straight-forward translation of what we do in meta-mathematics but the semantical notions (connected to infinite structures and model relation) do not have their counter-part in meta-mathematics.

To define the formal system of first order logic we choose first a set Var of variables which is an abbitraly countable set whose elements are called variables or variable symbols; as well as the following 6 logical and auxiliary symbols

(2.35) '$\equiv$', '$\wedge$', '$\neg$', '$\exists$', '(', ')', ','. model-a

These 7 symbols are just arbitrary but fixed sets distinct to each other and not among the variables in Var. We also assume that all the symbols to be introduced below are distinct from the symbols we introduced sofar.

A language $\mathcal{L}$, is a set of new symbols of the form

(2.36) $\{c_i : i \in I\} \,\dot\cup\, \{f_j : j \in J\} \,\dot\cup\, \{r_k : k \in K\}$ model-0

where c_i, f_j, r_k are called *costant, fuction and relation symbols* respectively. We assume that these symbols are pairwise different sets. To each f_j, $j \in J$ and r_k, $k \in K$, we attach their arity $m_j \in \omega \setminus 1$, $j \in J$ and $n_k \in \omega \setminus 1$ $k \in K$ respectively. Some or all of I, J, K may be empty sets. For example, this is the case for the language $\mathcal{L}_\varepsilon$ of the set theory which satisfies $I = \emptyset$, $J = \emptyset$ and K is a singleton; the unique relation symbol of $\mathcal{L}_\varepsilon$ is binary and we denote the symbol with ε.

For a language $\mathcal{L}$, the set of $\mathcal{L}$*-terms* and $\mathcal{L}$*-formulas* are specific finite sequences of symbols introduced at the beginning and those in $\mathcal{L}$ defined recursively as follows.

(2.37) The sequences of length 1 consisting of some variable $x \in \mathsf{Var}$ is an $\mathcal{L}$-term. For simplicity we denote such a term also by x term-0

(2.38) he sequences of length 1 consisting of the constant symbol c_i for some $i \in I$ is an $\mathcal{L}$-term. For simplicity we denote such a term also by c_i; term-0-0

(2.39) For $j \in J$, if $t_0, ..., t_{m_j-1}$ are $\mathcal{L}$-terms then $f_j(t_0, ..., t_{m_j-1})$ is also an $\mathcal{L}$-term.[19] term-1

[19] Here "$f_j(t_0, ..., t_{m_j-1})$" means the concatenation of the symbols and sequences f_j, '(', t_0, ',', t_1 etc.

(2.40) For $\mathcal{L}$-terms t_0, t_1, the sequence $t_0 \equiv t_1$ is an $\mathcal{L}$-formula; [20] formula-0

(2.41) For $k \in K$ and $\mathcal{L}$-terms $t_0, ..., t_{n_k-1}$, $r_k(t_0, ..., t_{n_k-1})$ is an $\mathcal{L}$-formula; formula-1

(2.42) If φ_0, φ_1 are $\mathcal{L}$-formulas then $(\varphi_0 \wedge \varphi_1)$ and $\neg\varphi_0$ are also $\mathcal{L}$-formulas; formula-2

(2.43) If φ is an $\mathcal{L}$-formula and $x \in \mathsf{Var}$ then $\exists x\, \varphi$ is also an $\mathcal{L}$-formula. formula-3

For $\mathcal{L}$-terms t and $\mathcal{L}$-formulas φ the set of *free variables* in t and φ (denoted here by $\mathit{freeVar}(t)$, $\mathit{freeVar}(\varphi) \in [\mathsf{Var}]^{<\aleph_0}$) is the finite set of variables defined recursively corresponding to the recursive definition of terms and formulas (2.37) $\sim$ (2.43):

(2.44) If an $\mathcal{L}$-term t is (the sequence of length 1 consisting of) a variable $x \in \mathsf{Var}$ then $\mathit{freeVar}(t) = \{x\}$; free-t-0

(2.45) If an $\mathcal{L}$-term t is (the sequence of length 1 consisting of) a constant symbol c_i for $i \in I$ then $\mathit{freeVar}(t) = \emptyset$; free-t-1

(2.46) If an $\mathcal{L}$-term t is of the form $f_j(t_0, ..., t_{m_j-1})$, for $j \in J$ then $\mathit{freeVar}(t) = \mathit{freeVar}(t_0) \cup \cdots \cup \mathit{freeVar}(t_{m_j-1})$. free-t-2

(2.47) If a $\mathcal{L}$-formula φ is of the form $t_0 \equiv t_1$ for some $\mathcal{L}$-terms t_0, t_1, then $\mathit{freeVar}(\varphi) = \mathit{freeVar}(t_0) \cup \mathit{freeVar}(t_1)$; free-f-0

(2.48) If a $\mathcal{L}$-formula φ is of the form $r_k(t_0, ..., t_{n_k-1})$ for $k \in K$ and $\mathcal{L}$-termus $t_0, ...,$ t_{n_k-1}, then $\mathit{freeVar}(\varphi) = \mathit{freeVar}(t_0) \cup \cdots \cup \mathit{freeVar}(t_{n_k-1})$; free-f-1

(2.49) If a $\mathcal{L}$-formula φ is of the form $(\varphi_0 \wedge \varphi_1)$ or $\neg\varphi_0$ for some $\mathcal{L}$-formulas φ_0 and φ_1, then $\mathit{freeVar}(\varphi) = \mathit{freeVar}(\varphi_0) \cup \mathit{freeVar}(\varphi_1)$ or $\mathit{freeVar}(\varphi) = \mathit{freeVar}(\varphi_0)$ respectively; free-f-2

(2.50) If a $\mathcal{L}$-formula φ is of the form $\exists x\, \varphi_0$ for an $x \in \mathsf{Var}$ and an $\mathcal{L}$-formula φ_0 then $\mathit{freeVar}(\varphi) = \mathit{freeVar}(\varphi_0) \setminus \{x\}$. free-f-3

For an $\mathcal{L}$-term t, if $\mathit{freeVar}(t) \subseteq \{x_0, ..., x_{n-1}\}$ where x_i, $i \in n$ are pairwise distinct variables in Var, we represent this inclusion by the equation $t = t(x_0, ..., x_{n-1})$. Similarly, for an $\mathcal{L}$-formula φ, if $\mathit{freeVar}(\varphi) \subseteq \{x_0, ..., x_{n-1}\}$, we write $\varphi = \varphi(x_0, ..., x_{n-1})$.

In the notation above we also allow the situation where the list of the variables x_0, $..., x_{n-}$ is void. In this case we write $t = t()$ and $\varphi = \varphi()$. An $\mathcal{L}$-formula φ is said to be an $\mathcal{L}$-*sentence* if $\varphi = \varphi()$ or equivalently $\mathit{freeVar}(\varphi) = \emptyset$.

Note that the "equations" above are not unique for each t or φ: if for example $t = t(x_0, ..., x_{n-1})$ and $\{x_0, ..., x_{n-1}\} \subseteq \{y_0, ..., y_{m-1}\}$ then we also have $t = t(y_0, ..., y_{m-1})$.

The expressions $t = t(x_0, ..., x_{n-1})$ and $\varphi = \varphi(x_0, ..., x_{n-1})$ are introduced here since they satisfy the following: If $t = t(x_0, ..., x_{n-1})$ and s is a subterm of t then we also have

[20] The expression $t_0 \equiv t_1$ is to be interpreted similarly to footnote (19). The expressions in (2.41), (2.42) etc. are also to be interpreted in a similar way.

$s = s(x_0, ..., x_{n-1})$; If $\varphi = \varphi(x_0, ..., x_{n-1})$ and ϕ is a subformula of φ then we also have $\psi = \psi(x_0, ..., x_{n-1})$.[21]

For a language $\mathcal{L}$ of the form (2.36), an *$\mathcal{L}$-structure* $\mathfrak{A}$ is a sequence of the form

(2.51) $\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}} \rangle_{i\in I,\, j\in J,\, k\in K}$ model-1

where A is a non-empty set and is called the *underlying set* of $\mathfrak{A}$. $c_i \in A$ for $i \in I$, $f_j^{\mathfrak{A}} : A^{m_j} \to A$ for $j \in J$ and $r_k^{\mathfrak{A}} \subseteq A^{n_k}$ for $k \in K$. $c_i^{\mathfrak{A}}$, $i \in I$ $f_j^{\mathfrak{A}}$, $j \in J$ $r_k^{\mathfrak{A}}$, $k \in K$ are *interperetations of the symbols c_i, $i \in I$ f_j, $j \in J$ r_k, $k \in K$ in $\mathfrak{A}$*.

For an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, ...\rangle$ and an $\mathcal{L}$-term $t = t(x_0, ..., x_{n-1})$ we define by recursion the interpretation $t^{\mathfrak{A}}_{x_0,...,x_{n-1}}$ of $t = t(x_0, ..., x_{n-1})$ in $\mathfrak{A}$ as a mapping from A^n to A recursively by the following:

For $a_0, ..., a_{n-1} \in A$,

(2.52) If t is (the sequence of length 1 consisting of) the variable x_ℓ (for some $\ell < n$) then $t^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}) = a_\ell$; term-2

(2.53) If t is (the sequence of length 1 consisting of) the constant symbol c_i (for some $i \in I$) then $t^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}) = c_i$; term-3

(2.54) If t is of the form $f_j(t_0, ..., t_{m_j-1})$ for some $j \in J$ and $\mathcal{L}$-terms $t_0, ..., t_{m_j-1}$, then term-4
$t^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})$
$= f_j^{\mathfrak{A}}({t_0}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}), ..., {t_{m_j-1}}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}))$.

It is easy to prove by induction on the construction of $\mathcal{L}$-terms t that the definition of $t^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})$ consistent with the choices of the list of variables $a_0, ..., a_{n-1}$.

For an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, ...\rangle$, an $\mathcal{L}$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $a_0, ..., a_{n-1} \in A$, we define by recursion when φ holds in $\mathfrak{A}$ with parameters $a_0, ..., a_{n-1}$ substituted for $x_0, ..., x_{n-1}$ respectively (notation: $\mathfrak{A} \models \varphi_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})$ or simply $\mathfrak{A} \models \varphi(a_0, ..., a_{n-1})$ if it is apparent which sequence of variables is chosen here) recursively by the following:

(2.55) If φ is of the form $t_0 \equiv t_1$ for some $\mathcal{L}$-terms t_0, t_1 then formula-4
$\mathfrak{A} \models \varphi_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})$ if and only if

$$ {t_0}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}) = {t_1}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}); $$

(2.56) If φ is of the form $r_k(t_0, ..., t_{n_k-1})$ for some $k \in K$ and $\mathcal{L}$-terms $t_0, ..., t_{n_k-1}$ then formula-5
$\mathfrak{A} \models \varphi_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})$ if and only if

$$ \langle {t_0}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1}), ..., {t_{n_k-1}}^{\mathfrak{A}}_{x_0,...,x_{n-1}}(a_0, ..., a_{n-1})\rangle \in r_k^{\mathfrak{A}}. $$

(2.57) If φ is of the form $(\varphi_0 \wedge \varphi_1)$ for some $\mathcal{L}$-formulas φ_0, φ_1 then formula-6

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[21] subterms (subformulas) of a term t (a formula φ) are such terms (formulas) which appear at some step of the inductive construction of t (φ) respectively. We leave it as an exercise to find a recursive definition of a function which gives the set of all subterms (the set of all subformulas) for a given term (formula) respectively.

$\mathfrak{A} \models \varphi_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$ if and only if $\mathfrak{A} \models {\varphi_0}_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$ and $\mathfrak{A} \models {\varphi_1}_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$;

(2.58) If φ is of the form $\neg\varphi_0$ for some $\mathcal{L}$-formula φ_0 then $\mathfrak{A} \models \varphi_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$ formula-7
if and only if $\mathfrak{A} \not\models {\varphi_0}_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$;

$(2.59)_a$ If φ is of the form $\exists x\, \varphi_0$ and x is not among $x_0,\dots,x_{n-1}$, then formula-8
$\mathfrak{A} \models \varphi_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$ if and only if there is $a \in A$ such that
$\mathfrak{A} \models {\varphi_0}_{x_0,\dots,x_{n-1},x}(a_0,\dots,a_{n-1},a)$;

$(2.59)_b$ If φ is of the form $\exists x\, \varphi_0$ and x is among $x_0,\dots,x_{n-1}$, say $x = x_i$, then
$\mathfrak{A} \models \varphi_{x_0,\dots,x_{n-1}}(a_0,\dots,a_{n-1})$ if and only if there is $a \in A$ such that
$\mathfrak{A} \models {\varphi_0}_{x_0,\dots,x_{n-1}}(a_0,\dots,\underset{\underset{i}{\sqcup}}{a},\dots,a_{n-1})$ where "$a_0,\dots,\underset{\underset{i}{\sqcup}}{a},\dots,a_{n-1}$" is the list of elements of A obtained from $a_0,\dots,a_{n-1}$ by replacing a_i with a.

For $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \dots\rangle$, $B \subseteq A$ is said to be *closed with respect to* $\mathcal{L}$ if $c_i^{\mathfrak{A}} \in B$ for all $i \in I$ and $f_j^{\mathfrak{A}}(a_0,\dots,a_{m_j-1}) \in B$ for all $j \in J$ and $a_0,\dots,a_{m_j-1} \in B$.

For $B \subseteq A$ closed with respect to $\mathcal{L}$,

(2.60) $\langle B, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}} \restriction (A)^{m_j}, r_k^{\mathfrak{A}} \cap (A)^{n_k}\rangle_{i\in I,\, j\in J,\, k\in K}$ model-2

is an $\mathcal{L}$-structue. This $\mathcal{L}$-structue is denoted by $\mathfrak{A} \restriction B$.

An $\mathcal{L}$-structure $\mathfrak{B}$ of the form $\mathfrak{A} \restriction B$ for a $B \subseteq A$ closed with respect to $\mathcal{L}$ is called a substructure of $\mathfrak{A}$. If $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ we write $\mathfrak{B} \subseteq \mathfrak{A}$.

A substructure $\mathfrak{B} = \langle B, \dots\rangle$ of an $\mathcal{L}$-structure $\mathfrak{A}$ is said to be an $\mathcal{L}$- *elementary substructure of* $\mathfrak{A}$ (notation: $\mathfrak{B} \prec \mathfrak{A}$) if

(2.61) $\mathfrak{B} \models \varphi(b_0,\dots,b_{n-1})$ if and only if $\mathfrak{A} \models \varphi(b_0,\dots,b_{n-1})$ for all $\mathcal{L}$-formulas $\varphi = \varphi(x_0,$ model-3
$\dots, x_{n-1})$ and $b_0,\dots,b_{n-1} \in B$.

For a language $\mathcal{L} = \{c_i : i \in I\} \,\dot\cup\, \{f_j : j \in J\} \,\dot\cup\, \{r_k : k \in K\}$ a lanugage $\mathcal{L}_0$ of the form $\mathcal{L}_0 = \{c_i : i \in I_0\} \,\dot\cup\, \{f_j : j \in J_0\} \,\dot\cup\, \{r_k : k \in K_0\}$ for some $I_0 \subseteq I$, $J_0 \subseteq J$ and $K_0 \subseteq K$ is called a sublanguage of $\mathcal{L}$.

For $\mathcal{L}$ and $\mathcal{L}_0$ as above, and for an $\mathcal{L}$-structue $\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}}\rangle_{i\in I,\, j\in J,\, k\in K}$, the structure

(2.62) $\mathfrak{A} \restriction \mathcal{L}_0 = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}}\rangle_{i\in I_0,\, j\in J_0,\, k\in K_0}$ model-3-0

is called the *reduction of* $\mathfrak{A}$ *in* $\mathcal{L}_0$. If $\mathfrak{A}_0 = \mathfrak{A} \restriction \mathcal{L}_0$, then $\mathfrak{A}$ is said to be an *expansion of* $\mathfrak{A}_0$ *in* $\mathcal{L}$.

Note that if $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures, and $\mathfrak{B} \prec \mathfrak{A}$, then for any sublanguage $\mathcal{L}_0$ of $\mathcal{L}$ we have $\mathfrak{B} \restriction \mathcal{L}_0 \prec \mathfrak{A} \restriction \mathcal{L}_0$.

Lemma 2.11 (Tarski-Vaught Test) *For $\mathcal{L}$-structures $\mathfrak{A} = \langle A, \dots\rangle$, $\mathfrak{B} = \langle B, \dots\rangle$ with* P-model-a
$\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B}$ is an $\mathcal{L}$-elementary substructure of $\mathfrak{A}$ if and only if the following condition holds:

(2.63) *for any $\mathcal{L}$-formula $\varphi = \varphi(x_0, ..., x_{n-1}, x)$ and $b_0, ..., b_{n-1} \in B$, if $\mathfrak{A} \models \exists x\, \varphi(b_0, ..., b_{n-1}, x)$, then there is $b \in B$ such that $\mathfrak{A} \models \varphi(b_0, ..., b_{n-1}, b)$.* model-4

Proof. It is easy to see that $\mathfrak{B} \prec \mathfrak{A}$ implies (2.63): Suppose $\mathfrak{B} \prec \mathfrak{A}$. If $\mathfrak{A} \models \exists x\, \varphi(b_0, ..., b_{n-1}, x)$ for $b_0, ..., b_{n-1} \in B$, then $\mathfrak{A} \models \exists x\, \varphi(b_0, ..., b_{n-1}, x)$ by elementarity. Thus (by the definition of $\models$ for existencial formulas $(2.59)_a$) there is $b \in B$ such that $\mathfrak{B} \models \varphi(b_0, ..., b_{n-1}, b)$. By elementarity it follows that $\mathfrak{A} \models \varphi(b_0, ..., b_{n-1}, b)$.

To see that (2.63) implies $\mathfrak{B} \prec \mathfrak{A}$, we proceed by indcution on the constructuion of $\mathcal{L}$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ that if all subformulas of φ satisfy (2.61) then φ also satisfies (2.61).

If φ is an atomic formula or of the form $(\varphi_0 \land \varphi_1)$ or $\neg\varphi_0$ this is easy to prove. So assume that φ is of the form $\exists x\, \varphi_0$. We consider the case that x is not in the list $x_0, ..., x_{n-1}$. Thus $\varphi_0 = \varphi_0(x_0, ..., x_{n-1}, x)$. The other case can be treated similarly with some notational changes.

Let $b_0, ..., b_{n-1} \in \mathfrak{B}$.

Suppose now $\mathfrak{B} \models \varphi(b_0, ..., b_{n-1})$. Then by $(2.59)_a$ there is $b \in B$ such that $\mathfrak{B} \models \varphi_0(b_0, ..., b_{n-1}, b)$. By induction hypothesis it follows that $\mathfrak{A} \models \varphi_0(b_0, ..., b_{n-1}, b)$.

Suppose $\mathfrak{A} \models \varphi(b_0, ..., b_{n-1})$, or equivalently $\mathfrak{A} \models \exists x\, \varphi_0(b_0, ..., b_{n-1}, x)$. Then, by (2.63), there is $b \in B$ such that $\mathfrak{A} \models \varphi_0(b_0, ..., b_{n-1}, b)$. By induction hypothesis, it follows that $\mathfrak{B} \models \varphi_0(b_0, ..., b_{n-1}, b)$. Thus $\mathfrak{B} \models \exists x\, \varphi_0(b_0, ..., b_{n-1}, x)$ or equivalently $\mathfrak{B} \models \varphi(b_0, ..., b_{n-1})$. ❑ (Lemma 2.11)

Let $\mathcal{L}$ be any language (of the first order logic) and let $\mathcal{L}^{\sqsubseteq}$ be the language obtained from $\mathcal{L}$ by adding a new binary relation symbol $\sqsubseteq$. For any $\mathcal{L}$-structure $\mathfrak{A}$ an expansion $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ to an $\mathcal{L}^{\sqsubseteq}$ structure is said to be a well-ordered expansion if $\sqsubseteq^{\tilde{\mathfrak{A}}}$ is a well-ordering on the underlying set A of $\mathfrak{A}$.

