Iterated forcing

— a (still quite incomplete) lecture note of a course on iterated forcing
Oct.–Nov. 2018, Katowice

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main updates: 19.01.03(Th19:43(JST)) 18.12.30(So00:28(JST)) 18.12.24(Mo00:51(JST)) 18.12.14(Fr05:33(CET)) 18.11.12(Mo16:18(CET)) 18.10.29(Mo11:19(CET)) 18.10.21(So13:26(CEST)) 18.09.13(Th10:21(JST)) 18.08.27(Mo02:16(JST))

2023 年 06 月 22 日 (23:30 (JST))[(0)]

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(0) The present text is a lecture note of the course on iterated forcing given by the author in Katowice in October–November 2018. The course is a continuation of the introductory course on forcing given by the author in 2017. The present note often refers the lecture note [Fuchino 2017] of that course which is also available in the internet (see References on p.104 ∼.).

The most up to date version of this text is downloadable as:

https://fuchino.ddo.jp/notes/iterated-forcing-katowice-2018.pdf

The present text as well as [Fuchino 2017] is still a work in progress. These texts will be constantly updated during and after the course in 2018. Any comments and questions in connection with these texts are appreciated.

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1 Preliminaries

1.1 Posets, generic filter and forcing relation prel

We work in the axiom system of Zermelo-Fraenkel set-theory with Axiom of Choice (ZFC) po
or some extension of this system. Sometimes however we have to step out from the axiom system and treat the axiom system from the point of view of meta-mathematics.

A non-empty set $\mathbb{P}$ with a (partial) preorder[(1)] $\leq_{\mathbb{P}}$ and a designated greatest element of $\mathbb{P}$ with respect to $\leq_{\mathbb{P}}$ which is denoted by $\mathbb{1}_{\mathbb{P}}$,[(2)] is called a *poset*. Elements of a poset $\mathbb{P}$ are often called conditions.

Two elements $\mathbb{p}, \mathbb{q} \in \mathbb{P}$ are said to be *compatible* (notation: $\mathbb{p} \top_{\mathbb{P}} \mathbb{q}$) if there is $\mathbb{r} \in \mathbb{P}$ with $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}$. $\mathbb{p}, \mathbb{q} \in \mathbb{P}$ are *incompatible* (notation: $\mathbb{p} \perp_{\mathbb{P}} \mathbb{q}$) if they are not compatible.

$D \subseteq \mathbb{P}$ is *dense* if, for any $\mathbb{p} \in \mathbb{P}$, there is $\mathbb{d} \in D$ such that $\mathbb{d} \leq_{\mathbb{P}} \mathbb{p}$. $D \subseteq \mathbb{P}$ is *open dense* if D is dense and downward closed (i.e., for any $\mathbb{d} \in D$, if $\mathbb{d}' \leq_{\mathbb{P}} \mathbb{d}$ then $\mathbb{d}' \in D$).

$F \subseteq \mathbb{P}$ is a *filter* on $\mathbb{P}$ if F is upward closed (i.e., for any $\mathbb{p}, \mathbb{q} \in \mathbb{P}$, if $\mathbb{q} \in F$ and $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ then $\mathbb{p} \in F$) and, for any $\mathbb{p}_0, \mathbb{p}_1 \in F$ there is $\mathbb{p}_2 \in F$ such that $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_0, \mathbb{p}_1$.

Suppose that M is a transitive set[(3)] and $\langle M, \in \rangle$ satisfies sufficiently large finite fragment of ZFC (we say simply that M is a transitive model of ZFC[(4)]).

For a poset $\mathbb{P} \in M$ a filter $\mathbb{G} \subseteq \mathbb{P}$ is said to be *$(M, \mathbb{P})$-generic*[(5)] if

(1.1) $\mathbb{G} \cap D \neq \emptyset$ holds for all dense $D \subseteq \mathbb{P}$ with $D \in M$. po-a-0

For a poset $\mathbb{P}$, a filter F on $\mathbb{P}$ is said to be *maximal* if it is $\subseteq$-maximal among filters on $\mathbb{P}$.

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(1) A binary relation $\leq$ on a set X is said to be preorder if $\leq$ is transitive and reflexive.

(2) This means that we have $\mathbb{p} \leq_{\mathbb{P}} \mathbb{1}_{\mathbb{P}}$ for all $\mathbb{p} \in \mathbb{P}$. Since the preorder $\leq_{\mathbb{P}}$ does not necessarily satisfy antisymmetry, there might be greatest elements of $\mathbb{P}$ other than $\mathbb{1}_{\mathbb{P}}$. We often write $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ to indicate the components of $\mathbb{P}$ and use the symbol '$\mathbb{P}$' to denote both the underlying set and the structure of the poset.