Suppose that $\tilde{\mathfrak{A}}$ is a well-ordered expansion of an $\mathcal{L}$-structure $\mathfrak{A}$ with underlying set A. For a subset X of A the *Skolem hull* $sk_{\tilde{\mathfrak{A}}}(X)$ (or simply $sk(X)$ if it is clear which $\tilde{\mathfrak{A}}$ is meant) of X is the closure of the set X with respect to all definable functions in $\tilde{\mathfrak{A}}$.[22] By definition $sk(X)$ is closed with respect to $\mathcal{L}$. Thus the restriction of $\mathfrak{A}$ on $sk(X)$ is an $\mathcal{L}$-substructure of $\mathfrak{A}$. We shall denote this substructure of $\mathfrak{A}$ also with $sk(X)$.

T-skolem

Lemma 2.12 *Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-structure with underlying set A and $\tilde{\mathfrak{A}}$ a well-ordered expansion of $\mathfrak{A}$. Then, for any $X \subseteq A$, $sk_{\tilde{\mathfrak{A}}}(X)$ is an elementary substructure of $\mathfrak{A}$.*

Proof. We check that $sk(X)$ satisfies the condition of Tarski-Vaught Test.

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[22] A function $h : A^m \to A$ is definable in a structuere $\mathfrak{A} = \langle A, ...\rangle$ if there is a formula $\varphi = \varphi(x_0, ..., x_m, y_0, ..., y_{n-1})$ in the language of the strucuture $\mathfrak{A}$ and $b_0, ..., b_{n-1} \in A$ such that $h = \{\langle\langle a_0, ..., a_{m-1}\rangle, a_m\rangle \,:\, M \models \varphi(a_0, ..., b_0, ...)\}$.

Suppose that $\varphi = \varphi(x_0, x_1, ..., x_n)$ is an $\mathcal{L}$-forumla and $b_1^*,..., b_n^* \in sk(X)$ be such that $\mathfrak{A} \models \exists x_0 \varphi(x_0, b_1^*, ..., b_n^*)$. We have to show that there is $a^* \in sk(X)$ such that $\mathfrak{A} \models \varphi(a^*, b_1^*, ..., b_n^*)$.

Let $f_\varphi : A^n \to A$ be the function defined by

(2.64)
$$f_\varphi(b_1, ..., b_n) = \begin{cases} \text{the } \sqsubseteq^{\tilde{\mathfrak{A}}}\text{-minimal } a \in A \text{ with } \mathfrak{A} \models \varphi(a, b_1, ..., b_n), & \\ & \text{if such } a \text{ exists;} \\ \text{the } \sqsubseteq^{\tilde{\mathfrak{A}}}\text{-minimal element of } A, & \text{otherwise.} \end{cases}$$

Since f_φ is definable in $\tilde{\mathfrak{A}}$, $sk(X)$ is closed with respect to f_φ. Thus, letting $a^* = f_\varphi(b_1, ..., b_n)$, we have $a^* \in sk(X)$ and $\mathfrak{A} \models \varphi(a^*, b_1^*, ..., b_n^*)$. ❑ (Lemma 2.12)

For a well-ordered expansion $\mathfrak{A}^{\sqsubseteq}$ of an $\mathcal{L}$-structure $\mathfrak{A}$ with the underlying set A and $X \subseteq A$, we have $|\, sk(X)\,| \leq \max\{|\,X\,|, |\,\mathcal{L}\,|, \aleph_0\}$.[23] In particular, for a countable $\mathcal{L}$ and infinite $X \subseteq A$, we have $|\, sk(X)\,| = |\,X\,|$. Thus we obtain:

Theorem 2.13 (Downward Löwenheim-Skolem Theorem) dwLoSko
For any $\mathcal{L}$-structure $\mathfrak{A}$ with the underlying set A and $X \subseteq A$ there is an elementary substructure $\mathfrak{B} = \langle B, ...\rangle$ of $\mathfrak{A}$ such that $X \subseteq B$ and $|\,B\,| \leq \max\{|\,\mathcal{L}\,|, |\,X\,|, \aleph_0\}$.

In particular if $\mathcal{L}$ is countable, then for any $\mathcal{L}$-structure $\mathfrak{A} = \langle A, ...\rangle$ and infinite $X \subseteq A$ there is an elementary substructure $\mathfrak{B} = \langle B, ...\rangle$ of $\mathfrak{A}$ such that $X \subseteq B$ and $|\,A\,| = |\,X\,|$. ❑

Suppose that $\langle \mathfrak{A}_\alpha : \alpha < \delta\rangle$ is an increasing chain of $\mathcal{L}$-structures (i.e, such a sequence that satisfies $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ for all $\alpha < \beta < \delta$). Let

(2.65) $\mathfrak{A}_\alpha = \langle A_\alpha,\ c_i^{\mathfrak{A}_\alpha},\ f_j^{\mathfrak{A}_\alpha},\ r_k^{\mathfrak{A}_\alpha}\rangle_{i\in I, j\in J, k\in K}$

for $\alpha < \delta$. The union of the structures $\mathfrak{A}_\alpha$, $\alpha < \delta$ can be defined then as

(2.66) $\bigcup_{\alpha<\delta} \mathfrak{A}_\alpha = \langle \bigcup_{\alpha<\delta} A_\alpha,\ c_i^{\mathfrak{A}_0},\ \bigcup_{\alpha<\delta} f_j^{\mathfrak{A}_\alpha},\ \bigcup_{\alpha<\delta} t_k^{\mathfrak{A}_\alpha}\rangle_{i\in I, j\in J, k\in K}$.

Theorem 2.14 (Union of Chains) (1) *Suppose that γ is a limit ordinal and $\langle \mathfrak{A}_\alpha : \alpha \leq \gamma\rangle$ is a sequence such that* union-of-chains

(2.67) $\mathfrak{A}_\alpha$, *$\alpha < \gamma$ are $\mathcal{L}$-structures;* model-5

(2.68) $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ *for all $\alpha < \beta \leq \gamma$;* model-6

(2.69) *If $\xi \leq \gamma$ is a limit then $\mathfrak{A}_\xi = \bigcup_{\alpha<\xi} \mathfrak{A}_\alpha$; and* model-7

(2.70) $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\alpha+1}$ *for all $\alpha < \gamma$.* model-8

Then we have $\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$ holds for all $\alpha < \beta \leq \gamma$.

(2) *If $\langle \mathfrak{A}_\alpha : \alpha < \gamma\rangle$ is an increasing chain of elementary submodels of $\mathfrak{A}$ and $\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$, then we have $\mathfrak{A}_\gamma \prec \mathfrak{A}$.*

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[23] $|\,\mathcal{L}\,|$ denotes the number of symbols the language $\mathcal{L}$ has.

Proof. (1): By induction on γ. It is enough to show that $\mathfrak{A}_\alpha \prec \mathfrak{A}_\gamma$ holds for all $\alpha < \gamma$.

If $\gamma = \gamma_0 + 1$ for $\gamma_0 = \bigcup \gamma$. Then $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\gamma_0}$ for $\alpha \leq \gamma_0$ by induction hypothesis and $\mathfrak{A}_{\gamma_0} \prec \mathfrak{A}_\gamma$ by (2.70). Since $\prec$ is a transitive relation it follows that $\mathfrak{A}_\alpha \prec \mathfrak{A}_\gamma$.

If γ is a limit, the we can show that $\mathfrak{A}_\alpha \prec \mathfrak{A}_\gamma$ holds for all $\alpha < \gamma$ by Tarski-Vaught Test Lemma 2.11 (Exercise).

(2): This can be also easily proved using Tarski-Vaught Test (Exercise). ❑ (Theorem 2.14)

Combination of Theorem 2.13 and Theorem 2.14, (2) brings most of the constructions of elementary submodels with some useful additional properties. The following two lemmas are such examples:

P-model-0

Lemma 2.15 *Suppose that $\mathfrak{A} = \langle A, ...\rangle$ is an $\mathcal{L}$-structure and κ is a regular cardinal with $\aleph_0, |\mathcal{L}| < \kappa \leq |A|$ and $\kappa \subseteq A$. Then for any $X \in [A]^{<\kappa}$, there is $\mathfrak{B} \prec \mathfrak{A}$ with $\mathfrak{B} = \langle B, ...\rangle$ such that $|B| < \kappa$, $X \subseteq B$ and $\kappa \cap B < \kappa$ (that is, $\kappa \cap B$ is a proper initial segment of κ).*

Proof. Let $\mathfrak{B}_n \prec \mathfrak{A}$ with $\mathfrak{B} = \langle B_n, ...\rangle$ for $n \in \omega$ be taken inductively such that

(2.71) $|B_n| < \kappa$; model-8-0

(2.72) $X \subseteq B_0$; model-8-1

(2.73) $B_n \subseteq B_{n+1}$; model-8-2

(2.74) $\sup(\kappa \cap B_n) \subseteq B_{n+1}$ model-8-3

for all $n \in \omega$. The sequence $\langle B_n : n \in \omega\rangle$ is an increasing chain by (2.73), $\sup(\kappa \cap B_n) < \kappa$ by (2.71) and since κ is a regular cardinal. Thus the construction is possible by Theorem 2.13. Let $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$. Letting $\mathcal{B} = \langle B, ...\rangle$, we have $B = \bigcup_{n\in\omega} B_n$ and hence $\kappa \cap B < \kappa$ by (2.74) and $X \subseteq B$ by (2.72). Thus, by Theorem 2.14, this $\mathfrak{B}$ is as desired.

❑ (Lemma 2.15)

P-model-1

Lemma 2.16 *Suppose that θ, κ and λ are regular cardinals such that $\kappa < \lambda < \theta$ and*

(2.75) *$|\alpha|^{<\kappa} < \lambda$ for all $\alpha < \lambda$.* model-9

Then, for any $X \in [\mathcal{H}(\theta)]^{<\lambda}$, there is $\mathfrak{M} \prec \langle \mathcal{H}(\theta), \in\rangle$ with $\mathfrak{M} = \langle M, \in\rangle$ such that

(2.76) *$X \subseteq M$, $|M| < \lambda$;* model-10

(2.77) *$\lambda \cap M < \lambda$; and* model-11

(2.78) *$[M]^{<\kappa} \subseteq M$.* model-12

Proof. Similarly to Theorem 2.15. ❑ (Lemma 2.16)

2.4 Lévy-Montague Reflection Theorem

refl-abs

A class $\mathbf{C}$ of ordinals is said to be *club* (*closed unbounded*) (in $\mathsf{On} = \{\alpha : \alpha \text{ is an ordinal}\}$) if

(2.79) for all limit $\alpha \in \mathsf{On}$ if $\mathbf{C} \cap \alpha$ is unbounded in α then $\alpha \in \mathbf{C}$ (closed), and club-1

(2.80) for all $\alpha \in \mathsf{On}$ there is $\beta \in \mathbf{C}$ with $\alpha < \beta$ (unbounded). club-2

By (2.80) a club $\mathbf{C} \subseteq \mathsf{On}$ is a proper class. The following is easy to prove:

T-refl-abs-1

Lemma 2.17 *If $\mathbf{C}$, $\mathbf{D} \subseteq \mathsf{On}$ are club then $\mathbf{C} \cap \mathbf{D}$ is also club.*

Proof. $\mathbf{C} \cap \mathbf{D}$ is closed: Suppose that $\alpha \cap (\mathbf{C} \cap \mathbf{D})$ is unbounded in $\alpha \in \mathsf{On}$. Them $\alpha \cap \mathbf{C}$ and $\alpha \cap \mathbf{D}$ are both unbunded in α. Since $\mathbf{C}$ and $\mathbf{D}$ are closed, it follows that $\alpha \in \mathbf{C}$ and $\alpha \in \mathbf{D}$. Thus $\alpha \in \mathbf{C} \cap \mathbf{D}$.

$\mathbf{C} \cap \mathbf{D}$ is unbounded: For an arbitrary $\alpha \in \mathsf{On}$, let $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_n < \beta_n < \alpha_{n+1} < \cdots$, $n \in \omega$ be such that $\alpha < \alpha_0$ $\alpha_n \in \mathbf{C}$ and $\beta_n \in \mathbf{D}$ for all $n \in \omega$. This is possible since $\mathcal{C}$ and $\mathcal{D}$ are both club. Let $\beta = \sup_{n \in \omega} \alpha_n = \sup_{n \in \omega} \beta_n$. Then $\alpha < \beta$. $\beta \in \mathbf{C}$ and $\beta \in \mathbf{D}$ since $\mathbf{C} \cap \beta$ and $\mathbf{D} \cap \beta$ are closed in β by the construction of η. Thus $\beta \in \mathbf{C} \cap \mathbf{D}$. ❏ (Lemma 2.17)

Any set a can be considered as an $\mathcal{L}_\varepsilon$-structure $\langle a, \in \cap\, a^2 \rangle$. This structure is simply denoted by $\langle a, \in \rangle$ or even more simply by a. An $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ is said to be *absolute* over the $\mathcal{L}_\varepsilon$-structure a if for any $b_0, ..., b_{n-1} \in a$ we have $\langle a, \in \rangle \models \varphi(b_0, ..., b_{n-1})$ if and only if $\varphi(b_0, ..., b_{n-1})$ holds. Note that we can only talk about absoluteness of concretely given formulas (due to Tarski's Undefinability Theorem).

The formulas in the next theorem are thus meta-mathematical formulas and the theorem is actually a meta-theorem, that is, a collection of theorems for each concretely given formula φ.

Theorem 2.18 (A. Lévy and R. Montague) *For any $\mathcal{L}_\varepsilon$-forumla φ, there is a club $\mathbf{C}_\varphi \subseteq \mathsf{On}$ such that* T-levy

(2.81) *φ is absolute over V_α for all $\alpha \in \mathbf{C}_\varphi$.* club-2-0

Proof. By induction on φ.

Let $\mathbf{X}_\varphi = \{\alpha \in \mathsf{On} : \varphi \text{ is absolute over } V_\alpha\}$

For an atomic φ, we have $\mathbf{X}_\varphi = \mathsf{On} \setminus \{\emptyset\}$. Thus $\mathbf{C}_\varphi = \mathbf{X}_\varphi$ will do.

For $\varphi = (\varphi_0 \wedge \varphi_1)$, $\mathbf{C}_\varphi = \mathbf{C}_{\varphi_0} \cap \mathbf{C}_{\varphi_1}$ will do by Lemma 2.17. For $\varphi = \neg\varphi_0$ we may take $\mathbf{C}_\varphi = \mathbf{C}_{\varphi_0}$.

Thus it is enough to show that the assertion of the theorem holds for any forumula φ where φ is of the form $\exists x\, \psi(x, x_0, ..., x_{n-1})$ and we already have a club $\mathbf{C}_\psi \subseteq \mathbf{X}_\psi$.

Let

(2.82) $\quad \mathbf{C}_\varphi = \{\alpha \in \mathbf{C}_\psi : \varphi \text{ is absolute over } V_\alpha\}$.

That is, $\mathbf{C}_\varphi = \mathbf{X}_\varphi \cap \mathbf{C}_\psi$. Since we have in particular $\mathbf{C}_\varphi \subseteq \mathbf{X}_\varphi$, it is enough to show that this $\mathbf{C}_\varphi$ is a club.

Ydashed-closed

Claim 2.18.1 $\mathbf{C}_\varphi$ *is closed.*

$\vdash$ Suppose that

(2.83) $\quad \delta \in \mathrm{Lim}$ and $\delta \cap \mathbf{C}_\varphi$ is cofinal in δ. club-2-1

We have to show that $\delta \in \mathbf{C}_\varphi$.

By $\mathbf{C}_\varphi \subseteq \mathbf{C}_\psi$,

(2.84) $\quad \delta \cap \mathbf{C}_\psi$ is also cofinal in δ. club-2-2

Since $\mathbf{C}_\psi$ is club, it follows that

(2.85) $\quad \delta \in \mathbf{C}_\psi$. club-2-3

To show that $\delta \in \mathbf{X}_\varphi$ let $a_0, ..., a_{n-1} \in V_\delta$. Then there is an $\alpha \in \delta \cap \mathbf{C}_\varphi$ such that $a_0, ..., a_{n-1} \in V_\alpha$ by (2.83). If $\varphi(a_0, ..., a_{n-1})$ holds, then $V_\alpha \models \varphi(a_0, ..., a_{n-1})$ since $\alpha \in \mathbf{X}_\varphi$. Thus there is an $a \in V_\alpha$ such that $V_\alpha \models \psi(a, a_0, ..., a_{n-1})$. As $\alpha \in \mathbf{C}_\psi$ we have $\psi(a, a_0, ..., a_{n-1})$. By (2.85) and $a, a_0, ..., a_{n-1} \in V_\delta$, it follows that $V_\delta \models \psi(a, a_0, ..., a_{n-1})$ and hence $V_\delta \models \varphi(a_0, ..., a_{n-1})$.

On the other hand, if $V_\delta \models \varphi(a_0, ..., a_{n-1})$, then there is an $a \in V_\delta$ such that $V_\delta \models \psi(a, a_0, ..., a_{n-1})$. By (2.85), it follows that $\psi(a, a_0, ..., a_{n-1})$. Thus $\exists x \psi(x, a_0, ..., a_{n-1})$ or $\varphi(a_0, ..., a_{n-1})$ holds.

This shows that $\delta \in \mathbf{C}_\varphi$. $\dashv$ (Claim 2.18.1)

Claim 2.18.2 $\mathbf{C}_\varphi$ *is unbounded.*

$\vdash$ For arbitrary $\alpha \in \mathsf{On}$, let α_i, $i \in \omega$ be an increasing sequence of ordinals defined by

(2.86) $\quad \alpha_0 = \min(\mathbf{C}_\psi \setminus \alpha + 1)$ and club-3

(2.87) $\quad \alpha_{i+1} = \sup\big(\{\min\{\beta \in \mathbf{C}_\psi \setminus \alpha_i : V_\beta \models \varphi(a_0, ..., a_{n-1})\} : a_0, ..., a_{n-1} \in V_{\alpha_i}\} \cup \{\alpha_i\}\big)$ club-4

where we assume here that $\min(\emptyset) = 0$.

For $\delta = \sup(\{\alpha_i : i \in \omega\})$, we have

(2.88) $\quad \alpha < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \delta$

by (2.86) and (2.87). In particular $\alpha < \delta$.

We show that $\delta \in \mathbf{C}_\varphi$.

$\alpha_i \in \mathbf{C}_\psi$, $i \in \omega$ by (2.86) and (2.87). Since $\mathbf{C}_\psi$ is club, it follows that

(2.89) $\quad \delta \in \mathbf{C}_\psi$. club-5

For $a_0, ..., a_{n-1} \in V_\delta$, there is $i \in \omega$ such that $a_0, ..., a_{n-1} \in V_{\alpha_i}$.

If $\varphi(a_0, ..., a_{n-1})$ holds then there is an a such that $\psi(a, a_0, ..., a_{n-1})$ and thus there is $\beta \in \mathbf{C}_\psi$ such that $a, a_0, ..., a_{n-1} \in V_\beta$. Let a^*, β^* be such a pair a, β such that β^* is minimal among such β's. By (2.87) we have $\beta^* \leq \alpha_{i+1} \leq \delta$. Hence $a^* \in V_\delta$. By (2.89)we have $V_\delta \models \psi(a^*, a_0, ..., a_{n-1})$. It follows that $V_\delta \models \exists x \psi(x, a_0, ..., a_{n-1})$, that is, $V_\delta \models \varphi(a_0, ..., a_{n-1})$.

If we have $V_\delta \models \varphi(a_0, ..., a_{n-1})$ on the other hand, then we can take an $a \in V_\delta$ such that $V_\delta \models \psi(a, a_0, ..., a_{n-1})$. By (2.89) , it follows that $\psi(a, a_0, ..., a_{n-1})$. Hence we have $\exists x\, \psi(x, a_0, ..., a_{n-1})$, that is, $\varphi(a_0, ..., a_{n-1})$ holds. $\dashv$ (Claim 2.18.2)

❑ (Theorem 2.18)

reflection

Corollary 2.19 *Suppose that $\varphi_0, ..., \varphi_{n-1}$ are $\mathcal{L}_\varepsilon$-formulas. Then there is a club $\mathbf{C} \subseteq \mathsf{On}$ such that, for all $\alpha \in \mathbf{C}$ and $i < n$, φ_i is absolute over V_α.*

For α as above, we say that V_α *reflects* $\varphi_0, ..., \varphi_{n-1}$ (simultaneously).

Proof. For each $i < n$, let $\mathbf{C}_{\varphi_i}$ be the club as in Theorem 2.18. $\mathbf{C} = \mathbf{C}_{\varphi_0} \cap \cdots \cap \mathbf{C}_{\varphi_{n-1}}$ is then also a club by Lemma 2.17. Clearly this $\mathbf{C}$ is as desired. ❑ (Corollary 2.19)

Note that, by the construction of $\mathbf{C}_\varphi$ in Theorem 2.18, V_α for $\alpha \in \mathbf{C}_\varphi$ reflects all subformulas of φ.

The "finite fragment of ZFC" in the following corollary is in the sense of meta-mathematics. The corollary is thus a theorem in ZFC for each fixed (concretely given) finite fragment of ZFC.

ctbl-trans-model

Corollary 2.20 *For any finite fragment T of* ZFC *there is a countable transitive model M of T.*

Proof. Let $\mathbf{C} \subseteq \mathsf{On}$ be the club as in Corollary 2.19 for the finite collection T of $\mathcal{L}_\varepsilon$-formulas. Let $\alpha \in \mathbf{C}$. Since φ holds (as an axiom of ZFC) for all $\varphi \in T$, we have $V_\alpha \models \varphi$. Thus $V_\alpha \models T$.

By Theorem 2.13 there is a countable elementary submodel M of V_α. We have $M \models \varphi$. By Theorem 2.10 there is a transitive N with $\langle M, \in \rangle \cong \langle N, \in \rangle$. This N is as desired.

❑ (Corollary 2.20)

Historically, Montague proved Theorem 2.18 (independently from Lévy) to show the non-finitely-axiomatizability of ZFC:

Corollary 2.21 (R. Montague) *Assuming that* ZFC *is consistent,* ZFC *is not logically equivalent to any concretely given finite theory in $\mathcal{L}_\varepsilon$.* montague

Proof. Suppose toward a contradiction that T is a concretely given finite theory in $\mathcal{L}_\varepsilon$ which is logically equivalent to ZFC. In particular each sentence φ in T is a theorem in ZFC. Let T_φ be a finite fragment of ZFC such that $T_\varphi \vdash \varphi$. Let $T^* = \bigcup_{\varphi \in T} T_\varphi$.[24] T^* is a finite fragment of ZFC. Hence, by Corollary 2.20, we can prove that there is a transitive set M such that $M \models T^*$ (or more accurately $M \models$ "$\ulcorner\ulcorner T^* \urcorner\urcorner$"). Now the meta-mathematical argument of the fact that T and hence T^* axiomatizes ZFC is translated into a proof of "$(\forall \varphi \in \ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner)(\ulcorner\ulcorner T^* \urcorner\urcorner \vdash \varphi)$" in ZFC. Thus $M \models$ "$\ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner$". By the Second Incompleteness Theorem, this is a contradiction to the assumption that ZFC is consistent. □ (Corollary 2.21)

2.5 Absoluteness of formulas over a transitive set

absoluteness

The Δ_0-formulas are $\mathcal{L}_\varepsilon$-formulas defined inductively as follows:

(2.90) All atomic formulas are Δ_0-formulas. abs-1

(2.91) If φ and ψ are Δ_0-formulas then $\neg\varphi$ and $(\varphi \wedge \psi)$ are also Δ_0-fromulas. abs-2

(2.92) If φ is a Δ_0-formla then $(\exists x \mathrel{\varepsilon} y)\varphi$ is also a Δ_0-formula.[25] abs-3

We shall call also an $\mathcal{L}_\varepsilon$-formula φ which is logically equivalent to a Δ_0-formula in the sense above a Δ_0-formula.[26]

For an $\mathcal{L}_\varepsilon$-theory T an $\mathcal{L}_\varepsilon$-formula φ is said to be Δ_0^T-formula if there is a Δ_0-formula φ_0 which is equivalent to φ in T, that is, if $T \vdash (\varphi \leftrightarrow \varphi_0)$ holds. [27] Thus the Δ_0-formulas in the extended definition are just $\Delta_0^\emptyset$-formulas.

For a fragment T of ZFC, a transitive set M with $M \models T$ (a transitive set model M of T in other words) and a formula $\varphi = \varphi(x_0, ..., x_{n-1})$ (sometimes φ has also some parameters from M) is said to be absolute over M (for transitive models of T) if, for any transitive set model N of T with $M \subseteq N$, we have

(2.93) $M \models \varphi(a_0, ..., a_{n-1}) \;\Leftrightarrow\; N \models \varphi(a_0, ..., a_{n-1})$ for all $a_0, ..., a_{n-1} \in M$. absoluteness-0

φ is upward absolute over M (for transitive models of T) if, for any transitive set model N of T with $M \subseteq N$, we have

(2.94) $M \models \varphi(a_0, ..., a_{n-1}) \;\Rightarrow\; N \models \varphi(a_0, ..., a_{n-1})$ for all $a_0, ..., a_{n-1} \in M$. absoluteness-1

(24) We build T^* outside ZFC. The set theoretic notation used here is merely because of convenience.

(25) For an $\mathcal{L}_\varepsilon$-formula φ, $(\exists x \mathrel{\varepsilon} y)\varphi$ is an abbreviation for the formula $\exists x\,(x \mathrel{\varepsilon} y \wedge \varphi)$. We also use often the abbreviation $(\forall x \mathrel{\varepsilon} y)\,\varphi$ for $\forall x\,(x \mathrel{\varepsilon} y \rightarrow \varphi)$. Note that $(\forall x \mathrel{\varepsilon} y)\,\varphi$ is logically equivalent to the $\mathcal{L}_\varepsilon$-formula $\neg(\exists x \mathrel{\varepsilon} y)\neg\varphi$.

(26) Note that the notion of Δ_0-formulas in this extended definition is closed under $\varphi \;\mapsto\; (\forall x \mathrel{\varepsilon} y)\,\varphi$ (see footnote (25)).

(27) We assume that the deduction system of the first order logic we employ here satisfies always $T \vdash \varphi \Leftrightarrow T \vdash \exists x_0 \cdots \exists x_{n-1} \varphi(x_0, ..., x_{n-1})$ for any theory T and formula φ and variables $x_0, ..., x_{n-1}$. If $\forall \vec{x}\,\varphi$ is the $\forall$-closure of the formula φ, we denote with $M \models \varphi$ the assertion $M \models \forall \vec{x}\,\varphi$.

φ is downward absolute over M (for transitive models of T) if, for any transitive set model N of T with $M \subseteq N$, we have

$$(2.95) \quad M \models \varphi(a_0, ..., a_{n-1}) \Leftarrow N \models \varphi(a_0, ..., a_{n-1}) \text{ for all } a_0, ..., a_{n-1} \in M. \qquad \text{absoluteness-2}$$

L-forcing-2-0

Lemma 2.22 *Suppose that M is a transitive set and $M \models T$. for a fragment T of* ZFC*. Then any Δ_0^T-formula φ in $\mathcal{L}_\varepsilon$ is absolute over M (for transitive models of T).*

Proof. We can prove easily the absoluteness of Δ_0-formulas (i.e. absoluteness of $\Delta_0^\emptyset$-formulas over any transitive set M (for transitive sets = transitive models of $\emptyset$)) by indcution on the construction (2.90), (2.91), (2.92) of Δ_0-formulas.

Suppose that M is transitive and $M \models T$. Let $\varphi = \varphi(x_0, ..., x_{n-1})$ be a Δ_0^T-formula and $\varphi_0 = \varphi_0(x_0, ..., x_{n-1})$ a Δ_0-formula (in the strict sense) such that $T \vdash (\varphi \leftrightarrow \varphi_0)$. Then for any $a_0, ..., a_{n-1} \in M$, we have

$$(2.96) \quad \begin{aligned} M \models \varphi(a_0, ..., a_{n-1}) \quad &\Leftrightarrow \quad M \models \varphi_0(a_0, ..., a_{n-1}); \text{ by } M \models T; \\ &\Leftrightarrow \quad \varphi_0(a_0, ..., a_{n-1}) \quad ; \text{ by the first part of the lemma;} \\ &\Leftrightarrow \quad \varphi(a_0, ..., a_{n-1}) \quad ; \text{ since } T \text{ is a fragment of } \mathsf{ZFC}. \end{aligned} \qquad \text{forcing-18-0}$$

The assertion of the Lemma follows easily from this. ❑ (Lemma 2.22)

Suppose that T is an $\mathcal{L}_\varepsilon$-theory. A formula φ in $\mathcal{L}_\varepsilon$ is called a Σ_1^T-formula if it is equivalent to a formula of the form $\exists\vec{x}\varphi_0$ in T where φ_0 is a Δ_0-formula. An $\mathcal{L}_\varepsilon$-formula φ is called a Π_1^T-formula if it is equivalent to a formula of the form $\forall\vec{x}\varphi_0$ in T where φ_0 is a Δ_0-formula.

Exercise 2.23 *If an $\mathcal{L}_\varepsilon$-forumula φ is a Σ_1^T formula then $\neg\varphi$ is a Π_1^T-formula. If an $\mathcal{L}_\varepsilon$-forumula φ is a Π_1^T formula then $\neg\varphi$ is a Σ_1^T-formula.*

An $\mathcal{L}_\varepsilon$-formula φ is called a Δ_1^T-fromula if it is both Σ_1^T-formula and Π_1^T-formula. If $T = \emptyset$ then we also write simply Σ_1, Π_1, Δ_1 instead of $\Sigma_1^\emptyset$, $\Pi_1^\emptyset$, $\Delta_1^\emptyset$.

L-forcing-2-1

Lemma 2.24 *Suppose that T is a fragment* ZFC *and M a transitive set with $M \models T$. Then any Δ_1^T-formula is absolute over M. (1) Any Σ_1^T-formula is upward absolute over M (for transitive models of T).*

(2) *Any Π_1^T-formula is downward absolute over M (for transitive models of T).*

(3) *Any Δ_1^T-formula is absolute over M (for transitive models of T).*

Proof.

❑ (Lemma 2.24)

A class $\mathbf{A}$ is said to be absolute over M if $\mathbf{A} = \{a : \varphi(a)\}$ and the firmula φ is absolute over M.

For a class $\mathbf{A} = \{a : \varphi(a)\}$, we denote $\mathbf{A}^M = \{a \in M : M \models \varphi(a)\}$. Thus the class $\mathbf{A}$ is absolute over M if and only if $\mathbf{A}^M = \mathbf{A} \cap M$. We say that a class function $\mathbf{F} : \mathsf{V} \to \mathsf{V}$ is absolute over M if and only if $\mathbf{F}^M : M \to M$ and $\mathbf{F}^M = \mathbf{F} \restriction M$.

L-forcing-2-2

Lemma 2.25 *Suppose that T is a sufficiently large fragment of* ZFC.

(1) *"$x \,\varepsilon\, trcl(y)$" is a Δ_1^T-class relation.*

(2) *"$x \equiv trcl(y)$" is a Δ_1^T-class relation.*

Proof. (1): We have

$$(2.97)\quad T \vdash \text{“}x \,\varepsilon\, trcl(y)\text{”} \;\leftrightarrow\; \exists f \exists n\,(\text{“}n \,\varepsilon\, \omega\text{”} \;\wedge\; \mathrm{dom}(f) \equiv n \\ \wedge\; (\forall m \,\varepsilon\, n)(\forall \ell \,\varepsilon\, m)(m \equiv \ell + 1 \to f(\ell) \,\varepsilon\, f(m)) \\ \wedge\; f(0) \equiv x \;\wedge\; (\exists m \,\varepsilon\, n) f(m) \equiv y)).$$

Note that "$n \,\varepsilon\, \omega$" is Δ_0: it can be formulated as "n is transitive and ε is a linear ordering on n with a maximal element and without any limit point" ("n is transitive" is also Δ_0. See below).

On the other hand we also have

$$(2.98)\quad T \vdash \text{“}x \,\varepsilon\, trcl(y)\text{”} \;\leftrightarrow\; \forall z\,((\text{“}z \text{ is transitive”} \;\wedge\; y \,\varepsilon\, z) \;\to\; x \,\varepsilon\, z).$$

Note that "z is transitive" is Δ_0^T: it can be formulated as $(\forall u \,\varepsilon\, z)(\forall v \,\varepsilon\, u)(v \,\varepsilon\, z)$.

(2):

$$(2.99)\quad T \vdash \text{“}x \equiv trcl(y)\text{”} \;\leftrightarrow \exists f \exists w \cdots (\text{“}w \equiv \omega\text{”} \;\wedge\; \mathrm{dom}(f) \equiv w \;\wedge\; \text{”}f(0) \equiv \{y\}\text{”} \\ \wedge\; (\forall \ell, m \,\varepsilon\, w)\,(\text{“}\ell + 1 \equiv m \to f(m) \equiv \bigcup f(\ell)\text{”}) \\ \wedge\; \text{“}x \equiv \bigcup f''w\text{”}).$$

Also

$$(2.100)\quad T \vdash \text{“}x \equiv trcl(y)\text{”} \;\leftrightarrow \forall f \forall w \cdots ((\text{“}w \equiv \omega\text{”} \;\wedge\; \mathrm{dom}(f) \equiv w \;\wedge\; \text{”}f(0) \equiv \{y\}\text{”} \\ \wedge\; (\forall \ell, m \,\varepsilon\, w)\,(\text{“}\ell + 1 \equiv m \to f(m) \equiv \bigcup f(\ell)\text{”})) \\ \to\; \text{“}x \equiv \bigcup f''w\text{”}).$$

□ (Lemma 2.25)

L-forcing-2-3

Lemma 2.26 *Suppose that T is a sufficiently large fragment of* ZFC *and M is a transitive set with $M \models T$. If $\mathbf{X}$ is a class absolute over M and $\mathbf{G}$ is a Σ_1^T-class with*

$$(2.101)\quad T \vdash \text{“}\mathbf{G} : \mathbf{X} \to \mathsf{V},$$ forcing-18-1

then $\mathbf{G}$ is absolute over M.

Proof. Let $\varphi_0 = \varphi_0(x_0, ..., x_{m-1}, y_0, ..., y_{n-1}, y)$ be a Δ_0^T-formula such that

$$(2.102)\quad T \vdash \mathbf{G}(y_0, ..., y_{n-1}) \equiv y \;\leftrightarrow\; \exists x_0 \cdots \exists x_{n-1}\, \varphi_0.$$ forcing-18-2

For $b_0, ..., b_{n-1} \in \mathbf{X}^M$ $(= \mathbf{X} \cap M)$ and $b \in M$, assume that $M \models \mathbf{G}(b_0, ..., b_{n-1}) \equiv b$. Then we have

$$(2.103)\quad M \models \exists x_0 \cdots \exists x_{m-1}\, \varphi_0(x_0, ..., x_{m-1}, b_0, ..., b_{n-1}, b).$$

Let $a_0, ..., a_{m-1} \in M$ be such that

$$(2.104)\quad M \models \varphi_0(a_0, ..., a_{m-1}, b_0, ..., b_{n-1}, b).$$

By Lemma 2.22, it follows that $\varphi_0(a_0, ..., a_{m-1}, b_0, ..., b_{n-1}, b)$ holds. Hence $\exists x_0 \cdots \exists x_{m-1}\, \varphi_0(x_0, ..., x_{m-1}, b_0, ..., b_{n-1}, b)$. That is, $\mathbf{G}(b_0, ..., b_{n-1}) = b$.

Assume now that $\mathbf{G}(b_0, ..., b_{n-1}) = b$ for $b_0, ..., b_{n-1} \in \mathbf{X}^M$ and $b \in M$. By (2.101) and since $M \models T$, there is $b' \in M$ such that $M \models \mathbf{G}(b_0, ..., b_{n-1}) \equiv b'$. By the argument as above it follows that $\mathbf{G}(b_0, ..., b_{n-1}) = b'$. Thus $b = b'$ and $M \models \mathbf{G}(b_0, ..., b_{n-1}) \equiv b$.

□ (Lemma 2.26)

T-forcing-2

Theorem 2.27 *Suppose that T is a sufficiently large fragment of* ZFC*, $\mathbf{X}$ a Δ_1^T-class and $\mathbf{R}$ a set-like well-founded relation on $\mathbf{X}$ such that both "$x \,\varepsilon\, trcl_{\mathbf{R}}(y)$" and "$x \equiv trcl_{\mathbf{R}}(y)$" are Δ_1^T-class relations. Suppose further that T proves that $\mathbf{G} : \mathcal{F} \times \mathbf{X} \to \mathsf{V}$ and $\mathbf{G}$ is Σ_1^T-class function. Then the mapping $\mathbf{F} : \mathbf{X} \to \mathsf{V}$ defined by* (2.19) *in Theorem 2.5,* (2) *for these $\mathbf{X}$ and $\mathbf{R}$, $\mathbf{G}$ is Δ_1^T.*

Proof. We have

$$(2.105)\quad T \vdash \mathbf{F}(x) \equiv y \;\leftrightarrow\; x \,\varepsilon\, \mathbf{X} \wedge \exists f \exists u \exists v\, (v \equiv \mathrm{dom}(f) \wedge v \equiv trcl_{\mathbf{R}}(u) \wedge x \,\varepsilon\, \mathrm{dom}(f) \wedge \forall (w \,\varepsilon\, v) f(w) \equiv \mathbf{G}(f \restriction trcl^-_{\mathbf{R}}(w), w) \wedge f(x) \equiv y).$$

On the other hand, we also have:

$$(2.106)\quad T \vdash \mathbf{F}(x) \equiv y \;\leftrightarrow\; x \,\varepsilon\, \mathbf{X} \wedge \forall f \forall u \forall v\, ((v \equiv \mathrm{dom}(f) \wedge v \equiv trcl_{\mathbf{R}}(u) \wedge x \,\varepsilon\, \mathrm{dom}(f) \wedge \forall (w \,\varepsilon\, v) f(w) \equiv \mathbf{G}(f \restriction trcl^-_{\mathbf{R}}(w), w)) \to f(x) \equiv y).$$

□ (Theorem 2.27)

2.6 Infinitary combinatorics

inf-comb

The method of elementary submodels is a very powerful tool to prove theorems in infinitary combinatorics in a quite uniform way. One of the advantages of the method is that we see often directly the meaning of the complex set of set-theoretic conditions involved in the assertion of the combinatorial statements in the proof using the method of elementary submodels.