(3) a set X is said to be transitive if for any $x \in X$ and $y \in x$ we always have $y \in X$.

(4) Actually the finite fragment depends largely on the context in which it appears. For more details see Section 1 in [Fuchino 2017].

(5) In the literature, it is also said that "$\mathbb{G}$ is a generic filter on $\mathbb{P}$ over M", "$\mathbb{G}$ is M-generic over $\mathbb{P}$", etc.

Lemma 1.1 *Suppose that $\mathbb{G}$ is a $(M, \mathbb{P})$-generic filter for a transitive model M of* ZFC *and a poset $\mathbb{P} \in M$. If $\mathbb{p} \in \mathbb{P}$ is such that* P-prel-a-0

$$\mathbb{p} \top_{\mathbb{P}} \mathbb{q} \text{ for all } \mathbb{q} \in \mathbb{G} \tag{1.2}$$ prel-a-a-0

then $\mathbb{p} \in \mathbb{G}$. In particular, any $(M, \mathbb{P})$-generic filter is a maximal filter on $\mathbb{P}$.

Proof. Let $D = \{\mathbb{r} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \text{ or } \mathbb{q} \perp_{\mathbb{P}} \mathbb{p}\}$. Then $D \in M$ and D is a dense subset of M. Hence there is $\mathbb{q}^* \in \mathbb{G} \cap D$. By (1.2), we have $\mathbb{q}^* \leq_{\mathbb{P}} \mathbb{p}$. Since $\mathbb{G}$ is a filter it follows that $\mathbb{p} \in \mathbb{G}$.

Suppose that $\mathbb{G} \subsetneqq \mathbb{G}'$ and $\mathbb{G}'$ is a filter. Let $\mathbb{p} \in \mathbb{G}' \setminus \mathbb{G}$. Since $\mathbb{G}'$ is a filter, $\mathbb{p}$ should satisfy (1.2). By the first part of the present Lemma, it follows that $\mathbb{p} \in \mathbb{G}$. This is a contradiction to the choice of $\mathbb{p}$. ❑ (Lemma 1.1)

Generic filters are also "complete" in the following sense:

Lemma 1.2 *Suppose that M is a transitive model of* ZFC *and $\mathbb{P} \in M$ is a poset such that $M \models$ "$\mathbb{P}$ is the positive elements $\mathbb{B}^+$ of a complete Boolean algebra".*[6] *For an $(M, \mathbb{P})$-generic filter $\mathbb{G}$, if $S \in M$ is a subset of $\mathbb{P}$ and $S \subseteq \mathbb{G}$, then $(\prod^{\mathbb{B}} S)^M \in \mathbb{G}$.* P-prel-a-1

Proof. In M, let

$$D = \{\mathbb{r} \in \mathbb{P} : \mathbb{r} \leq_{\mathbb{P}} \textstyle\prod^{\mathbb{B}} S \text{ or } \mathbb{r} \perp_{\mathbb{P}} \mathbb{p} \text{ for some } \mathbb{p} \in S\}. \tag{1.3}$$

Then $M \models$ "D is dense in $\mathbb{P}$". Let $\mathbb{r}^* \in \mathbb{G} \cap D$. Then we must have $\mathbb{r}^* \leq_{\mathbb{P}} (\prod^{\mathbb{B}} S)^M$ (since otherwise $\mathbb{G}$ would have an element which is incompatible with an element of $S \subseteq \mathbb{G}$). Since $\mathbb{G}$ is a filter, it follows that $(\prod^{\mathbb{B}} S)^M \in \mathbb{G}$. ❑ (Lemma 1.2)

Lemma 1.3 *If $\mathbb{P}$ is atomless (i.e. if, for any $\mathbb{p} \in \mathbb{P}$, there are $\mathbb{q}_0, \mathbb{q}_1 \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{q}_0 \perp_{\mathbb{P}} \mathbb{q}_1$), then no $(M, \mathbb{P})$-generic filter $\mathbb{G}$ is an element of M.* P-prel-0

Proof. See Lemma 3.1 in [Fuchino 2017]. ❑ (Lemma 1.3)

Lemma 1.3 says that $M[\mathbb{G}]$ defined below (see (1.16)) for any $(M, \mathbb{P})$-generic $\mathbb{G}$ is a non-trivial extension of M, if $\mathbb{P}$ is atomless. This in mind, we shall also call a poset $\mathbb{P}$ *non-trivial* if $\mathbb{P}$ is atomless.