We use this method in this section to prove the Delta System Lemma and its generalizations as well as some theorems concerning ststionary sets.

A set $\mathcal{F}$ is said to be a Δ-system with the root r if for any two distinct a, $b \in \mathcal{F}$, we have $a \cap b = r$.

Theorem 2.28 (Δ-System Lemma) *For an uncountable regular cardinal κ and a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of finite sets, there are $I \in [\kappa]^\kappa$.*[28] *and a set r, such that $\{a_\alpha : \alpha \in I\}$ is a Δ-system with the root r.* P-inf-comb-0

Proof. Without loss of generality, we may assume that $a_\alpha \in [\kappa]^{<\aleph_0}$ for all $\alpha < \kappa$[29].

Let θ be a regular cardinal such that $\langle a_\alpha : \alpha < \kappa\rangle \in \mathcal{H}(\theta)$. By Lemma 2.15, there is an $M \prec \mathcal{H}(\theta)$ (that is, $\langle M, \in\rangle \prec \langle \mathcal{H}(\theta), \in\rangle$) such that

(2.107) $|\,M\,| < \kappa$; inf-comb-a-0

(2.108) $\langle a_\alpha : \alpha < \kappa\rangle \in M$; and inf-comb-a-1

(2.109) $\kappa \cap M < \kappa$. inf-comb-a-2

Let $r = a_{\alpha^*} \cap M$. Since r is finite and $r \subseteq M$, we have $r \in M$ by elementarity[30].

Cl-inf-comb-1

Claim 2.28.1 $\mathcal{H}(\theta) \models \forall\alpha < \kappa\exists\beta < \kappa\,(\alpha < \beta \wedge a_\beta \cap \beta \equiv r)$.

$\vdash$ By elementarity of M, it is enough to prove that $M \models \forall\alpha < \kappa\exists\beta < \kappa\,(\alpha < \beta \wedge a_\beta \cap \beta \equiv r)$ holds.

Suppose that $\alpha \in M$ is such that $M \models \alpha_0 < \kappa$. Then $\alpha_0 < \alpha^*$. We have $\mathcal{H}(\theta) \models \alpha_0 < \alpha^* \wedge a_{\alpha^*} \cap \alpha^* = r$. Thus $\mathcal{H}(\theta) \models \exists < \kappa(\alpha_0 < \beta \wedge a_\beta \cap \beta = r)$. By elementarity of M, it follows that $M \models \exists < \kappa(\alpha_0 < \beta \wedge a_\beta \cap \beta = r)$ $\dashv$ (Claim 2.28.1)

By Claim 2.28.1, we can construct inductively a strictly increasing sequence ξ_α, $\alpha < \kappa$ in κ in $\mathcal{H}(\theta)$ (or in V) such that

(2.110) $\xi_\alpha > \sup\{\sup(a_{\xi_\beta}) : \beta < \alpha\}$ and

(2.111) $a_{\xi_\alpha} \cap \xi_\alpha = r$.

$I = \{\xi_\alpha : \alpha < \kappa\}$ is then as desired. ❑ (Theorem 2.28)

Theorem 2.29 (Generalized Δ-System Lemma) *Suppose that κ and λ with $\kappa < \lambda$ are regular cardinals such that* P-inf-comb-1

(2.112) *for all $\alpha < \lambda$ we have $|\,[\alpha]^{<\kappa}\,| < \lambda$.*[31] inf-comb-0

If $\langle a_\alpha : \alpha < \lambda\rangle$ is a sequence of sets of size $< \kappa$, then there is $I \in [\lambda]^\lambda$ and r such that $\{a_\alpha : \alpha \in I\}$ is a Δ-system with the root r.

---

(28) We use the notation: $[X]^\mu = \{x \subseteq X : |\,x\,| = \mu\}$ and $[X]^{<\mu} = \{x \subseteq X : |\,x\,| < \mu\}$.

(29) We may assume this since $|\bigcup\{a_\alpha : \alpha < \kappa\}| \leq \kappa$ holds.

(30) Suppose that r has n elements. We have $\mathcal{H}(\theta) \models \forall x_0 \cdots \forall x_{n-1}\exists y\forall z\,(z \mathrel{\varepsilon} y \leftrightarrow \bigvee\!\!\!\bigvee_{i<n} z \equiv x_i)$. By elementarity of M (i.e., by $M \prec \mathcal{H}(\theta)$), we also have $M \models \forall x_0 \cdots \forall x_{n-1}\exists y\forall z\,(z \mathrel{\varepsilon} y \leftrightarrow \bigvee\!\!\!\bigvee_{i<n} z \equiv x_i)$.

(31) If λ satisfies (2.112), we say that λ is v $< \kappa$-incaccessible.

Remark. If $\kappa = \omega$ then $|\,[\alpha]^{<\omega}\,| = |\,\alpha\,| + \aleph_0 < \lambda$ holds for all $\alpha < \lambda$ and thus the condition (2.112) is always satisfied. Thus Theorem 2.29 is actually a generalization of Theorem 2.28.

Proof. This theorem can be proved similarly to Theorem 2.28 using Lemma 2.16 in place of Lemma 2.15. $\square$ (Theorem 2.29)

3 Generic filters and generic extensions

generic

In this section, we shall see some more technical details of the construction of M^* and M^{**} of Section 1.

In ZFC, a set $\mathbb{P}$ with a binary relation $\leq_{\mathbb{P}}$ is said to be a *preordering* if $\leq_{\mathbb{P}}$ is transitive and reflexive, i.e.

(3.1) $p \leq_{\mathbb{P}} q$ and $q \leq_{\mathbb{P}} r$ imply $p \leq_{\mathbb{P}} r$ for all p, q, $r \in \mathbb{P}$; po-1

(3.2) $p \leq_{\mathbb{P}} p$ for all $p \in \mathbb{P}$. po-2

A preordering $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a *poset* (or a *forcing notion*) if $\mathbb{P}$ has a greatest element(32), i.e. such an element $p \in \mathbb{P}$ that, for all $q \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ holds. For a poset $\mathbb{P}$ we fix a maximal element and denote it by $\mathbb{1}_{\mathbb{P}}$. We also write $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ to indicate explicitly the preordering and the specified greatest element of $\mathbb{P}$.

A subset $\mathbb{G}$ of a poset $\mathbb{P}$ is a *filter* if

(3.3) $\mathbb{1}_{\mathbb{P}} \in \mathbb{G}$; po-3

(3.4) For any $p, q \in \mathbb{P}$, if $p \in \mathbb{G}$ and $p \leq_{\mathbb{P}} q$, then $q \in \mathbb{P}$; po-4

(3.5) For any p, $q \in \mathbb{G}$, there is $r \in \mathbb{G}$ such that $r \leq_{\mathbb{P}} p$, q. po-5

A subset D of a poset $\mathbb{P}$ is said to be *dense* if, for any $p \in \mathbb{P}$, there is some $q \in D$ such that $q \leq_{\mathbb{P}} p$.

We fix a sufficiently large finite fragment ZFC_0 of ZFC. Actually we can fix ZFC_0 only when we have written down all the following arguments in this article. This is like labels in a LaTeX file, whose value is set to be "???" in the first run of LaTeX on the source file. In the second run the values are fixed as some sequence of symbols.(33)

In any case what we can produce as a mathematical theory is a finite object so that after we have written down a discourse we can fix ZFC_0 as a sufficiently large finite set containing all the axioms of ZFC which were used in the proofs in the present manuscript and proofread the written material from the beginning again to check if it makes sense.

(32) We have to talk about "a" greatest element since, by the absence of anti-symmetry of $\leq_{\mathbb{P}}$, there may be more than one greatest elements.

(33) The situation with LaTeX is actually worse since, for each natural number n, the is a LaTeX source file which we have to compile more than n times to get the correct final dvi file (Exercise).

Suppose now that M is a countable transitive set with $\langle M, \in \rangle \models \mathsf{ZFC}_0$ [(34)] and $\mathbb{P} \in M$ is a poset with $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$.[(35)]

Note that for a transitive M, $\mathbb{P} \in M$ is a poset if and only if $M \models$ "$\mathbb{P}$ is a poset". A filter $\mathbb{G} \subseteq \mathbb{P}$ is said to be $(M, \mathbb{P})$-generic if, for any dense $D \subseteq \mathbb{P}$ with $D \in M$, we have $\mathbb{G} \cap D \neq \emptyset$.

In most of the cases, an $(M, \mathbb{P})$-generic filter is not an element of M. To formulate this more precisely, we need the following definitions. For a poset $\mathbb{P}$ and p, $q \in \mathbb{P}$, p and q are *compatible* (in $\mathbb{P}$, notation: $p \top_{\mathbb{P}} q$) if there is $r \in \mathbb{P}$ such that $r \leq_{\mathbb{P}} p$, q. Otherwise p and q are said to be *incompatible* (in $\mathbb{P}$, notation: $p \perp_{\mathbb{P}} q$). A poset $\mathbb{P}$ is said to be *atomless* if, for any $p \in \mathbb{P}$, there are q, $q' \in \mathbb{P}$ such that q, $q' \leq_{\mathbb{P}} p$ and, q and q' are incompatible.

P-atomless-0

Lemma 3.1 *For a transitive model M of ZFC_0 and a poset $\mathbb{P} \in M$, if $\mathbb{P}$ is atomless and $\mathbb{G}$ is a $(\mathbb{P}, M)$-generic filter, then $\mathbb{G} \notin M$.*

Proof. Since M is transitive we have $\mathbb{P} \subseteq M$. Suppose that $\mathbb{G} \subseteq \mathbb{P} \subseteq M$ is a filter and $\mathbb{G} \in M$. We show that $\mathbb{G}$ is not $(M, \mathbb{P})$-generic. Let $D = \mathbb{P} \setminus \mathbb{G}$. Then we have $D \in M$[(36)]. D is dense in $\mathbb{P}$: For any $p \in \mathbb{P}$, let q, $r \in \mathbb{P}$ be such that q, $r \leq_{\mathbb{P}} p$ and, q and r are incompatible. By (3.5), it is impossible that both of q and r are in $\mathbb{G}$. Say $q \notin \mathbb{G}$. Then we have $q \leq_{\mathbb{P}} p$ and $q \in D$.

Thus $\mathbb{G}$ is not $(M, \mathbb{P})$-generic since $\mathbb{G} \cap D = \emptyset$. ❏ (Lemma 3.1)

An $(M, \mathbb{P})$-generic filter does exist for a countable transitive model M of ZFC_0:

existence-of-generic

Lemma 3.2 *If M is a countable transitive model of ZFC_0, then for any poset $\mathbb{P} \in M$ and $p \in \mathbb{P}$ there is an $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $p \in \mathbb{G}$.*

Proof. Since M is countable the set $\mathbf{D} = \{D \in M : D \text{ is a dense subset of } \mathbb{P}\}$ is countable as well. Let $\mathbf{D} = \{D_n : n \in \omega\}$. Let $\langle p_n : n \in \omega \rangle$ be a descending sequence (with respect to $\leq_{\mathbb{P}}$) of elements of $\mathbb{P}$ such that $p_0 \leq_{\mathbb{P}} p$ and $p_n \in D_n$ for all $n \in \omega$. The construction of such a sequence is possible since each D_n $(n \in \omega)$ is dense in $\mathbb{P}$.

Let

$$(3.6) \qquad \mathbb{G} = \{p \in \mathbb{P} : p_n \leq_{\mathbb{P}} p \text{ for some } n \in \omega\}.$$

Then $\mathbb{G}$ is a $(M, \mathbb{P})$-generic filter. ❏ (Lemma 3.2)

For a poset $\mathbb{P}$, we define the class $\mathsf{V}^{\mathbb{P}}$ recursively[(37)] by

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(34) If M is a countable transitive set with $\langle M, \in \rangle \models T$, we shall also say that M is a countable transitive model of T.

(35) If we say $\mathbb{P} \in M$ for a poset with $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$, then we mean $\mathbb{P} \in M$, $\leq_{\mathbb{P}} \in M$ and $\mathbb{1}_{\mathbb{P}} \in M$ where the last condition follows from the first if M is transitive.

(36) Here, we assume that the instance of the Axiom of Separation, which is needed to prove that $D = \{p \in \mathbb{P} : p \notin \mathbb{G}\}$ is a set, is in ZFC_0.

(3.7) $\quad \underset{\sim}{x} \in \mathsf{V}^{\mathbb{P}} \;\Leftrightarrow\;$ all elements of $\underset{\sim}{x}$ are of the form $\langle \underset{\sim}{y}, p\rangle$ where $\underset{\sim}{y} \in \mathsf{V}^{\mathbb{P}}$ and $p \in \mathbb{P}$. po-6

Elements of $\mathsf{V}^{\mathbb{P}}$ are called $\mathbb{P}$-names. As we already did above, $\mathbb{P}$-names are denoted by alphabets with undertilde[38].

For a transitive model M of ZFC_0, and a poset $\mathbb{P} \in M$, let

(3.9) $\quad M^{\mathbb{P}} = (\mathsf{V}^{\mathbb{P}})^M = \{x \in M \,:\, M \models x \text{ is a } \mathbb{P}\text{-name}\}.$ po-7

By the definition of $\mathbb{P}$-names, the statement "x is a $\mathbb{P}$-name" is *absolute* over M, i.e., for any $x \in M$, x is a $\mathbb{P}$-name if and only if $M \models x$ is a $\mathbb{P}$-name. [39] Thus we have $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$.

For a poset $\mathbb{P}$ and a filter $\mathbb{G} \subseteq \mathbb{P}$, we define the interpretation $\underset{\sim}{x}^{\mathbb{G}}$ of a $\mathbb{P}$-name $\underset{\sim}{x}$ under $\mathbb{G}$ recursively[40] by:

(3.11) $\quad \underset{\sim}{x}^{\mathbb{G}} = \{\underset{\sim}{y}^{\mathbb{G}} \,:\, \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x} \text{ for some } p \in \mathbb{G}\}.$ po-7-0

For a transitive M, a poset $\mathbb{P} \in M$ and a filter $\mathbb{G} \subseteq M$, let

(3.12) $\quad M[\mathbb{G}] = \{\underset{\sim}{x}^{\mathbb{G}} \,:\, \underset{\sim}{x} \in M^{\mathbb{P}}\}.$ po-8-0

$M[\mathbb{G}]$ is called the generic extension of M by $\mathbb{G}$.

For a set x let $\check{x}$ be the $\mathbb{P}$-name defined recursively by

(3.13) $\quad \check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle \,:\, y \in x\}.$ [41] po-xo9-0

For a poset $\mathbb{P}$, let

(3.14) $\quad \underset{\sim}{\mathbb{G}}_{\mathbb{P}} = \{\langle \check{p}, p\rangle \,:\, p \in \mathbb{P}\}.$ po-10-0

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[37] The recursive definition can be accomplished by applying Theorem 2.5 (2) to define the function $\mathbf{F}$ which should become the characteristic function of $\mathsf{V}^{\mathbb{P}}$. A $\in$-recursion on the set-like well-founded relation $\in$ will do. Note that if $\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}$ then $\underset{\sim}{y} \in trcl^-(\underset{\sim}{x})$.

More precisely, in terms of Theorem 2.5, (2), we let $\mathbf{X} = \mathsf{V}$ and $\mathbf{R} = \{\langle x, y\rangle \,:\, x \in y\}$. For class function $\mathbf{G} : \mathcal{F} \times \mathbf{X} \to 2$ defined by

(3.8) $\quad \mathbf{G}(f, x) = 1 \;\Leftrightarrow\; \operatorname{dom}(f) = trcl^-(x)$, and
for all $y \in x$ there are y' and $p \in \mathbb{P}$ such that $f(y') = 1$ and $y = \langle y', p\rangle$

for all $\langle f, x\rangle \in \mathcal{F} \times \mathbf{X}$. Then, for the class function $\mathbf{F}$ as in Theorem 2.5, (2), for this $\mathbf{G}$, $\mathsf{V}^{\mathbb{P}}$ can be defined to be $\{x \,:\, \mathbf{F}(x) = 1\}$.

[38] There are also texts in which $\mathbb{P}$-names are denoted by dotted alphabets, like $\dot{x}$, $\dot{y}$, $\dot{z}$, etc.

[39] Actually, "x is a $\mathbb{P}$-name" is $\Delta_1^{\mathsf{ZFC}_0}$, see Theorem 2.27. Again we assume here that all instances of the Axiom of Separation needed to make this hold are included in ZFC_0.

[40] This recursion can be carried out by letting $\mathbf{X} = \mathsf{V}^{\mathbb{P}}$, $\mathbf{R} = \in \cap (\mathsf{V}^{\mathbb{P}})^2$ and defining $\mathbf{G} : \mathcal{F} \times \mathsf{V}^{\mathbb{P}} \to \mathsf{V}$ as in Theorem 2.5, (2) by

$$(3.10) \quad \mathbf{G}(f, x) = \begin{cases} y & \text{if } \operatorname{dom}(f) = trcl^-(x) \cap \mathsf{V}^{\mathbb{P}} \text{ and } y = \{f(z) \,:\, \text{there is } p \in \mathbb{G} \text{ such that } \langle z, p\rangle \in x\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The class function $\mathbf{F}$ in Theorem 2.5, (2) for this $\mathbf{G}$ gives the mapping $\underset{\sim}{x} \mapsto \underset{\sim}{x}^{\mathbb{G}}$ satifsying (3.11).

[41] Note that $\check{\emptyset} = \emptyset$.

Clearly $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ is a $\mathbb{P}$-name. If $\mathbb{P} \in M$ then $\underset{\sim}{\mathbb{G}}_{\mathbb{P}} \in M^{\mathbb{P}}$. We shall simply write $\underset{\sim}{\mathbb{G}}$ instead of $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$, if it is clear which poset $\mathbb{P}$ is meant.

L-0

Lemma 3.3 (1) *For any filter $\mathbb{G}$ on a poset $\mathbb{P}$, and any set x, we have $(\check{x})^{\mathbb{G}} = x$.*

(2) *For any filter $\mathbb{G}$ on a poset $\mathbb{P}$, we have $\underset{\sim}{\mathbb{G}}^{\mathbb{G}} = \mathbb{G}$.*

Proof. (1): By $\in$-induction on x: We have to show that if $(\check{y})^{\mathbb{G}} = y$ for all $y \in x$ then $(\check{x})^{\mathbb{G}} = x$.

So suppose that $(\check{y})^{\mathbb{G}} = y$ for all $y \in x$. By (3.11) and since $\mathbb{1}_{\mathbb{P}} \in \mathbb{G}$ for any filter $\mathbb{G}$, we have $\check{x}^{\mathbb{G}} = \{\check{y}^{\mathbb{G}} : y \in x\} = \{y : y \in x\} = x$.

(2): By (3.11), (3.14) and (1), we have

$$\underset{\sim}{\mathbb{G}}^{\mathbb{G}} = \{\check{p}^{\mathbb{G}} : p \in \mathbb{G}\} = \{p : p \in \mathbb{G}\} = \mathbb{G}. \tag{3.15}$$

❑ (Lemma 3.3)

L-1

Lemma 3.4 *Suppose that M is a transitive model of ZFC_0 and $\mathbb{P} \in M$ is a poset. Suppose further that $\mathbb{G}$ is a filter on $\mathbb{P}$. Then:*

(1) *$M[\mathbb{G}]$ is a transitive set.*

(2) *$M \subseteq M[\mathbb{G}]$ and $\mathbb{G} \in M[\mathbb{G}]$.*

(3) *For all $\underset{\sim}{x} \in M^{\mathbb{P}}$, we have $rank(\underset{\sim}{x}^{\mathbb{G}}) \leq rank(\underset{\sim}{x})$.*

(4) $\mathsf{On}^{M[\mathbb{G}]} = \mathsf{On} \cap M[\mathbb{G}] = \mathsf{On} \cap M = \mathsf{On}^M$.

(5) $|\,M[\mathbb{G}]\,| = |\,M\,|$.

Proof. (1): Suppose that $x \in M[\mathbb{G}]$. By the definition (3.12) of $M[\mathbb{G}]$, there is a $\mathbb{P}$-name $\underset{\sim}{x} \in M$ such that $x = \underset{\sim}{x}^{\mathbb{G}}$. If $y \in x$ then by (3.11), $y = \underset{\sim}{y}^{\mathbb{G}}$ for some $\underset{\sim}{y} \in \mathsf{V}^{\mathbb{P}}$ and $p \in \mathbb{G}$ such that $\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}$. Since $\underset{\sim}{y} \in trcl(\underset{\sim}{x}) \subseteq M$ by the transitivity of M, it follows that $\underset{\sim}{y} \in M$ and thus $y \in M[\mathbb{G}]$.

(2): For all $x \in M$ $\check{x} \in M$ by $M \models \mathsf{ZFC}_0$. Thus $x = \check{x}^{\mathbb{G}} \in M[\mathbb{G}]$ for all $x \in M$. Since $M \models \mathsf{ZFC}_0$ and $\mathbb{P} \in M$, we have $\underset{\sim}{\mathbb{G}}_{\mathbb{P}} \in M$ and thus $\mathbb{G} = (\underset{\sim}{\mathbb{G}}_{\mathbb{P}})^{\mathbb{G}} \in M[\mathbb{G}]$.

(3): This can be proved by induction of $rank(\underset{\sim}{x})$. Suppose that the inequality is established for all names $\underset{\sim}{y}$ with $rank(\underset{\sim}{y}) < rank(\underset{\sim}{x})$. Then

$$\begin{aligned} rank(\underset{\sim}{x}^{\mathbb{G}}) &= \sup\{rank(\underset{\sim}{y}^{\mathbb{G}}) + 1 : \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x} \text{ and } p \in \mathbb{G}\}\ . \\ &\leq \sup\{rank(\underset{\sim}{y}) + 1 : \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x} \text{ and } p \in \mathbb{G}\} \\ &\leq \sup\{rank(\underset{\sim}{y}) + 1 : \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}\} \\ &\leq rank(\underset{\sim}{x}) \end{aligned} \tag{3.16}$$

(4): Since On is Δ_0 and, M and $M[\mathbb{G}]$ are transitive, we have $\mathsf{On}^M = \mathsf{On} \cap M$ and $\mathsf{On}^{M[\mathbb{G}]} = \mathsf{On} \cap M[\mathbb{G}]$ by Lemma 2.22.

Note that $\mathsf{On} \cap M$, $\mathsf{On} \cap M[\mathbb{G}] \in \mathsf{On}$ since M and $M[\mathbb{G}]$ are transitive (for $M[\mathbb{G}]$ this is shown in (1)).

Since $M \subseteq M[\mathbb{G}]$ by (2), we have $\mathsf{On} \cap M \subseteq \mathsf{On} \cap M[\mathbb{G}]$. Thus $\mathsf{On} \cap M \leq \mathsf{On} \cap M[\mathbb{G}]$.

For $\alpha \in M[\mathbb{G}]$, if $\alpha = \underset{\sim}{x}^{\mathbb{G}}$ for a $\underset{\sim}{x} \in M^{\mathbb{P}} \subseteq M$, then by (3) and since $M \models \mathsf{ZFC}_0$, we have

$$\alpha = \mathit{rank}(\alpha) = \mathit{rank}(\underset{\sim}{x}^{\mathbb{G}}) \leq \mathit{rank}(\underset{\sim}{x}) < \mathsf{On} \cap M. \tag{3.17}$$

Thus $\mathsf{On} \cap M[\mathbb{G}] \leq \mathsf{On} \cap M$.

(5): $|\,M[\mathbb{G}]\,| \geq |\,M\,|$ by (2). On the other hand, $|\,M[\mathbb{G}]\,| \leq |\,M^{\mathbb{P}}\,| \leq |\,M\,|$ by $M^{\mathbb{P}} \subseteq M$.

❑ (Lemma 3.4)

To prove the next two Theorems we need a very much sophisticated and profound technical tool called "forcing relation" which will be introduced in the following section. For now we assume these Theorem and going to see how the ideas sketched in Section 1 can be accomplished.

generic-ZFC

Theorem 3.5 *If T is a finite fragment of* ZFC*, there is a large enough fragment T^* of* ZFC *containing T such that, if M is a transitive model of T^*, $\mathbb{P} \in M$ a poset and $\mathbb{G}$ an $(M, \mathbb{P})$-generic filter, Then $M[\mathbb{G}] \models T$.* ❑

A poset $\mathbb{P}$ is said to satisfy the ccc (countable chain condition) if any pairwise incompatible subset[(42)] $A \subseteq \mathbb{P}$ is countable.

ccc

Theorem 3.6 *For a finite fragment T of* ZFC *containing* ZFC_0 *and T^* as in Theorem 3.5, suppose that M is a transitive model of T^*, $\mathbb{P} \in M$ is a poset with*

$$M \models \text{“}\mathbb{P} \textit{ satisfies the ccc”} \tag{3.18}$$

and $\mathbb{G}$ an $(M, \mathbb{P})$-generic filter. Then we have $\mathrm{Card}^M = \mathrm{Card}^{M[\mathbb{G}]}$. [(43)] ❑

For $\kappa \in \mathsf{On}$, we regard

$$\mathrm{Fn}(\kappa, 2) = \{p \,:\, p : x \to 2 \text{ for some } x \in [\kappa]^{<\aleph_0}\}. \tag{3.19}$$ [(44)]

---

(42) $A \subseteq \mathbb{P}$ is said to be pairwise incompatible if any distinct $p, q \in \mathbb{P}$ are incompatible in $\mathbb{P}$. A pairwise incompatible subset A of a poset $\mathbb{P}$ is also called an antichain in $\mathbb{P}$.

(43) For a class $\mathbf{C}$ introduced by an $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x)$ as $\mathbf{C} = \{x \,:\, \varphi(x)\}$ we denote with $\mathbf{C}^M$ the set $\{x \in M \,:\, M \models \varphi(x)\}$. For a transitive M if $M \models$ "α is an ordinal then α is always really an ordinal. However, in general, $M \models$ "α is a cardinal" does not imply that α is a cardinal. In fact for large enough fragment T, if M is a countable model of ZFC_0 then M thinks there is $\aleph_1$. But what M thinks is $\aleph_1$ $(= (\aleph_1)^M)$ is actually a countable ordinal since $\mathsf{On} \cap M$ is a countable ordinal and hence also $(\aleph_1)^M \in \mathsf{On} \cap M$.

as a partial ordering with the order $\leq$ defined by

$$(3.20) \quad p \leq q \;\Leftrightarrow\; p \subseteq q$$

for all p, $q \in \mathrm{Fn}(\kappa, 2)$. $\emptyset$ is the maximal element of this partially ordered set and thus we obtain the poset

$$(3.21) \quad (\mathrm{Fn}(\kappa, 2), \leq, \emptyset)$$

which we also denote simply by $\mathrm{Fn}(\kappa, 2)$. The proof of the following Lemma is not very much involved but we shall also postpone its proof.

cohen-ccc

Lemma 3.7 *For any $\kappa \in \mathsf{On}$ the poset $\mathrm{Fn}(\kappa, 2)$ satisfies the ccc.* ❑

L-2

Lemma 3.8 *Suppose that M is a transitive model of ZFC_0 and $\kappa \in \mathsf{On} \cap M$ $(= \mathsf{On}^M)$ is a limit ordinal. Let $\mathbb{P}$ be the poset $\mathrm{Fn}(\kappa, 2)$. Then for any $(M, \mathbb{P})$-generic filter $\mathbb{G}$, letting $g = \bigcup \mathbb{G}$, we have*

(1) *$g : \kappa \to 2$;*

(2) *Letting $a_\gamma = \{n \in \omega : g(\gamma + n) = 1\}$ for all limit ordinal $\gamma < \kappa$, a_γ, $\gamma \in \mathrm{Lim} \cap \kappa$ are pairwise distinct.*

Proof. (1): Note that $\kappa \subseteq M$. $\bigcup \mathbb{G}$ is a mapping since any p, $q \in \mathbb{G}$ are compatible ($\Leftrightarrow$ compatible as functions).

For an arbitrary $\alpha \in \kappa$, the set

$$(3.22) \quad D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\}$$

is dense in $\mathbb{P}$ and $D \in M$.[45] Hence, by genericity of $\mathbb{G}$, there is $p \in \mathbb{G} \cap D$. Thus $\alpha \in \mathrm{dom}(p) \subseteq \mathrm{dom}(g)$.

(2): Let γ, $\gamma' \in \mathrm{Lim} \cap \kappa$ with $\gamma \neq \gamma'$. We show that $a_\gamma \neq a_{\gamma'}$. Let

$$(3.23) \quad D_{\gamma,\gamma'} = \{p \in \mathbb{P} : \text{there is } n \in \omega \text{ such that } \gamma + n,\ \gamma' + n \in \mathrm{dom}(p) \text{ and } p(\gamma + n) \neq p(\gamma' + n).\}$$

It is easy to see that $D_{\gamma,\gamma'}$ is dense in $\mathbb{P}$ and $D_{\gamma,\gamma'} \in M$. By genericity of $\mathbb{G}$, there is $p \in \mathbb{G} \cap D_{\gamma,\gamma'}$. Since $p \subseteq \bigcup \mathbb{G} = g$, it follows that a_γ and $a_{\gamma'}$ are different. ❑ (Lemma 3.8)

With these preparations we can show that there are T^* and M^* as in (1.1) for each given finite fragment T of ZFC.

Let T be a finite fragment of ZFC. We may assume that T contains all the axioms of ZFC_0. Let T^* be the extension of T in Theorem 3.5. Let M be a countable transitive model of T^* and let $\kappa \in \mathsf{On}$ be such that $M \models \kappa = \aleph_2$.[46] Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Since $\mathbb{P} \in M$

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(44) For a set X we denote with $[X]^{<\aleph_0}$ the set of all finite subsets of X. More generally $[X]^{<\kappa}$ denotes the set of all subsets of X of cardinality $< \kappa$. $[X]^{\leq\kappa}$, $[X]^{\kappa}$, $[X]^{\geq\kappa}$ etc. are defined similarly.

(45) This is also the place where we have to assume that a certain instance of the Axiom of Separation is included in ZFC_0.

(46) Here we are assuming that ZFC_0 contains enough axioms to prove the assertion "There is $\aleph_2$".

there is an $(M, \mathbb{P})$-generic filter $\mathbb{G}$ by Lemma 3.2. By the choice of T^*, $M[\mathbb{G}] \models T$ and by Lemma 3.7 and Theorem 3.6, We have $\mathrm{Card}^M = \mathrm{Card}^{M[\mathbb{G}]}$. In particular, $M[\mathbb{G}] \models \kappa = \aleph_2$. Since

(3.24) $\quad M[\mathbb{G}] \models$ "there are at least κ distinct subsets of ω"

by Lemma 3.8, (2), it follows that $M[\mathbb{G}] \models \neg\mathsf{CH}$.

For the construction of M^{**} in (1.3), we proceed similarly with the poset $\mathbb{P} \in M$ with $M \models \mathbb{P} = \mathrm{Fn}(\omega_1, 2, \aleph_1)$ where

(3.25) $\quad \mathrm{Fn}(\omega_1, 2, \aleph_1) = \{p : p : x \to 2 \text{ for some } x \in [\omega_1]^{<\aleph_1}\}$

with the partial order defined in the same way as for $\mathrm{Fn}(\kappa, 2)$:

(3.26) $\quad p \leq q \;\Leftrightarrow\; q \subseteq p$

for p, $q \in \mathrm{Fn}(\omega_1, 2, \aleph_1)$ and the maximal element $\emptyset$.

That the generic extension $M[\mathbb{G}]$ by this $\mathbb{P}$ is as desired can be seen in the following Lemma:

L-3

Lemma 3.9 *Suppose that M is a transitive model of ZFC_0. Let $\mathbb{P}$ be the poset defined by $M \models \mathbb{P} = \mathrm{Fn}(\omega_1, 2, \aleph_1)$ and $\mathbb{G}$ an $(M, \mathbb{P})$-generic filter. Letting $g = \bigcup \mathbb{G}$, we have*

(0) $(\omega_1)^M = (\omega_1)^{M[\mathbb{G}]}$.

(1) $g : \omega_1^M \to 2$

(2) $(\mathcal{P}(\omega))^M = (\mathcal{P}(\omega))^{M[\mathbb{G}]}$.

(3) *There is an $f \in M$ which is is a surjection from ω_1^M to $\mathcal{P}(\omega)^M$.*

Proof. (0) and (2): hold since $\mathrm{Fn}(\omega_1, 2, \aleph_1)$ is σ-closed. The proof of (0) and (2) can be done fairly easily using the forcing relation and some of its basic properties (see Theorem 4.19).

(1): can be proved similarly to Lemma 3.8, (1) (Exercise).

(3): For a limit $\gamma \in \mathrm{Lim} \cap \omega_1^M$, let a_γ be defined as before by

(3.27) $\quad a_\gamma = \{n \in \omega : g(\gamma + n) = 1\}$

By (2) $a_\gamma \in \mathcal{P}(\omega)^M$ for all limt $\gamma < \omega_1^M$. By density argument we can prove that

(3.28) $\quad f : \omega_1^M \to \mathcal{P}(\omega)^M;\ \alpha \mapsto a_\gamma$ where γ is the αth limit ordinal in $(\omega_1)^M$

is a surjection.

For this, we only need the following fact: in M, let

(3.29) $$D_a = \{p \in \mathbb{P} : \text{there is } \gamma < \omega_1 \text{ such that } \{\gamma + n : n \in \omega\} \subseteq \mathrm{dom}(p) \text{ and } \{n \in \omega : p(\gamma + n) = 1\} = a\}$$

for each $a \in \mathcal{P}(\omega)^M$.

Then $D_a \in M$ and D_a is dense in $\mathbb{P}$. $p \in \mathbb{G} \cap D_a$ "forces" $f(\gamma) = a$ for $\gamma \in \mathrm{dom}(p)$ as in the definition of D_a. ❑ (Lemma 3.9)

4 Forcing

forcing

4.1 Forcing relation and Forcing Theorem

forcing-rel

For each (concretely given) $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ we define the relation (*forcing relation*) $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" where $\mathbb{P}$ should be a poset, $p \in \mathbb{P}$ and $\underset{\sim}{a}_0,..., \underset{\sim}{a}_{n-1}$ $\mathbb{P}$-names. We read "$p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"" as "$p$ forces $\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$ (in $\mathbb{P}$)".

We define the forcing relation in the following (4.3), (4.6) $\sim$ (4.9) by induction on φ.

For a poset $\mathbb{P}$ and $p \in \mathbb{P}$, a set $D \subseteq \mathbb{P}$ is said to be *dense below* p if, for any $q \leq_{\mathbb{P}} p$, there is $r \in D$ such that $r \leq_{\mathbb{P}} q$.

L-forcing-a

Lemma 4.1 (1) *If $D \subseteq \mathbb{P}$ is dense below $p \in \mathbb{P}$ and $q \leq_{\mathbb{P}} p$, then D is dense below q.*

(2) *If $D_0 \subseteq \mathbb{P}$ is dense below p and $D_1 \subseteq \mathbb{P}$ is dense below q for each $q \in D_0$ then D_1 is dense below p.*

The relevance of this notion can be seen in the following lemma.

For a poset $\mathbb{P}$ and $p \in \mathbb{P}$, let

$$\mathbb{P} \downarrow p = \{q \in \mathbb{P} \,:\, q \leq_{\mathbb{P}} p\}. \tag{4.1}$$ forcing-a-0

For $S \subseteq \mathbb{P}$ we also write $S \downarrow p$ to denote $S \cap (\mathbb{P} \downarrow p)$.

$\mathbb{P} \downarrow p$ can be considered to be a poset with the maximal element p. $D \subseteq \mathbb{P}$ is then dense below p if and only if $D \downarrow p = D \cap (\mathbb{P} \downarrow p)$ is dense in the poset $\mathbb{P} \downarrow p$

In the following we shall often say simply that "M is a model of ZFC" instead of saying that "M is a model of a sufficiently large finite fragment of ZFC".

L-forcing-0

Lemma 4.2 *For a transitive model of ZFC and a poset $\mathbb{P} \in M$ let $\mathbb{G}$ be an $(M, \mathbb{P})$-generic filter. If $p \in \mathbb{G}$ and $D \in M$ with $D \subseteq \mathbb{P}$ is dense below p then there is $q \leq_{\mathbb{P}} p$ such that $q \in \mathbb{G} \cap D$.*

Proof. In M, let

$$\tilde{D} = (D \cap \mathbb{P} \downarrow p) \cup \{q \in \mathbb{P} \,:\, p \perp_{\mathbb{P}} q\}. \tag{4.2}$$ forcing-a

Then $\tilde{D}$ is dense in $\mathbb{P}$. By genericity there is $q \in \mathbb{G} \cap \tilde{D}$. Since p and q are compatible as elements of $\mathbb{G}$, it follows that $q \leq_{\mathbb{P}} p$ and $q \in D$ by the definition of $\tilde{D}$. ❑ (Lemma 4.2)

$$p \Vdash_{\mathbb{P}} \text{"}\, \underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1 \,\text{"} \;:\Leftrightarrow \tag{4.3}$$ forcing-0

(α) for all $\langle \underset{\sim}{b}_0, p_0 \rangle \in \underset{\sim}{a}_0$,
$\{q \in \mathbb{P} \,:\, q \not\leq_{\mathbb{P}} p_0 \ \lor\ \exists \langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1\ (q \leq_{\mathbb{P}} p_1 \ \land\ q \Vdash_{\mathbb{P}} \text{"}\, \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \,\text{"})\}$
is dense below p, and

(β) for all $\langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1$,
$\{q \in \mathbb{P} \,:\, q \not\leq_{\mathbb{P}} p_1 \ \lor\ \exists \langle \underset{\sim}{b}_0, p_0 \rangle \in \underset{\sim}{a}_0\ (q \leq_{\mathbb{P}} p_0 \ \land\ q \Vdash_{\mathbb{P}} \text{"}\, \underset{\sim}{b}_1 \equiv \underset{\sim}{b}_0 \,\text{"})\}$
is dense below p.