For a poset $\mathbb{P}$, $D \subseteq \mathbb{P}$ is said to be *predense* if

$$D{\downarrow} = \{\mathbb{s} \in \mathbb{P} : \mathbb{s} \leq_{\mathbb{P}} \mathbb{d} \text{ for some } \mathbb{d} \in D\} \tag{1.4}$$ prel-a-a-1

is dense in $\mathbb{P}$. Note that if $D \subseteq \mathbb{P}$ is dense then it is predense. Note also that if $D{\downarrow}$ is dense then it is open dense.

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[6] When we say that $\mathbb{P}$ is the positive elements of a Boolean algebra $\mathbb{B}$, we mean that $\mathbb{P} = \mathbb{B}^+ = \mathbb{B} \setminus \{\mathbb{0}_{\mathbb{B}}\}$ and the preorder $\leq_{\mathbb{P}}$ of $\mathbb{P}$ coincides with the ordering $\leq_{\mathbb{B}}$ of the Boolean algebra $\mathbb{B}$. We show later that this condition is not very much restrictive in connection with the generic extensions (which we are going to introduce in (1.16).

$A \subseteq \mathbb{P}$ is said to be an *antichain* if any two distinct elements of A are incompatible in $\mathbb{P}$. $A \subseteq \mathbb{P}$ is said to be a *maximal antichain* if A is an antichain and maximal with respect to $\subseteq$ among the antichains of $\mathbb{P}$.

Lemma 1.4 (1) *$D \subseteq \mathbb{P}$ is predense if and only if,* P-forcing-eq-3

(1.5) *for all $\mathbb{p} \in \mathbb{P}$, there is $\mathbb{d} \in D$ such that $\mathbb{p} \top_{\mathbb{P}} \mathbb{d}$.* forcing-eq-5-0

(2) *$A \subseteq \mathbb{P}$ is a maximal antichain if and only if A is an antichain and,*

(1.6) *for all $\mathbb{p} \in \mathbb{P}$, there is $\mathbb{a} \in A$ such that $\mathbb{p} \top_{\mathbb{P}} \mathbb{a}$.* forcing-eq-5-1

(3) *$A \subseteq \mathbb{P}$ is maximal antichain if and only if A is an antichain and predense in $\mathbb{P}$.*

(4) *If $D \subseteq \mathbb{P}$ is dense, then any antichain $A \subseteq D$ in $\mathbb{P}$, which is maximal with respect to $\subseteq$ among antichains of $\mathbb{P}$ which are subsets of D, is a maximal antichain in $\mathbb{P}$.*

Proof. (1): Suppose that $D \subseteq \mathbb{P}$ is predense. For $\mathbb{p} \in \mathbb{P}$, let $\mathbb{r} \in D{\downarrow}$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ and let $\mathbb{d} \in D$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{d}$. Then $\mathbb{p} \top_{\mathbb{P}} \mathbb{d}$. Thus (1.5) holds.

Assume now that (1.5) holds. For an arbitrary $\mathbb{p} \in \mathbb{P}$, let $\mathbb{d} \in D$ be such that $\mathbb{p} \top_{\mathbb{P}} \mathbb{d}$. Then there is $\mathbb{r} \in D{\downarrow}$ such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$.

(2): Suppose that $A \subseteq \mathbb{P}$ is an antichain. We have to show that A is a maximal antichain in $\mathbb{P}$ if and only if (1.6) holds.

Suppose that A does not satisfy (1.6). Then there is $\mathbb{p}^* \in \mathbb{P}$ such that $\mathbb{p}^* \perp_{\mathbb{P}} \mathbb{a}$ holds for all $\mathbb{a} \in A$. Then $A \cup \{\mathbb{p}^*\} \supsetneq A$ is an antichain in $\mathbb{P}$. Thus A is not a maximal antichain.

Suppose now that A is not a maximal antichain then there is an antichain A^* in $\mathbb{P}$ such that $A^* \supsetneq A$. Let $\mathbb{p}^* \in A^* \setminus A$. Then $\mathbb{p}^* \perp_{\mathbb{P}} \mathbb{a}$ holds for all $\mathbb{a} \in A$. Thus A does not satisfy (1.6).

(3): This follows form (1) and (2).

(4): Suppose that $A \subseteq D$ is an antichain in $\mathbb{P}$. If D is a maximal antichain then D is clearly maximal with respect to $\subseteq$ among antichains of $\mathbb{P}$ which are subsets of D.