The actual definition of (4.3) can be done in connection with Theorem 2.5, (2) by defining

$$\mathbf{X} = \{\langle \underset{\sim}{a}_0, \underset{\sim}{a}_1\rangle \,:\, \underset{\sim}{a}_0 \text{ and } \underset{\sim}{a}_1 \text{ are } \mathbb{P}\text{-names}\}, \tag{4.4}$$ forcing-1

$$\mathbf{R} = \{\langle x, y\rangle \in \mathbf{X}^2 \,:\, \mathit{rank}(x) < \mathit{rank}(y)\}. \tag{4.5}$$ forcing-2

$\mathbf{R}$ is well-founded and set-like by Lemma 2.4. We then take an appropriate $\mathbf{G}$ which introduces the class function $\mathbf{F} : \mathbf{X} \to {}^{\mathbb{P}}2$ whose value is a function assigning the truth value of $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1$" to each $p \in \mathbb{P}$.

$$p \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{a}_0 \,\varepsilon\, \underset{\sim}{a}_1\text{”} \;:\Leftrightarrow \tag{4.6}$$ forcing-3
$$\{q \in \mathbb{P} \,:\, \exists\langle \underset{\sim}{b}, r\rangle \in \underset{\sim}{a}_1\ (q \leq_{\mathbb{P}} r \ \wedge\ q \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{b} \equiv \underset{\sim}{a}_0\text{”})\} \text{ is dense below } p.$$

$$p \Vdash_{\mathbb{P}} \text{“}\Big(\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}) \wedge \psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\Big)\text{”} \;:\Leftrightarrow \tag{4.7}$$ forcing-4
$$p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”} \ \text{ and } \ p \Vdash_{\mathbb{P}} \text{“}\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}.$$

$$p \Vdash_{\mathbb{P}} \text{“}\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”} \;:\Leftrightarrow\; q \nVdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”} \text{ for all } q \leq_{\mathbb{P}} p. \tag{4.8}$$ forcing-5

$$p \Vdash_{\mathbb{P}} \text{“}\exists x \varphi(x, \underset{\sim}{a}_1, ..., \underset{\sim}{a}_n)\text{”} \;:\Leftrightarrow \tag{4.9}$$ forcing-6
$$\{r \in \mathbb{P} \,:\, \exists \underset{\sim}{a}_0 \in V^{\mathbb{P}}\ (r \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, \underset{\sim}{a}_1, ..., \underset{\sim}{a}_n)\text{”})\} \text{ is dense below } p.$$

L-forcing-1

Lemma 4.3 *The following are equivalent:*

(a) $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

(b) $r \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" *for all* $r \leq_{\mathbb{P}} p$.

(c) $\{r \in \mathbb{P} \,:\, r \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"$\}$ *is dense below* p.

Proof. (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial.

We prove (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) by induction on φ.

Case I. φ is of the form $x_0 \,\varepsilon\, x_1$.

(a) $\Rightarrow$ (b): Assume $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1$". Then, for any $q \leq_{\mathbb{P}} p$ the sets in (4.3), (α) and (β) are dense below q by Lemma 4.1, (1). Hence $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1$".

(c) $\Rightarrow$ (a): Assume that

$$D^* = \{r \in \mathbb{P} \,:\, r \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1\text{”}\} \tag{4.10}$$ forcing-6-0

is dense below p. By (4.3),

$$
\begin{aligned}
(4.11)\quad D^* = \{r \in \mathbb{P} : (\alpha')\ & \text{for all } \langle \underset{\sim}{b}_0, p_0\rangle \in \underset{\sim}{a}_0,\\
& \{s \in \mathbb{P} : s \not\leq_{\mathbb{P}} p_0 \text{ or}\\
& \qquad \exists \langle \underset{\sim}{b}_1, p_1\rangle \in \underset{\sim}{a}_1 (s \leq_{\mathbb{P}} p_1 \wedge s \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \,\text{”})\}\\
& \text{is dense below } r; \text{ and}\\
(\beta')\ & \text{for all } \langle \underset{\sim}{b}_1, p_1\rangle \in \underset{\sim}{a}_1,\\
& \{s \in \mathbb{P} : s \not\leq_{\mathbb{P}} p_1 \text{ or}\\
& \qquad \exists \langle \underset{\sim}{b}_0, p_0\rangle \in \underset{\sim}{a}_0 (s \leq_{\mathbb{P}} p_0 \wedge s \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{b}_1 \equiv \underset{\sim}{b}_0 \,\text{”})\}\\
& \text{is dense below } r\}.
\end{aligned}
$$

forcing-6-1

By Lemma 4.1, (2), it follows from (4.10) and (4.11) that

$$
\begin{aligned}
(4.12)\qquad (\alpha\text{'})\ & \text{for all } \langle \underset{\sim}{b}_0, p_0\rangle \in \underset{\sim}{a}_0,\\
& \quad \{s \in \mathbb{P} : s \not\leq_{\mathbb{P}} p_0 \ \vee\ \exists \langle \underset{\sim}{b}_1, p_1\rangle \in \underset{\sim}{a}_1\ (s \leq_{\mathbb{P}} p_1 \ \wedge\ s \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \,\text{”})\}\\
& \text{is dense below } p, \text{ and}\\
(\beta')\ & \text{for all } \langle \underset{\sim}{b}_1, p_1\rangle \in \underset{\sim}{a}_1,\\
& \quad \{s \in \mathbb{P} : s \not\leq_{\mathbb{P}} p_1 \ \vee\ \exists \langle \underset{\sim}{b}_0, p_0\rangle \in \underset{\sim}{a}_0\ (s \leq_{\mathbb{P}} p_0 \ \wedge\ s \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \,\text{”})\}\\
& \text{is dense below } p.
\end{aligned}
$$

forcing-6-2

By (4.3), this simply means that $p \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1 \,\text{”}$.

Case II: φ is of the form $x_0 \,\varepsilon\, x_1$.

(a) $\Rightarrow$ (b): If $p \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{a}_0 \,\varepsilon\, \underset{\sim}{a}_1 \,\text{”}$, then by (4.6), $\{q \in \mathbb{P} : \exists \langle \underset{\sim}{b}, r\rangle \in \underset{\sim}{a}_1\ (q \leq_{\mathbb{P}} r \ \wedge\ q \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{b} \equiv \underset{\sim}{a}_0 \,\text{”})\}$ is dense below p. By Lemma 4.1, (1), it follows that this set is dense below any $q \leq_{\mathbb{P}} p$. Thus we have $q \Vdash_{\mathbb{P}} \text{“}\, \underset{\sim}{a}_0 \,\varepsilon\, \underset{\sim}{a}_1 \,\text{”}$.

(c) $\Rightarrow$ (a): Similarly to the Case I by Lemma 4.1, (1).

Case III: φ is of the form $\varphi = (\varphi_0 \wedge \varphi_1)$ and the lemma holds for φ_0 and φ_1.

(a) $\Rightarrow$ (b): Suppose that $p \Vdash_{\mathbb{P}} \text{“}\, (\varphi_0 \wedge \varphi_1) \,\text{”}$. Then, by (4.7), we have $p \Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}$ and $p \Vdash_{\mathbb{P}} \text{“}\, \varphi_1 \,\text{”}$. By the induction hypothesis, it follows that $q \Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}$ and $q \Vdash_{\mathbb{P}} \text{“}\, \varphi_1 \,\text{”}$ for all $q \leq_{\mathbb{P}} p$. Again by (4.7), it follows that $q \Vdash_{\mathbb{P}} \text{“}\, (\varphi_0 \wedge \varphi_1) \,\text{”}$ for all $q \leq_{\mathbb{P}} p$.

(c) $\Rightarrow$ (a): Suppose that

$$
(4.13)\quad \{r \in \mathbb{P} : r \Vdash_{\mathbb{P}} \text{“}\, (\varphi_0 \wedge \varphi_1) \,\text{”}\} = \{r \in \mathbb{P} : r \Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{” and } r \Vdash_{\mathbb{P}} \text{“}\, \varphi_1 \,\text{”}\}
$$

forcing-8-0

is dense below p. Then $\{r \in \mathbb{P} : r \Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}\}$ and $\{r \in \mathbb{P} : r \Vdash_{\mathbb{P}} \text{“}\, \varphi_1 \,\text{”}\}$ are both dense below p. By the induction hypothesis, it follows that $p \Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}$ and $p \Vdash_{\mathbb{P}} \text{“}\, \varphi_1 \,\text{”}$. By (4.7), it follows that $p \Vdash_{\mathbb{P}} \text{“}\, (\varphi_0 \wedge \varphi_1) \,\text{”}$.

Case IV: φ is of the form $\neg\varphi_0$ and the lemma holds for φ_0.

(a) $\Rightarrow$ (b): Suppose that $p \Vdash_{\mathbb{P}} \text{“}\, \neg\varphi_0 \,\text{”}$ then, by (4.8), $r \not\Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}$ for all $r \leq_{\mathbb{P}} p$. Thus we have $r \not\Vdash_{\mathbb{P}} \text{“}\, \varphi_0 \,\text{”}$ for all $r \leq_{\mathbb{P}} q$ for $q \leq_{\mathbb{P}} p$. Again by (4.8), it follows that $q \Vdash_{\mathbb{P}} \text{“}\, \neg\varphi_0 \,\text{”}$ for all $q \leq_{\mathbb{P}} p$.

(c) $\Rightarrow$ (a): Suppose that $D_0 = \{r \in \mathbb{P} : r \Vdash_{\mathbb{P}} \text{“}\neg\varphi_0\text{”}\}$ is dense below p. For each $r \in D_0$ the set $D_1 = \{q \in \mathbb{P} : q \not\Vdash_{\mathbb{P}} \text{“}\varphi_0\text{”}\}$ is dense below r (actually $\mathbb{P} \downarrow r \subseteq D_0$ by (4.8)). Thus by Lemma 4.1, (2), D_1 is dense below p.

Claim 4.3.1 *$q \not\Vdash_{\mathbb{P}} \text{“}\varphi_0\text{”}$ for all $q \leq_{\mathbb{P}} p$.*

$\vdash$ Suppose that there is a $q_0 \leq_{\mathbb{P}} p$ such that $q_0 \Vdash_{\mathbb{P}} \text{“}\varphi_0\text{”}$. Then, by the density of D_1 below p, there is $q_1 \in D_1$ with $q_1 \leq_{\mathbb{P}} q_0$. Since $q_1 \not\Vdash_{\mathbb{P}} \text{“}\varphi_0\text{”}$, this is a contradition to the induction hypothesis. $\dashv$ (Claim 4.3.1)

By (4.8), it follows that $p \Vdash_{\mathbb{P}} \text{“}\neg\varphi_0\text{”}$.

Case V: φ is of the form $\exists x \varphi_0(x, x_0, ..., x_{n-1})$ and the lemma holds for φ_0.

(a) $\Rightarrow$ (b): This is clear by the definition (4.9) of the forcing relation for existential formulas and Lemma 4.1, (1).

(c) $\Rightarrow$ (a) follows from (4.9) and Lemma 4.1, (2). ❑ (Lemma 4.3)

The following is a part of Lemma 4.3. We formulate it as a separate lemma since it is used quite often:

L-forcing-2-4

Lemma 4.4 *If $p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}$ and $q \leq_{\mathbb{P}} p$ then $q \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}$.* ❑

The following Theorem 4.5, 4.7 and Corollary 4.8 are often called *Forcing Theorem*.

T-forcing-0

Theorem 4.5 *Let $\varphi = \varphi(x_0, ..., x_{n-1})$ be an $\mathcal{L}_\varepsilon$-formula. For any transitive model of ZFC_0 and a poset $\mathbb{P} \in M$, let $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$.*[47] *Then, for any $(M, \mathbb{P})$-generic filter $\mathbb{G}$, we have:*

(i) *If $M \models \text{“}p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}\text{”}$, for some $p \in \mathbb{G}$, then $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.*

(ii) *If $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$, then there is a $p \in \mathbb{G}$ such that $M \models \text{“}p \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}\text{”}$.*

Proof. We prove (i) and (ii) simultaneously by induction on φ. In the following, most of the arguments are carried out in M.[48]

Case I: φ is of the form $x_0 \equiv x_1$. We prove (i) and (ii) for this case by induction on the set-like well-founded relation $\mathbf{R}$ of (4.5).

Suppose that $\underset{\sim}{a}_0$ and $\underset{\sim}{a}_1$ are $\mathbb{P}$-names and (i) and (ii) hold for $\varphi = x_0 \equiv x_1$ and for all pairs of $\mathbb{P}$-names $\langle \underset{\sim}{b}_0, \underset{\sim}{b}_1 \rangle$ with $\langle \underset{\sim}{b}_0, \underset{\sim}{b}_1 \rangle \mathrel{\mathbf{R}} \langle \underset{\sim}{a}_0, \underset{\sim}{a}_1 \rangle$. We have to show that (i) and (ii) also hold for $\langle \underset{\sim}{a}_0, \underset{\sim}{a}_1 \rangle$.

---

[47] We denote with $M^{\mathbb{P}}$ the subset $\left(V^{\mathbb{P}}\right)^M$ of M.

[48] This means we often say “In M, ...” instead of saying “$M \models$ ...”. In particular, if we say something like “In M, let $p \Vdash_{...} \text{“}...\text{”}$” this actually means $M \models \text{“}p \Vdash_{...} \text{“}...\text{”}\text{”}$. Note that the forcing relation is not absolute over M in general so that “$M \models p \Vdash_{...} \text{“}...\text{”}$” is not necessarily equivalent to “$p \Vdash_{...} \text{“}...\text{”}$”.

(i): Suppose that $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter, $p \in \mathbb{G}$ and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1$". For $b \in (\underset{\sim}{a}_0)^{\mathbb{G}}$ there is $\langle \underset{\sim}{b}_0, p' \rangle \in \underset{\sim}{a}_0$ such that $p' \in \mathbb{G}$ and $(\underset{\sim}{b}_0)^{\mathbb{G}} = b$. Let $p'' \in \mathbb{G}$ be such that $p'' \leq_{\mathbb{P}} p, p'$. The set

$$D = \{q \leq_{\mathbb{P}} p'' : \exists \langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1 \ (q \leq_{\mathbb{P}} p_1 \ \wedge \ q \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \text{”})\} \tag{4.14}$$ forcing-9

(constructed) in M is dense below p'' by (4.3), (α) and Lemma 4.1, (1). By Lemma 4.2, there is $q \in \mathbb{G} \cap D$ with $q \leq_{\mathbb{P}} p''$. By the definition of D, we can find $\langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1$ such that $q \leq_{\mathbb{P}} p_1$ and $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1$". We have $p_1 \in \mathbb{G}$ since $q \leq_{\mathbb{P}} p_1$. By the induction hypothesis, it follows that $(\underset{\sim}{b}_0)^{\mathbb{G}} = (\underset{\sim}{b}_1)^{\mathbb{G}}$ and $b = (\underset{\sim}{b}_0)^{\mathbb{G}} = (\underset{\sim}{b}_1)^{\mathbb{G}} \in (\underset{\sim}{a}_1)^{\mathbb{G}}$. Since b was arbitrary, this shows that $(\underset{\sim}{a}_0)^{\mathbb{G}} \subseteq (\underset{\sim}{a}_1)^{\mathbb{G}}$. The same argument applied to (4.3), (β) proves $(\underset{\sim}{a}_0)^{\mathbb{G}} \supseteq (\underset{\sim}{a}_1)^{\mathbb{G}}$. Thus $(\underset{\sim}{a}_0)^{\mathbb{G}} = (\underset{\sim}{a}_1)^{\mathbb{G}}$.

(ii): We show the contraposition: We assume that $p \nVdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1$" for all $p \in \mathbb{G}$ and show that $(\underset{\sim}{a}_0)^{\mathbb{G}} \not\equiv (\underset{\sim}{a}_1)^{\mathbb{G}}$.

(In M), let

$$\begin{aligned} D = \{p \in \mathbb{P} : \ & p \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{a}_0 \equiv \underset{\sim}{a}_1 \text{” or} \\ & (\alpha^*) \text{ for some } \langle \underset{\sim}{b}_0, p_0 \rangle \in \underset{\sim}{a}_0 \text{ with } p \leq_{\mathbb{P}} p_0 \text{ we have that, for all} \\ & \qquad \langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1 \text{ and } q \leq_{\mathbb{P}} p, \text{ if } q \leq_{\mathbb{P}} p_1, \text{ then } q \nVdash_{\mathbb{P}} \text{“} \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \text{” or} \\ & (\beta^*) \text{ for some } \langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1 \text{ with } p \leq_{\mathbb{P}} p_1 \text{ we have that, for all} \\ & \qquad \langle \underset{\sim}{b}_0, p_0 \rangle \in \underset{\sim}{a}_0 \text{ and } q \leq_{\mathbb{P}} p, \text{ if } q \leq_{\mathbb{P}} p_0, \text{ then } q \nVdash_{\mathbb{P}} \text{“} \underset{\sim}{b}_0 \equiv \underset{\sim}{b}_1 \text{”}\}. \end{aligned} \tag{4.15}$$ forcing-10

D is dense in $\mathbb{P}$ by (4.3) and Lemma 4.3. By the genericity of $\mathbb{G}$, there is a $p^* \in \mathbb{G} \cap D$. Since $p^* \nVdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \not\equiv \underset{\sim}{a}_1$" by assumption, p^* should satisfy at least one of (α^*) and (β^*). Let us assume that (α^*) holds. The case that (β^*) holds can be treated similarly.

Let $\langle \underset{\sim}{b}_0^*, p_0^* \rangle \in \underset{\sim}{a}_0$ be such that $p^* \leq_{\mathbb{P}} p_0^*$ and, for all $\langle \underset{\sim}{b}_1, p_1 \rangle \in \underset{\sim}{a}_1$, and $q \leq_{\mathbb{P}} p^*$, if $q \leq_{\mathbb{P}} p_1$ then $q \nVdash_{\mathbb{P}}$ "$\underset{\sim}{b}_0^* \equiv \underset{\sim}{b}_1$". By the induction hypothesis, we have that $(\underset{\sim}{b}_0^*)^{\mathbb{G}} \neq (\underset{\sim}{b}_1)^{\mathbb{G}}$ for all $(\underset{\sim}{b}_1)^{\mathbb{G}} \in (\underset{\sim}{a}_1)^{\mathbb{G}}$. On the other hand, since $(\underset{\sim}{b}_0^*)^{\mathbb{G}} \in (\underset{\sim}{a}_0)^{\mathbb{G}}$ by $p_0^* \in \mathbb{G}$, we have $(\underset{\sim}{a}_0)^{\mathbb{G}} \neq (\underset{\sim}{a}_1)^{\mathbb{G}}$.

Case II: φ is of the form $x_0 \varepsilon x_1$.

(i): Suppose that $p \in \mathbb{G}$ and $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \ \varepsilon \ \underset{\sim}{a}_1$". By (4.6), and Lemma 4.2 there is $q \in \mathbb{G} \downarrow p$ and $\langle \underset{\sim}{b}, r \rangle \in \underset{\sim}{a}_1$ such that $q \leq_{\mathbb{P}} r$ and $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{a}_0$". Since $r \in \mathbb{G}$, we have $(\underset{\sim}{b})^{\mathbb{G}} \in (\underset{\sim}{a}_1)^{\mathbb{G}}$. By Case I, we have $(\underset{\sim}{b})^{\mathbb{G}} = (\underset{\sim}{a})^{\mathbb{G}}$. Thus $(\underset{\sim}{a}_0)^{\mathbb{G}} \in (\underset{\sim}{a}_1)^{\mathbb{G}}$.