Suppose that A is not a maximal antichain. Then, by (2), there is $\mathbb{p}^*$ such that $\mathbb{p}^* \perp_{\mathbb{P}} \mathbb{a}$ holds for all $\mathbb{a} \in A$. Let $\mathbb{d}^* \in D$ be such that $\mathbb{d}^* \leq_{\mathbb{P}} \mathbb{p}^*$. Then $\mathbb{d}^* \perp_{\mathbb{P}} \mathbb{a}$ holds for all $\mathbb{a} \in A$. Thus, letting $A^* = A \cup \{\mathbb{d}^*\}$. $A \subsetneq A^* \subseteq D$ and A^* is an antichain.

❑ (Lemma 1.4)

Lemma 1.5 *For a transitive model M of* ZFC*, a poset $\mathbb{P} \in M$, and $\mathbb{G} \subseteq \mathbb{P}$ the following are equivalent:* P-prel-1

(a) *$\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter.*

(b) *$\mathbb{G}$ is pairwise compatible, upward closed and satisfies (1.1).*

(c) *$\mathbb{G}$ is pairwise compatible, upward closed and*

(1.7) $\mathbb{G} \cap D \neq \emptyset$ *for all open dense $D \subseteq \mathbb{P}$ such that $D \in M$.* po-a-0-0

(d) *$\mathbb{G}$ is pairwise compatible, upward closed and*

(1.8) $\mathbb{G} \cap D \neq \emptyset$ *for all predense $D \subseteq \mathbb{P}$ such that $D \in M$.* po-a-0-1

(e) *$\mathbb{G}$ is pairwise compatible, upward closed and*

(1.9) $\mathbb{G} \cap A \neq \emptyset$ *for all maximal antichain $A \subseteq \mathbb{P}$ such that $A \in M$.* po-a-0-2

Proof. (b) $\Rightarrow$ (a): Suppose that $\mathbb{G} \subseteq \mathbb{P}$ satisfies the conditions in (b) and $\mathbb{p}$, $\mathbb{q} \in \mathbb{G}$. We have to show that there is $\mathbb{r} \in \mathbb{G}$ such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}$.

In M, let

(1.10) $D = \{\mathbb{r} \in \mathbb{P} : \mathbb{r} \perp_{\mathbb{P}} \mathbb{p} \text{ or } \mathbb{r} \perp_{\mathbb{P}} \mathbb{q} \text{ or } \mathbb{r} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}\}$.[7] po-a-1

Claim 1.5.1 *D is dense in $\mathbb{P}$.*

$\vdash$ Suppose that $\mathbb{s} \in \mathbb{P}$. If $\mathbb{s} \perp_{\mathbb{P}} \mathbb{p}$ then $\mathbb{s} \in D$. Otherwise there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{s}, \mathbb{p}$. If $\mathbb{r} \perp_{\mathbb{P}} \mathbb{q}$ then $\mathbb{r} \in D$. Otherwise there is $\mathbb{r}' \leq_{\mathbb{P}} \mathbb{r}, \mathbb{q}$. Then $\mathbb{r}' \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}$ and hence $\mathbb{r}' \in D$. $\dashv$ (Claim 1.5.1)

Since $\mathbb{G}$ satisfies (1.1). There is $\mathbb{r} \in \mathbb{G} \cap D$. Since $\mathbb{G}$ is pairwise compatible, $\mathbb{r} \in D$ must be established by $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}$.

(a) $\Rightarrow$ (c): Easy.

(c) $\Rightarrow$ (b): Suppose that $\mathbb{G} \subseteq \mathbb{P}$ satisfies the conditions in (c). We have to show that $\mathbb{G}$ satisfies (1.1). Suppose that $D \in M$ is such that $D \subseteq \mathbb{P}$ and D is dense in $\mathbb{P}$. In M, let $\tilde{D} = \{\mathbb{r} \in \mathbb{P} : \mathbb{r} \leq_{\mathbb{P}} \mathbb{d} \text{ for some } \mathbb{d} \in D\}$. Then $\tilde{D}$ is open dense. Thus by $\mathbb{G} \models$(1.7), there is $\mathbb{r} \in \mathbb{G} \cap \tilde{D}$. Let $\mathbb{d} \in D$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{d}$. Since $\mathbb{G}$ is upward closed $\mathbb{d} \in \mathbb{G}$.

(c) $\Rightarrow$ (d): Assume that $\mathbb{G} \subseteq \mathbb{P}$ is pairwise compatible upward closed and satisfies (1.7). We show that $\mathbb{P}$ satisfies (1.8).

Suppose that $D \subseteq \mathbb{P}$ is predense then $D\downarrow$ as defined in (1.4) is open dense. Hence, by (1.7), there is $\mathbb{r} \in D$ such that $\mathbb{r} \in \mathbb{G} \cap D\downarrow$. Let $\mathbb{d} \in D$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{d}$. Since $\mathbb{G}$ is upward closed, we have $\mathbb{d} \in \mathbb{G} \cap D$.

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[7] Here, we assume that the fragment of ZFC M satisfies is large enough to guarantee that there is such D in M.

(d) $\Rightarrow$ (e): This implication is easy to show since every maximal antichain in $\mathbb{P}$ is predense by Lemma 1.4, (3).

(e) $\Rightarrow$ (b): Assume that $\mathbb{G}$ is pairwise compatible, upward closed and satisfies (1.9). To prove that $\mathbb{G}$ satisfies (1.1), let $D \subseteq \mathbb{P}$ be a dense subset. Then an antichain $A \subseteq D$ maximal among antichains which are subsets of D is a maximal antichain in $\mathbb{P}$ (Lemma 1.4, (4)). By (1.9), there is $\mathbb{d} \in \mathbb{G} \cap A \subseteq \mathbb{G} \cap D$. ❑ (Lemma 1.5)

The following is a special case of the Rasiowa-Sikorski lemma:

Lemma 1.6 *If M is a countable transitive model of* ZFC*, then, for any poset $\mathbb{P} \in M$ and $\mathbb{p} \in \mathbb{P}$, there is an $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$.* P-prel-2

Proof. Since M is countable, the set $\mathcal{D} = \{D \in M : D \text{ is a dense subset of } \mathbb{P}\}$ is countable as well.

Let $\mathcal{D} = \{D_n : n \in \omega\}$ and let $\langle \mathbb{p}_n : n \in \omega \rangle$ be a descending sequence (with respect to $\leq_{\mathbb{P}}$) of elements of $\mathbb{P}$ such that $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p}_n \in D_n$ for all $n \in \omega$. The construction of such a sequence is possible since each D_n $(n \in \omega)$ is dense in $\mathbb{P}$.

Let

$$\mathbb{G} = \{\mathbb{q} \in \mathbb{P} : \mathbb{p}_n \leq_{\mathbb{P}} \mathbb{r} \text{ for some } n \in \omega\}. \tag{1.11}$$

Then $\mathbb{G}$ is a $(M, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$. ❑ (Lemma 1.6)

For a poset $\mathbb{P}$, the class $\mathsf{V}^{\mathbb{P}}$ of all $\mathbb{P}$-names $\mathsf{V}^{\mathbb{P}}$ is defined recursively[8] by

$$\underset{\sim}{x} \in \mathsf{V}^{\mathbb{P}} \;\Leftrightarrow\; \text{all elements of } \underset{\sim}{x} \text{ are of the form } \langle \underset{\sim}{y}, \mathbb{p} \rangle \text{ where } \underset{\sim}{y} \in \mathsf{V}^{\mathbb{P}} \text{ and } \mathbb{p} \in \mathbb{P}. \tag{1.12}$$ po-a-2

If M is a transitive model of ZFC and $\mathbb{P} \in M$, then we have $(\mathsf{V}^{\mathbb{P}})^M = \mathsf{V}^{\mathbb{P}} \cap M$ (see [Fuchino 2017], p.33). We write $M^{\mathbb{P}} = (\mathsf{V}^{\mathbb{P}})^M$.

For a transitive model M of ZFC, a poset $\mathbb{P} \in M$ and an $(M, \mathbb{P})$-generic filter $\mathbb{G}$, the interpretation $\underset{\sim}{x}^{\mathbb{G}}$ of $\underset{\sim}{x} \in M^{\mathbb{P}}$ by $\mathbb{G}$ is defined recursively by

$$\underset{\sim}{x}^{\mathbb{G}} = \{\underset{\sim}{y}^{\mathbb{G}} : \langle \underset{\sim}{y}, \mathbb{p} \rangle \in \underset{\sim}{x} \text{ for some } \mathbb{p} \in \mathbb{G}\}. \tag{1.13}$$ prel-0

$\underset{\sim}{x}^{\mathbb{G}}$ is also denoted by $\underset{\sim}{x}[\mathbb{G}]$.