(ii): Suppose that $p \nVdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \ \varepsilon \ \underset{\sim}{a}_1$" for all $p \in \mathbb{G}$. By (4.6) and Lemma 4.3, the set (in M)

$$\begin{aligned} D = \{p \in \mathbb{P} : \ & p \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{a}_0 \ \varepsilon \ \underset{\sim}{a}_1 \text{” or for every } q \leq_{\mathbb{P}} p \text{ and} \\ & \langle \underset{\sim}{b}, r \rangle \in \underset{\sim}{a}_1, \text{ if } q \leq_{\mathbb{P}} r \text{ then } q \nVdash_{\mathbb{P}} \text{“} \underset{\sim}{b} \equiv \underset{\sim}{a}_0 \text{”}\} \end{aligned} \tag{4.16}$$ forcing-11

is dense in $\mathbb{P}$. By the genericity of $\mathbb{G}$, there is $p^* \in \mathbb{G} \cap D$. Since $p^* \not\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}_0 \in \underset{\sim}{a}_1$" by assumption, we have,

(4.17) for every $q \leq_{\mathbb{P}} p$ and $\langle \underset{\sim}{b}, r \rangle \in \underset{\sim}{a}_1$, if $q \leq_{\mathbb{P}} r$ then $q \not\Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{a}_0$". forcing-12

By Case I, it follows that, for every $b \in (\underset{\sim}{a}_1)^{\mathbb{G}}$, $b \neq (\underset{\sim}{a}_0)^{\mathbb{G}}$. Thus $(\underset{\sim}{a}_0)^{\mathbb{G}} \notin (\underset{\sim}{a}_1)^{\mathbb{G}}$.

Case III: φ is of the form $\varphi = (\varphi_0 \wedge \varphi_1)$ and, (i) and (ii) hold for φ_0 and φ_1.

(i): Suppose that $p \in \mathbb{G}$ and $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Then, by (4.7),

(4.18) $p \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $p \Vdash_{\mathbb{P}}$ "$\varphi_1(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". forcing-13

By the induction hypothesis, it follows that

(4.19) $M[\mathbb{G}] \models \varphi_0((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ and $M[\mathbb{G}] \models \varphi_1((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. forcing-14

Thus $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.

(ii): Suppose that $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. Then, $M[\mathbb{G}] \models \varphi_0((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ and $M[\mathbb{G}] \models \varphi_1((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. By the induction hypothesis, there are p_0, $p_1 \in \mathbb{G}$ such that $p_0 \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $p_1 \Vdash_{\mathbb{P}}$ "$\varphi_1(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Let $p \in \mathbb{G}$ be such that $p \leq_{\mathbb{P}} p_0, p_1$. By Lemma 4.3, we have $p \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $p \Vdash_{\mathbb{P}}$ "$\varphi_1(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Thus, by (4.7), $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

Case IV: φ is of the form $\neg\varphi_0$ and, (i) and (ii) hold for φ_0.

(i): Suppose that $p \in \mathbb{G}$ and $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Then, by (4.8),

(4.20) $q \not\Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" for all $q \leq_{\mathbb{P}} p$. forcing-15

There is no $r \in \mathbb{G}$ with $r \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" [If there were such $r \in \mathbb{G}$ then $r' \in \mathbb{G}$ with $r' \leq_{\mathbb{P}} r, p$ would contradict with (4.20) by Lemma 4.3]. By (ii) for φ_0, it follows that $M[\mathbb{G}] \not\models \varphi_0((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. That is, $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.

(ii): Suppose that $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. That is,

(4.21) $M[\mathbb{G}] \not\models \varphi_0((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.

By (i) for φ_0, it follows that $p \not\Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" for all $p \in \mathbb{G}$.

In M, let

(4.22) $D = \{p \in \mathbb{P} : p \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" or forcing-16
$q \not\Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" for all $q \leq_{\mathbb{P}} p\}$.

Claim 4.5.1 *D is dense in $\mathbb{P}$.*

$\vdash$ Suppose $p \in \mathbb{P}$. If $p \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then $p \in D$. Otherwise $p \not\Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and hence, by Lemma 4.3, the set $\{r \in \mathbb{P} : r \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"$\}$ is not dense below p. Thus, there is some $p' \leq_{\mathbb{P}} p$ such that there is no $r \leq_{\mathbb{P}} p'$ with $r \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". That is, $p' \in D$. $\dashv$ (Claim 4.5.1)

Let $p^* \in \mathbb{G} \cap D$. By the assumption, the second half of the condition in the definition of $p^* \in D$ should hold. That is, $q \nVdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$" for all $q \leq_{\mathbb{P}} p^*$. Thus $q^* \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$".

Case V: φ is of the form $\exists x \varphi_0(x, x_0, \dots, x_{n-1})$ and (i) and (ii) hold for φ_0.

(i): Suppose that $p \in \mathbb{G}$ and $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". Then, by (4.9),

(4.23) $D = \{q \in \mathbb{P} \,:\,$ there is a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $q \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}, \underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$"$\}$ forcing-17

is dense below p. Let $q \leq_{\mathbb{P}} p$ be such that $q \in \mathbb{G} \cap D$ and let $\underset{\sim}{a}$ be a $\mathbb{P}$-name such that $q \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}, \underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". By (i) for φ_0, it follows that $M[\mathbb{G}] \models \varphi_0(\underset{\sim}{a}^{\mathbb{G}}, (\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. Thus $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.

(ii): Suppose that $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. Then there is a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $M[\mathbb{G}] \models \varphi_0(\underset{\sim}{a}^{\mathbb{G}}, (\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$. By (ii) for φ_0, there is a $p \in \mathbb{G}$ such that $p \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}, \underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". By Lemma 4.3, it follows that $q \Vdash_{\mathbb{P}}$ "$\varphi_0(\underset{\sim}{a}, \underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$" for all $q \leq_{\mathbb{P}} p$. By (4.9) it follows that $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". ❑ (Theorem 4.5)

L-forcing-2

Lemma 4.6 *For an $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, \dots, x_{n-1})$, poset $\mathbb{P}$, $\mathbb{P}$-names $\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}$ and $p \in \mathbb{P}$, if $p \nVdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$" then there is $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$".*

Proof. Suppose that $p \nVdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". Then by Lemma 4.3,

(4.24) $D = \{r \in \mathbb{P} \,:\, r \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$"$\}$ forcing-18

is not dense below p. Thus there is $q \leq_{\mathbb{P}} p$ such that $D \cap \mathbb{P} \downarrow q = \emptyset$. By (4.8) it follows that $q \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$". ❑ (Lemma 4.6)

T-forcing-3

Theorem 4.7 *Let M be a transitive model of* ZFC *and $\mathbb{P} \in M$ a poset such that*

(4.25) *for all $p \in \mathbb{P}$ there is an $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $p \in \mathbb{G}$.* forcing-18-3

Then, for any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ and $\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$, we have $M \models$ "$p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$"" if and only if we have $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ for all $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $p \in \mathbb{G}$.

Proof. If $M \models$ "$p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$"" then $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ for all $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $p \in \mathbb{G}$ by Theorem 4.5.

If $p \nVdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$" in M, then there is $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$" in M by Lemma 4.6. By (4.25) there is $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $q \in \mathbb{G}$. Since $p \in \mathbb{G}$, $M[\mathbb{G}] \models \neg\varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, \dots, (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ by Theorem 4.5. ❑ (Theorem 4.7)

T-forcing-1

Corollary 4.8 *Let M be a countable transitive model of* ZFC *and $\mathbb{P} \in M$ a poset. Then, for any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$, we have $M \models$ "$p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"" if and only if we have $M[\mathbb{G}] \models \varphi((\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$ for all $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $p \in \mathbb{G}$.*

Proof. By Lemma 3.2 and Theorem 4.7. ❑ (Corollary 4.8)

4.2 $M[\mathbb{G}]$ is a model of ZFC

MG

In this subsection we prove Theorem 3.5.

We assume that M is a transitive model of a sufficiently large fragment of ZFC and $\mathbb{G}$ an $(M, \mathbb{P})$-generic filter. We prove that $M[\mathbb{G}] \models \varphi$ for all axioms φ of ZFC where the meaning of "sufficiently large'' varies depending on φ.

We have already shown in Lemma 3.4 that $M[\mathbb{G}]$ is transitive and $\mathsf{On} \cap M[\mathbb{G}] = \mathsf{On} \cap M$. In particular $\omega+1 \subseteq M[\mathbb{G}]$. From this it follows that M satisfies Axiom of Extensionality, Axiom of Empty Set and Axiom of Infinity.

To see that $M[\mathbb{G}]$ satisfies the Pairing Axiom it is enough to show that $M[\mathbb{G}]$ is closed with respect to the pairing operation[(49)].

For $\mathbb{P}$-names $\underset{\sim}{a}$ and $\underset{\sim}{b}$ we define

(4.26) $\quad \mathrm{up}_{\mathbb{P}}(\underset{\sim}{a}, \underset{\sim}{b}) = \{\langle \underset{\sim}{a}, \mathbb{1}_{\mathbb{P}}\rangle, \langle \underset{\sim}{b}, \mathbb{1}_{\mathbb{P}}\rangle\},$ forcing-19

(4.27) $\quad \mathrm{op}_{\mathbb{P}}(\underset{\sim}{a}, \underset{\sim}{b}) = \mathrm{up}(\mathrm{up}(\underset{\sim}{a}, \underset{\sim}{a}), \mathrm{up}(\underset{\sim}{a}, \underset{\sim}{b})).$ forcing-20

We drop the subscript $\mathbb{P}$ and simply write $\mathrm{up}(\underset{\sim}{a}, \underset{\sim}{b})$ and $\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b})$ in place of $\mathrm{up}_{\mathbb{P}}(\underset{\sim}{a}, \underset{\sim}{b})$ and $\mathrm{op}_{\mathbb{P}}(\underset{\sim}{a}, \underset{\sim}{b})$, if it is clear which $\mathbb{P}$ is meant,

$\mathrm{up}(\underset{\sim}{a}, \underset{\sim}{b})$ and $\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b})$ are standard $\mathbb{P}$-names of pairs and ordered pairs of sets corresponding to $\underset{\sim}{a}$ and $\underset{\sim}{b}$:

L-forcing-3

Lemma 4.9 *Suppose that $\underset{\sim}{a}$, $\underset{\sim}{b} \in M^{\mathbb{P}}$. Then*

(1) $\mathrm{up}(\underset{\sim}{a}, \underset{\sim}{b}) \in M^{\mathbb{P}}$ *and* $(\mathrm{up}(\underset{\sim}{a}, \underset{\sim}{b}))^{\mathbb{G}} = \{\underset{\sim}{a}^{\mathbb{G}}, \underset{\sim}{b}^{\mathbb{G}}\}$.

(2) $\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b}) \in M^{\mathbb{P}}$ *and* $(\mathrm{op}(\underset{\sim}{a}, \underset{\sim}{b}))^{\mathbb{G}} = \langle \underset{\sim}{a}^{\mathbb{G}}, \underset{\sim}{b}^{\mathbb{G}}\rangle$.

Note that the genericity of $\mathbb{G}$ is not needed in the proof of Lemma 4.9 and hence the lemma holds for any filter $\mathbb{G}$ over $\mathbb{P}$.

It follows immediately form Lemma 4.9 that $M[\mathbb{G}]$ is a model of Pairing Axiom.

For any $\mathbb{P}$-name $\underset{\sim}{a}$ let

(4.28) $\quad \mathrm{ps}(\underset{\sim}{a}) = \{\langle \underset{\sim}{b}, \mathbb{1}_{\mathbb{P}}\rangle : \underset{\sim}{b}$ is a $\mathbb{P}$-name and $\mathrm{dom}(\underset{\sim}{b}) \subseteq \mathrm{dom}(\underset{\sim}{a})\}.$ forcing-21

L-forcing-4

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(49) Note that the $\mathcal{L}_\varepsilon$-formula $x \equiv \{y, z\}$ is Δ_0 and hence absolute over the transitive $\in$-structure $M[\mathbb{G}]$ by Lemma 2.22.

Lemma 4.10 *Suppose that* $\underset{\sim}{a} \in M^{\mathbb{P}}$. *Then* $(\mathrm{ps}(\underset{\sim}{a}))^M \in M^{\mathbb{P}}$ *and,*

$$(4.29) \quad (\mathrm{ps}(\underset{\sim}{a}))^{\mathbb{G}} \supseteq \mathcal{P}(\underset{\sim}{a}^{\mathbb{G}}) \cap M[\mathbb{G}].$$

Proof. Suppose that $x \in M[\mathbb{G}]$ such that $M[\mathbb{G}] \models x \subseteq \underset{\sim}{a}^{\mathbb{G}}$. In M, let $\underset{\sim}{x} \in M^{\mathbb{P}}$ be a $\mathbb{P}$-name of x and[(50)] let $p_0 \in \mathbb{G}$ be such that $p_0 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{x} \subseteq \underset{\sim}{a}$ " (there is such p_0 by the Forcing Theorem 4.5,(ii)). Let

$$(4.30) \quad \underset{\sim}{y} = \{\langle \underset{\sim}{b}, q\rangle : q \leq_{\mathbb{P}} p_0,\ \underset{\sim}{b} \in \mathrm{dom}(\underset{\sim}{a}) \text{ and } q \Vdash_{\mathbb{P}} \text{“ } \underset{\sim}{b} \in \underset{\sim}{x} \text{ ”}\}.$$ forcing-21-0

Since $\mathrm{dom}(\underset{\sim}{y}) \subseteq \mathrm{dom}(\underset{\sim}{a})$, we have $\underset{\sim}{y}^{\mathbb{G}} \in (\mathrm{ps}(\underset{\sim}{a}))^{\mathbb{G}}$. Thus the following claim finishes the proof:

Claim 4.10.1 $\underset{\sim}{y}^{\mathbb{G}} = \underset{\sim}{x}^{\mathbb{G}}$.

$\vdash$ Suppose that $z \in \underset{\sim}{y}^{\mathbb{G}}$. Then there is $\langle \underset{\sim}{b}, q\rangle \in \underset{\sim}{y}^{\mathbb{G}}$ such that $q \in \mathbb{G}$ and $\underset{\sim}{b}^{\mathbb{G}} = z$. By the definition of $\underset{\sim}{y}$, we have $q \Vdash_{\mathbb{P}}$ " $\underset{\sim}{b} \in \underset{\sim}{x}$ ". Thus $z = \underset{\sim}{b}^{\mathbb{G}} \in \underset{\sim}{x}^{\mathbb{G}}$.

Suppose now that $z \in \underset{\sim}{x}^{\mathbb{G}}$. Then there is $\langle \underset{\sim}{b}, q\rangle \in \underset{\sim}{x}$ such that $q \in \mathbb{G}$ and $z = \underset{\sim}{b}^{\mathbb{G}}$. We have $q \Vdash_{\mathbb{P}}$ " $\underset{\sim}{b} \in \underset{\sim}{x}$ ". Let $q' \in \mathbb{G}$ be such that $q' \leq_{\mathbb{P}} p_0, q$. Then $\langle \underset{\sim}{b}, q'\rangle \in \underset{\sim}{y}$ by the definition of $\underset{\sim}{y}$. Thus $z = \underset{\sim}{b}^{\mathbb{G}} \in \underset{\sim}{y}^{\mathbb{G}}$. $\dashv$ (Claim 4.10.1)

$\square$ (Lemma 4.10)

Lemma 4.10 together with the fact that $M[\mathbb{G}]$ satisfies the separation axiom proved below implies that $M[\mathbb{G}]$ models the Power Set Axiom.

To prove that the Separation Axiom for a formula $\varphi = \varphi(x, x_0, ..., x_{n-1})$ holds in $M[\mathbb{G}]$, let $\underset{\sim}{b}$, $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ be $\mathbb{P}$-names in M. We have to show that there is a $\mathbb{P}$-name $\underset{\sim}{c}$ such that

$$(4.31) \quad M[\mathbb{G}] \models \forall x\,(x \in \underset{\sim}{c}^{\mathbb{G}} \ \leftrightarrow\ (x \in \underset{\sim}{b}^{\mathbb{G}} \wedge \varphi(x, (\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}}))).$$ forcing-22

In M, let

$$(4.32) \quad \underset{\sim}{c} = \{\langle \underset{\sim}{u}, q\rangle : q \leq_{\mathbb{P}} p \text{ and } \langle \underset{\sim}{u}, p\rangle \in \underset{\sim}{b} \text{ for some } p \in \mathbb{P}; \text{ and } q \Vdash_{\mathbb{P}} \text{“ } \varphi(\underset{\sim}{u}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}) \text{ ”}\}.$$ forcing-23

Actually $\underset{\sim}{c} \in M$ (since M satisfies the instance of the Axiom of Separation needed to prove the existence of such $\underset{\sim}{c}$ satisfying (4.32)) and it is a $\mathbb{P}$-name.

Lemma 4.11 $\underset{\sim}{c}$ *as above satisfies* (4.31).

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(50) This means simply that $\underset{\sim}{x}$ is a $\mathbb{P}$-name in M such that $\underset{\sim}{x}^{\mathbb{G}} = x$.

Proof. We work in $M[\mathbb{G}]$ and show that (4.31) holds.

If $d \in M[\mathbb{G}]$ and $d \in \underset{\sim}{c}^{\mathbb{G}}$, there are $q \in \mathbb{G}$ and $\mathbb{P}$-name $\underset{\sim}{d}$ such that $\underset{\sim}{d}^{\mathbb{G}} = d$ and $\langle \underset{\sim}{d}, q\rangle \in \underset{\sim}{c}$. By the definition of $\underset{\sim}{c}$, we have $M \models$ "$q \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{d}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"". By Theorem 4.5, (i), it follows that $M[\mathbb{G}] \models \varphi(\underbrace{\underset{\sim}{d}^{\mathbb{G}}}_{=d}, (\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})$.

Conversely, suppose that $d \in M[\mathbb{G}]$ satisfies

$$(4.33) \quad M[\mathbb{G}] \models (d \in \underset{\sim}{b}^{\mathbb{G}} \wedge \varphi(d, (\underset{\sim}{a}_0)^{\mathbb{G}}, ..., (\underset{\sim}{a}_{n-1})^{\mathbb{G}})).$$ forcing-24

Then there is $\langle \underset{\sim}{u}, p\rangle \in \underset{\sim}{b}$ such that $p \in \mathbb{G}$ and $\underset{\sim}{u}^{\mathbb{G}} = d$. By Theorem 4.5, (ii), there is $q \in \mathbb{G}$ such that

$$(4.34) \quad M \models \text{“}q \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{u}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{””}.$$ forcing-25

Letting $r \in \mathbb{G}$ be such that $r \leq_{\mathbb{P}} p$, q, we have

$$(4.35) \quad M \models \text{“}r \Vdash_{\mathbb{P}} \text{“}(\underset{\sim}{u} \in \underset{\sim}{b} \wedge \varphi(\underset{\sim}{u}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}))\text{””}.$$ forcing-26

By the definition of $\underset{\sim}{c}$, it is $\langle \underset{\sim}{u}, r\rangle \in \underset{\sim}{a}$. Thus $M[\mathbb{G}] \models d = \underset{\sim}{u}^{\mathbb{G}} \in \underset{\sim}{c}^{\mathbb{G}}$. ❑ (Lemma 4.11)

4.3 Cardinals in generic extensions

card-in-gen-ext

Recall that an *antichain* in a poset $\mathbb{P}$ is a subset A of $\mathbb{P}$ whose elements are pairwise incompatible. An antichain A in $\mathbb{P}$ is *maximal* if it is maximal with respect to $\subseteq$ among the antichains in $\mathbb{P}$. An antichain A is *maximal below* $p \in \mathbb{P}$ if $A \subseteq \mathbb{P} \downarrow p$ and it is maximal in $\mathbb{P} \downarrow p$. Under AC, any antichain $A \subseteq \mathbb{P}$ is extended to a maximal antichain. Actually this assertion is equivalent to AC under ZF.

In the argument below this fact is used essentially.