For a transitive model M of ZFC and a poset $\mathbb{P} \in M$ the standard $\mathbb{P}$-name $\check{x}_{\mathbb{P}}$ (or simply $\check{x}$ if it is clear which $\mathbb{P}$ is meant[9]) of $x \in M$ is defined recursively by

$$\check{x}_{\mathbb{P}} = \{\langle \check{y}_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle : y \in x\}. \text{ [10]} \tag{1.14}$$ prel-1

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[8] In spite of apparent circulation, the definition is legitimate. For more details about recursive definition, see Section 2.2 in [Fuchino 2017].

[9] We shall also denote the check name $\check{x}_{\mathbb{P}}$ by $\surd_{\mathbb{P}}(x)$. This notation is in need in particular, if x is represented by some term consisting of several letters.

[10] Note that $\check{\emptyset}_{\mathbb{P}} = \emptyset$.

For a poset $\mathbb{P}$, the *standard name for a generic filter $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ over $\mathbb{P}$* (written often simply as $\underset{\sim}{\mathbb{G}}$ if it is clear which $\mathbb{P}$ is meant) is defined by

$$\underset{\sim}{\mathbb{G}}_{\mathbb{P}} = \{\langle \check{\mathbb{p}}_{\mathbb{P}}, \mathbb{p}\rangle \,:\, \mathbb{p} \in \mathbb{P}\}. \tag{1.15}$$ po-0

Note that, for a transitive model M of ZFC, if $\mathbb{P} \in M$ then, for $x \in M$, $(\check{x}_{\mathbb{P}})^M = \check{x}_{\mathbb{P}}$.[(11)] Hence we have $\check{x}_{\mathbb{P}} \in M^{\mathbb{P}}$ for all $x \in M$. Similarly we have $(\underset{\sim}{\mathbb{G}}_{\mathbb{P}})^M = \underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ [(11)] and hence $\underset{\sim}{\mathbb{G}}_{\mathbb{P}} \in M^{\mathbb{P}}$.

$\check{x}_{\mathbb{P}}$ is actually a "name" of x and $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ a "name" of the generic filter:

Lemma 1.7 *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ a poset and $\mathbb{G}$ a filter on $\mathbb{P}$.* P-prel-3

(1) *For any $x \in M$, we have $(\check{x}_{\mathbb{P}})[\mathbb{G}] = x$.*

(2) *For any poset $\mathbb{P}$ and $(M, \mathbb{P})$-generic filter $\mathbb{G}$, we have $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}[\mathbb{G}] = \mathbb{G}$.*

Proof. See Lemma 3.3 in [Fuchino 2017]. ❑ (Lemma 1.7)

For a transitive model of ZFC, poset $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic filter $\mathbb{G}$, let

$$M[\mathbb{G}] = \{\underset{\sim}{x}[\mathbb{G}] \,:\, \underset{\sim}{x} \in M^{\mathbb{P}}\}. \tag{1.16}$$ prel-2

$M[\mathbb{G}]$ is called the *generic extension of M by $\mathbb{G}$*.

Lemma 1.8 *Suppose that M is a transitive model of ZFC, $\mathbb{P} \in M$ a poset and $\mathbb{G}$ a filter on $\mathbb{P}$. Then:* P-prel-4

(1) *$M[\mathbb{G}]$ is a transitive set.*

(2) *$M \subseteq M[\mathbb{G}]$ and $\mathbb{G} \in M[\mathbb{G}]$.*

(3) *For all $\underset{\sim}{x} \in M^{\mathbb{P}}$, we have $\mathrm{rank}(\underset{\sim}{x}^{\mathbb{G}}) \leq \mathrm{rank}(\underset{\sim}{x})$.*

(4) $\mathsf{On}^{M[\mathbb{G}]} = \mathsf{On} \cap M[\mathbb{G}] = \mathsf{On} \cap M = \mathsf{On}^M$.

(5) $|\, M[\mathbb{G}] \,| = |\, M \,|$.

Proof. (2): follows from Lemma 1.7. For the rest, see the proof of Lemma 3.4 in [Fuchino 2017]. ❑ (Lemma 1.8)

Theorem 1.9 *If M is a transitive model of ZFC, $\mathbb{P} \in M$ a poset and $\mathbb{G}$ an $(M, \mathbb{P})$-generic filter, then $M[\mathbb{G}]$ is a transitive model of ZFC. Furthermore, $M[\mathbb{G}]$ is the $\subseteq$-minimal transitive model N of ZFC with $M \subseteq N$ and $\mathbb{G} \in M$.*[(12)] ❑ P-prel-5

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(11) This follows from the absoluteness of the corresponding notions over a transitive model. See [Fuchino 2017], Section 2.5.

(12) The real statement of the Theorem is as follows: For any finite fragment T of ZFC, we can find a finite fragment T^* of ZFC extending T such that for any transitive model M of T^*, poset $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic filter $\mathbb{G}$, $M[\mathbb{G}]$ is a transitive model of T. Furthermore, if T is sufficiently large, then $M[\mathbb{G}]$ is the $\subseteq$-minimal transitive model N of T with $M \subseteq N$ and $\mathbb{G} \in M$.

Theorem 1.9 is proved by using the *forcing relation*. For each $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ we introduced in [Fuchino 2017] the forcing relation $\cdot \Vdash \cdot$ "$\varphi(\cdots)$" which is a class relation (definable in ZFC) (by a rather complicated $\mathcal{L}_\varepsilon$-formula see (4.3),(4.6) $\sim$ (4.9) in [Fuchino 2017]) which satisfies the properties formulated in the following Lemma 1.11 and Theorem 1.12.

For $\mathbb{p} \in \mathbb{P}$, $D \subseteq \mathbb{P}$ is said to be dense below $\mathbb{p}$ if $D \cap (\mathbb{P} \downarrow \mathbb{p})$ is dense in the poset $\mathbb{P} \downarrow \mathbb{p}$ where $\mathbb{P} \downarrow \mathbb{p}$ is the poset with the underlying set $\mathbb{P} \downarrow \mathbb{p} = \{\mathbb{q} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}\}$, the preorder $\leq_{\mathbb{P}} \cap (\mathbb{P} \downarrow \mathbb{p})^2$ (which is denoted by $\leq_{\mathbb{P}}$ for simplicity) and the designated maximal element $\mathbb{1}_{\mathbb{P}\downarrow\mathbb{p}} = \mathbb{p}$. "Open dense below $\mathbb{p}$", "predense below $\mathbb{P}$", "maximal antichain below $\mathbb{p}$" etc. are defined similarly.

Exercise 1.10 (1) *For poset $\mathbb{P}$ and $\mathbb{p} \in \mathbb{P}$, if $\mathcal{D} \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ then $\mathcal{D} \cap \mathbb{P} \downarrow \mathbb{p}$ is dense in $\mathbb{P} \downarrow \mathbb{p}$.* *ex-prel-a*

(2) *For a poset $\mathbb{P} \in M$, and $\mathbb{p} \in \mathbb{P}$, if $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$, then, for any $D \in M$ such that D is predense below $\mathbb{p}$, we have $\mathbb{G} \cap D \neq \emptyset$.*

Hints. For (2), let $\tilde{D} = \{\mathbb{r} : \mathbb{r} \in D \text{ or } \mathbb{r} \perp_{\mathbb{P}} \mathbb{d} \text{ for all } \mathbb{d} \in D\}$. Then $\tilde{D}$ is predense in $\mathbb{P}$. ❑ (Exercise 1.10)

Lemma 1.11 (Forcing Lemma) (Lemma 4.3 in [Fuchino 2017]) *Suppose that $\mathbb{P}$ is a poset, $\mathbb{p} \in \mathbb{P}$ and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ are $\mathbb{P}$-names.* *P-prel-5-0* *forcing-l*

(1) *If $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$, then $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".*

(2) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" $\Leftrightarrow$ $\{\mathbb{q} \leq_{\mathbb{P}} \mathbb{p} : \mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"$\}$ *is dense below* $\mathbb{p}$. ❑

Note that by (1) of Lemma 1.11 and by Lemma 1.4 applied to $\mathbb{P} \downarrow \mathbb{p}$, "... is dense below $\mathbb{p}$" in (2) may be replaced by "... is open dense below $\mathbb{p}$" or "... is predense below $\mathbb{p}$".

By (1) and (2) of Lemma 1.11, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(\cdots)$" if and only if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\cdots)$" holds for all $\mathbb{p} \in \mathbb{P}$. We shall also simply write $\Vdash_{\mathbb{P}}$ "$\varphi(\cdots)$" in place of $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(\cdots)$".