L-ccc-a

Lemma 4.12 *For a poset $\mathbb{P}$ and antichain $A \subseteq \mathbb{P}$ A is maximal if and only if, for any $p \in \mathbb{P}$, there is $q \in A$ which is compatible with p.*

An antichain $A \subseteq \mathbb{P} \downarrow p$ is maximal below p if and only if, for any $p' \leq_{\mathbb{P}} p$, there is $q \in A$ which is compatible with p'.

Proof. If A is not maximal then there is an antichain $A' \subseteq \mathbb{P}$ with $A \subsetneqq A'$. Any $p \in A' \setminus A$ is incompatible with all $q \in A$.

If there is a $p \in \mathbb{P}$ such that p is incompatible with all $q \in A$ then $A' = A \cup \{p\}$ is an antichain which is a proper extension of A. Thus A is not maximal.

The second part of the assertion is just a reformulation of the first in the poset $\mathbb{P} \downarrow p$.

❑ (Lemma 4.12)

$D \subseteq \mathbb{P}$ is *open* if it is downward closed (with respect to $\leq_{\mathbb{P}}$), that is, if we have $q \in D$ for all $p \in D$ and $q \leq_{\mathbb{P}} p$.

L-ccc-0

Lemma 4.13 *Suppose that $\mathbb{P}$ is a poset, $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ and $p \in \mathbb{P}$. The following are equivalent:*

(a) *D is dense below p.*

(b) *There is a maximal antichain A below p such that $A \subseteq D$.*

Proof. (a) $\Rightarrow$ (b): Suppose that D is dense below p let A be maximal among the antichains of $\mathbb{P} \downarrow p$ which are subsets of D. We show that A is a maximal antichain below p. In particular A is a maximal antichain in the poset $D \downarrow p \cup \{p\}$.

Suppose that $p' \leq_{\mathbb{P}} p$ then there is $p'' \leq_{\mathbb{P}} p'$ with $p'' \in D$ by density of D. By the maximality of A and Lemma 4.12, there is $q \in A$ such that q is compatible with p''. But then q is also compatible with p'. By Lemma 4.12, this shows that A is a maximal antichain below p.

(b) $\Rightarrow$ (a): Suppose that $A \subseteq D$ is a maximal antichain below p. Let

$$(4.36) \quad D' = \{q \in \mathbb{P} \,:\, q \leq_{\mathbb{P}} q' \text{ for some } q' \in A\}.$$ ccc-a

Since D is open, we have $D' \subseteq D$. We show that D' is dense below p. From this it follows that D is also dense beleow p.

Let $p' \leq p$. Since A is maximal antichain below p, there is $q \in A$ which is compatible with p'. Let $r \leq_{\mathbb{P}} q, p'$. Then $r \in D'$. This shows that D' is dense below p. ❑ (Lemma 4.13)

L-ccc-1

Lemma 4.14 *Suppose that $\mathbb{P}$ is a poset and $p \in \mathbb{P}$. For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, if there is a maximal antichain $A \subseteq \mathbb{P}$ below p such that $q \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" for all $q \in A$, then $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".*

Proof. Let $D = \{q \in \mathbb{P} \downarrow p \,:\, q \Vdash_{\mathbb{P}} \text{"}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{"}\}$. By Lemma 4.3, D is open. Thus, by Lemma 4.13, D is dense below p. Again by Lemma 4.3, it follows that $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". ❑ (Lemma 4.14)

L-ccc-2

Lemma 4.15 *Suppose that M and N are transitive models of ZFC with $M \subseteq N$ and $\mathsf{On} \cap M = \mathsf{On} \cap N$.* [51] *Then the following are equivalent:*

(a) $\{\kappa \in M \,:\, M \models$ "*κ is a regular cardinal*"$\}$
$= \{\kappa \in M \,:\, M \models$ "*κ is a regular cardinal*"$\}$.

(b) $\mathrm{Card}^M = \mathrm{Card}^N$.

(c) *For all $\alpha \in \mathsf{On} \cap M$, $(cf(\alpha))^M = (cf(\alpha))^N$.*

Proof. The implication "(b) $\Rightarrow$ (c) $\Rightarrow$ (a)" is trivial (For "(c) $\Rightarrow$ (a)" note that κ is a regular cardinal if and only if $cf(\kappa) = \kappa$).

For the implication "(a) $\Rightarrow$ (b)", suppose that $\mathrm{Card}^M \neq \mathrm{Card}^N$. That is, $\mathrm{Card}^M \supsetneqq \mathrm{Card}^N$. Let $\kappa = \min(\mathrm{Card}^M \setminus \mathrm{Card}^N)$.

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[51] That is, M is an inner model of N.

We show that $M \models$ "κ is a regular cardinal". This implies the negation of (a). [52]

Suppose otherwise and, working further in M, suppose that $\kappa = \bigcup S$ where S is a set of cardinals $< \kappa$ of order-type $\lambda < \kappa$. By the minimality of κ, elements of S remain cardinals in N. It follows that κ is a cardinal in N as a limit of cardinals. This is a contradiction to the choice of κ. ❑ (Lemma 4.15)

The following theorem is apparently slightly weaker than Theorem 3.6 (in Theorem 3.6 the countability of M is not assumed). We show later that Theorem 3.6 easily follows from the following theorem. Note that similarly as before, if we say "a model of ZFC" here then it actually means "a model of an appropriate sufficiently large finite fragment of ZFC".

T-ccc-0

Theorem 4.16 *Suppose that M is a countable transitive model of* ZFC *and*

(4.37) $M \models$ "*$\mathbb{P}$ is a ccc poset*"

for a poset $\mathbb{P} \in M$. Then for any $(M, \mathbb{P})$-generic $\mathbb{G}$, $\mathrm{Card}^{M[\mathbb{G}]} = \mathrm{Card}^M$.

Proof. We work in M. Suppose that $\mathbb{P}$ is a ccc poset.

By Forcing Theorem and Lemma 4.15 it is enouegh to prove the following:

(4.38) For any regular cardinal κ we have $\Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is a regular cardinal". ccc-0

Suppose otherwise: Let κ be a regular cardinal and $\not\Vdash_{\mathbb{P}}$ "κ is a regular cardinal". Then there are $p \in \mathbb{P}$, $\lambda < \kappa$ and $\mathbb{P}$-name $\underset{\sim}{f}$ such that

(4.39) $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{\lambda} \to \check{\kappa}$ and $\underset{\sim}{f}\,''\check{\lambda}$ is cofinal in $\check{\kappa}$". ccc-1

For each $\alpha \in \lambda$, let $A_\alpha \subseteq \mathbb{P} \downarrow p$ be a maximal antichain such that for each $q \in A_\alpha$ there is (a unique) $\beta^\alpha_q < \kappa$ such that $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{\alpha}) \equiv \check{\beta}^\alpha_q$". n For $\alpha < \lambda$, let $\gamma_\alpha = \sup\{\beta^\alpha_q : q \in A_\alpha\}$. Each A_α is countable by the ccc of $\mathbb{P}$. Thus we have $\gamma_\alpha < \kappa$ since κ is regular. By Lemma 4.14 and Lemma 4.3 it follows that

(4.40) $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{\alpha}) \leq \check{\gamma}_\alpha$". ccc-2

Now let $g : \lambda \to \kappa$ be defined by $g(\alpha) = \gamma_\alpha$ for $\alpha < \lambda$.

Now stepping out of M, let $\mathbb{G}$ be an $(M, \mathbb{G})$-generic filter with $p \in \mathbb{G}$.

Claim 4.16.1 $M[\mathbb{G}] \models$ *"$\underset{\sim}{f}^{\mathbb{G}}(\alpha) \leq g(\alpha)$ for all $\alpha < \lambda$".*

$\vdash$ By (4.40) and the definition of g (and the Forcing Theorem). $\dashv$ (Claim 4.16.1)

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(52) Our proof actually shows that κ is a successor cardinal.

Since $M[\mathbb{G}] \models$ "$\underset{\sim}{f}^{\mathbb{G}}{}''\lambda$ is cofinal in κ" by (4.39) (and the Forcing Theorem), we have $M[\mathbb{G}] \models$ "$g''\lambda$ is cofinal in κ". It follows that $M \models$ "$g''\lambda$ is cofinal in κ" by absoluteness.(53) This is a contradiction to the regularity of κ in M. $\square$ (Theorem 4.16)

For a regular uncountable cardinal κ a poset $\mathbb{P}$ has the κ-cc if any antichain in $\mathbb{P}$ has cardinality $< \kappa$. Thus $\mathbb{P}$ has the ccc if and only if it has the $\aleph_1$-cc. The following theorem can be proved in the same way as Theorem 4.16:

T-ccc-1

Theorem 4.17 *Suppose that M is a countable transitive model of* ZFC *and*

(4.41) $\quad M \models$ "*κ is a regular uncountable carinal and $\mathbb{P}$ is a κ-ccc poset*"

for a poset $\mathbb{P} \in M$. Then for any $(M, \mathbb{P})$-generic $\mathbb{G}$, $\mathrm{Card}^{M[\mathbb{G}]} \setminus \kappa = \mathrm{Card}^M \setminus \kappa$ and, for any λ and μ with $\kappa \leq \lambda, \mu$, if $M \models$ "$cf(\lambda) = \mu$" then $M[\mathbb{G}] \models$ "$cf(\lambda) = \mu$". $\square$

For an uncountable regular cardinal κ, a poset $\mathbb{P}$ is said to be $< \kappa$-closed if any descending sequence $\langle p_\xi : \xi < \gamma \rangle$ of length $\gamma < \kappa$ has a lower bound in $\mathbb{P}$. $< \omega_1$-closed posets are also called σ-closed.

L-forcing-5

Lemma 4.18 *Suppose that κ is an uncountable regular cardinal and $\mathbb{P}$ is a κ-closed poset. If $\gamma < \kappa$ and D_α, $\alpha < \gamma$ are open dense in $\mathbb{P}$, then $\bigcap_{\alpha<\gamma} D_\alpha$ is also open dense in $\mathbb{P}$.*

Proof. Suppose that $p \in \mathbb{P}$. Then let $\langle p_\alpha : \alpha \leq \gamma \rangle$ be a decreasing sequence (with respect to $\leq_{\mathbb{P}}$) of elements in $\mathbb{P}$ such that $p_0 \leq_{\mathbb{P}} p$ and $p_\alpha \in D_\alpha$ for all $\alpha < \gamma$. The construction is possible since $\mathbb{P}$ is κ-closed and D_α's are closed. Since D_α's are open, it follows that $p_\gamma \in \bigcap_{\alpha<\gamma} D_\gamma$. $\square$ (Lemma 4.18)

T-closed-0

Theorem 4.19 *Suppose that M is a countable transitive model of* ZFC *and*

(4.42) $\quad M \models$ "*κ is a regular uncountable carinal and $\mathbb{P}$ is a $< \kappa$-closed poset*"

for a poset $\mathbb{P} \in M$. Then for any $(M, \mathbb{P})$-generic $\mathbb{G}$,

(4.43) $\quad ({}^{\kappa>}\mathsf{On})^M = ({}^{\kappa>}\mathsf{On})^{M[\mathbb{G}]}$. closed-0

In particular

(4.44) $\quad \mathrm{Card}^{M[\mathbb{G}]} \cap (\kappa+1) = \mathrm{Card}^M \cap (\kappa+1)$. closed-1

Proof. By Lemma 3.4, (2) and (4), $({}^{\kappa>}\mathsf{On})^M \subseteq ({}^{\kappa>}\mathsf{On})^{M[\mathbb{G}]}$. Suppose that $f \in ({}^{\kappa>}\mathsf{On})^{M[\mathbb{G}]}$. We have to show $f \in M$.

Let $\gamma < \kappa$ be such that $f : \gamma \to \mathsf{On}$. Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name such that $f = \underset{\sim}{f}^{\mathbb{G}}$. By Forcing Theorem (Theorem 4.5, (ii)), there is $p \in \mathbb{G}$ such that $M \models$ "$p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{\gamma} \to \mathsf{On}$"".

Working in M, for $\alpha < \gamma$, let

(4.45) $\quad D_\alpha = \{q \leq_{\mathbb{P}} p : q \text{ desides } \underset{\sim}{f}(\check{\alpha})\}$. [54] closed-2

Claim 4.19.1 *D_α is open dense for all $\alpha < \gamma$.*

$\vdash$ Moving to $M[\mathbb{G}]$, there is $\beta \in \mathsf{On}^{M[\mathbb{G}]} = \mathsf{On}^M$ such that $M[\mathbb{G}] \models$ "$\underset{\sim}{f}^{\mathbb{G}}(\alpha) \equiv \beta$". By Forcing Theorem (Theorem 4.5, (ii)) there is $q \in \mathbb{G}$ such that $M \models$ "$q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{\alpha}) \equiv \check{\beta}$"". Remember that $p \in \mathbb{G}$. Thus there is an $r \leq_{\mathbb{P}} p$, q. $r \leq p$ and $r \in D_\alpha$.

D_α is dense by Lemma 4.4. $\dashv$ (Claim 4.19.1)

By Lemma 4.18, $D = \bigcap_{\alpha<\gamma} D_\alpha$ is open dense. It follows that there is $q \in D \cap \mathbb{G}$. In M, let $f : \gamma \to \mathsf{On}$ be defined by

(4.46) $\quad f(\alpha) = $ the ordinal β such that $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{\alpha}) \equiv \check{\beta}$".

$f \in M$ and $f = \underset{\sim}{f}^{\mathbb{G}}$. $\square$ (Theorem 4.19)

4.4 Further properties of the forcing relation

further-prop

The following Lemma is used very often without any mention.

Lemma 4.20 *Suppose that $\mathbb{P}$ is a poset, $p \in \mathbb{P}$, $\varphi = \varphi(x_0, ..., x_{n-1})$, $\psi = \psi(x_0, ..., x_{n-1})$ $\mathcal{L}_\varepsilon$-formulas and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ $\mathbb{P}$-names. Suppose that $p \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $(\varphi \to \psi)$ holds.* [55] forcing-implication

Then we have $p \Vdash_{\mathbb{P}}$ "$\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

Proof. Let $\chi \in \mathsf{On}$ be sufficiently large and such that V_χ reflects a large enough finite collection of $\mathcal{L}_\varepsilon$-formulas (see Corollary 2.19, in particular V_χ should satisfy a large enough finite fragment of ZFC). Let $M \prec V_\chi$ be countable such that p, $\mathbb{P}$, $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}, \dots \in M$. Let M_0 be the Mostowski collapse of M and let f be the collapsing function $f : M \overset{\cong}{\to} M_0$. By elementarity and isomorphism, we have

(4.47) $\quad M_0 \models$ "$f(\mathbb{P})$ is a poset, $f(p) \in f(\mathbb{P})$, $f(\underset{\sim}{a}_0), ..., f(\underset{\sim}{a}_{n-1})$ are $\mathbb{P}$-names and $f(p) \Vdash_{\mathbb{P}}$ "$\varphi(f(\underset{\sim}{a}_0), ..., f(\underset{\sim}{a}_{n-1}))$""

For any $(M_0, f(\mathbb{P}))$-generic filter $\mathbb{G}$ with $f(p) \in$, we have

(4.48) $\quad M_0[\mathbb{G}] \models \varphi((f(\underset{\sim}{a}_0))^{\mathbb{G}}, ..., (f(\underset{\sim}{a}_{n-1}))^{\mathbb{G}})$

by the Forcing Theorem (Theorem 4.5, (1)).

By the choice of χ, $M_0[\mathbb{G}]$ satisfies a large enough portion of ZFC to prove $(\varphi \to \psi)$. Thus $M_0[\mathbb{G}] \models (\varphi \to \psi)$ and hence

(4.49) $\quad M_0[\mathbb{G}] \models \psi((f(\underset{\sim}{a}_0))^{\mathbb{G}}, ..., (f(\underset{\sim}{a}_{n-1}))^{\mathbb{G}})$.

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(53) Note that $\varphi(g, \lambda, \kappa) = $ "$g''\lambda$ is cofinal in κ" is Δ_0.

(54) "q decides $\underset{\sim}{f}(\check{\alpha})$" means that there is a (unique) $\xi_q^\alpha \in \mathsf{On}$ such that $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{\alpha}) = \xi_q^\alpha$".

(55) This formally means that $\mathsf{ZFC} \vdash (\varphi \to \psi)$.

By Forcing Theorem (Corollary 4.8), it follows that

$$M_0 \models \text{“} f(p) \Vdash_{\mathbb{P}} \text{“} \psi(f(\underset{\sim}{a}_0), \dots, f(\underset{\sim}{a}_{n-1})) \text{””}. \tag{4.50}$$

By the isomorphism f this is equivalent to

$$M \models \text{“} p \Vdash_{\mathbb{P}} \text{“} \psi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}) \text{””}. \tag{4.51}$$

Finally, by the elementarity $M \prec V_\chi$ and the absoluteness over V_χ, this is equivalent to $p \Vdash_{\mathbb{P}} \text{“} \psi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}) \text{”}$. □ (Lemma 4.20)

For the following theorem we need AC. Actually we can even prove that the assertion of the following theorem is equivalent to AC over ZF.

Theorem 4.21 (Maximal Principle) *Suppose that $\mathbb{P}$ is a poset, $p \in \mathbb{P}$, $\varphi = \varphi(x_0, \dots, x_{n-1}, x)$ an $\mathcal{L}$-formula and $\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}$ $\mathbb{P}$-names.* maximality-principle

If $p \Vdash_{\mathbb{P}} \text{“} \exists x\, \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, x) \text{”}$, then there is a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $p \Vdash_{\mathbb{P}} \text{“} \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, \underset{\sim}{a}) \text{”}$.

Proof. Let

$$D = \{q \in \mathbb{P} \restriction p : \text{there is a } \mathbb{P}\text{-name } \underset{\sim}{a} \text{ such that } q \Vdash_{\mathbb{P}} \text{“} \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, \underset{\sim}{a}) \text{”}\}. \tag{4.52}$$ mp-0

By (4.9), D is dense below p.

Let $A \subseteq D$ be a maximal antichain below p (see Lemma 4.13). For each $r \in A$ choose a $\mathbb{P}$-name $\underset{\sim}{a}_r$ such that

$$r \Vdash_{\mathbb{P}} \text{“} \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, \underset{\sim}{a}_r) \text{”}. \tag{4.53}$$ mp-0-0

Let

$$\underset{\sim}{a} = \{\langle \underset{\sim}{b}, r' \rangle : \langle \underset{\sim}{b}, q \rangle \in \underset{\sim}{a}_r \text{ for some } r \in A \text{ and } q \in \mathbb{P}, \text{ and } r' \leq_{\mathbb{P}} r, q\}. \tag{4.54}$$ mp-1

Cl-mp-0

Claim 4.21.1 *For each $r \in A$ we have $r \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{a} \equiv \underset{\sim}{a}_r \text{”}$.*

Proof. By definition of $\underset{\sim}{a}$ and the definition of the forcing relation for euqations (4.3). ⊣ (Claim 4.21.1)

Cl-mp-1

Claim 4.21.2 *For each $r \in A$ we have $r \Vdash_{\mathbb{P}} \text{“} \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, \underset{\sim}{a}) \text{”}$*

Proof. By (4.53), Claim 4.21.1 and Lemma 4.20. ⊣ (Claim 4.21.2)

By Lemma 4.14, it follows that $p \Vdash_{\mathbb{P}} \text{“} \varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1}, \underset{\sim}{a}) \text{”}$. Thus this $\underset{\sim}{a}$ is as desired. □ (Theorem 4.21)

5 Some further applications of simple forcing constructions

appl

[to be written later.]

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