Theorem 1.12 (Forcing Theorem) (Theorem 4.5 in [Fuchino 2017]) *For any transitive model of ZFC and a poset $\mathbb{P} \in M$, let $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$.*[13] *Then, for any $(M, \mathbb{P})$-generic filter $\mathbb{G}$, we have:* *forcing-th*

(1) *If $M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"", for some $\mathbb{p} \in \mathbb{G}$, then $M[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$.*

(2) *If $M[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$, then there is a $\mathbb{p} \in \mathbb{G}$ such that $M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"".* ❑

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[13] We denote with $M^{\mathbb{P}}$ the subset $\left(V^{\mathbb{P}}\right)^M$ of M. Since the relation "$\underset{\sim}{a}$ is a $\mathbb{P}$ name" is Δ_1^{ZF} (see

Theorem 1.13 (2. Forcing Theorem) (Theorem 4.7 in [Fuchino 2017]) *Let M be a transitive model of* ZFC *and $\mathbb{P} \in M$ a poset such that* *P-prelim-7-th*

(1.17) *for all $\mathbb{p} \in \mathbb{P}$ there is an $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$.* *prel-2-0*

Then, for any $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$, we have
$M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1})$" " if and only if we have $M[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], \ldots, \underset{\sim}{a}_{n-1}[\mathbb{G}])$ for all $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$.

Corollary 1.14 (Corollary 4.8 in [Fuchino 2017]) *Let M be a countable transitive model of* ZFC *and $\mathbb{P} \in M$ a poset. Then, for any $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1} \in M^{\mathbb{P}}$, we have* *P-prel-8*
$M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1})$" " if and only if we have $M[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], \ldots, \underset{\sim}{a}_{n-1}[\mathbb{G}])$ for all $(M, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$.

Proof. By Lemma 1.6 and Theorem 1.13. ❑ (Corollary 1.14)

By Forcing Theorem 1.12 (and by the M, M^*-argument in the proof of Lemma 1.16), (the first part of) Theorem 1.9 is equivalent to the following property of the forcing relation:

Theorem 1.15 *For every poset $\mathbb{P}$, we have $\Vdash_{\mathbb{P}}$ "ψ" holds for each axiom ψ of* ZFC. *forcing-models-ZFC*

Everything about the forcing relation can be derived from the properties of forcing relation formulated in the Forcing Lemma (Lemma 1.11) and the Forcing Theorem (Theorem 1.12) and Theorem 1.15. This follows from Theorem 1.17 below. In particular, the 2. Forcing Theorem is derivable form these properties. We shall also check this below.

In [Fuchino 2017], the following Lemma is an easy consequence of definition of the forcing relation and Lemma 1.11. We prove it here from Lemma 1.11 and the Forcing Theorem (Theorem 1.12).

Lemma 1.16 *For a poset $\mathbb{P}$, $\mathbb{p} \in \mathbb{P}$, $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, \ldots, x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1}$, if $\mathbb{p} \not\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1})$", then there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1})$".* *P-prel-9*

Proof. Suppose $\mathbb{p} \not\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1})$".
Let M be a countable $\in$-model of a sufficiently large fragment of ZFC such that

(1.18) all the $\mathcal{L}_\varepsilon$-formulas which appear in the following argument are absolute over M and prel-3

(1.19) M contains everything needed. In particular, $\mathbb{P}$, $\mathbb{p}$, $\underset{\sim}{a}_0, \ldots, \underset{\sim}{a}_{n-1} \in M$. prel-4

We can find such M by virtue of Lévy-Montague Absoluteness Theorem (Theorem 2.18 in [Fuchino 2017]) and Downward Löwenheim-Skolem Theorem (Theorem 2.13 in [Fuchino 2017]).

[Fuchino 2017], Theorem 2.27), $M^{\mathbb{P}} = \{x \in M : M \models$ "x is a $\mathbb{P}$-name"$\}$. Because of this fact, elements of $M^{\mathbb{P}}$ are called $\mathbb{P}$-names in M.

Let $i : M \overset{\cong}{\to} M^*$ be the Mostowski collapse of M (in particular, M^* is countable transitive model of (a sufficiently large fragment of) ZFC, see Theorem 2.10 in [Fuchino 2017] for Mostowski collapse). Let $\mathbb{P}^* = i(\mathbb{P})$, $\mathbb{p}^* = i(\mathbb{p})$, $\underset{\sim}{a}_0^* = i(\underset{\sim}{a}_0), \dots, \underset{\sim}{a}_{n-1}^* = i(\underset{\sim}{a}_{n-1})$. Since i is an isomorphism and since $M \models$ "$\mathbb{p} \not\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \dots, \underset{\sim}{a}_{n-1})$"", we have $M^* \models$ "$\mathbb{p}^* \not\Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$""

Working in M^*, let $\mathbb{r}^* \leq_{\mathbb{P}^*} \mathbb{p}^*$ be such that

(1.20) for all $\mathbb{r} \leq_{\mathbb{P}^*} \mathbb{r}^*$ we have $\mathbb{r} \not\Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$". prel-4-0

Note that there is such $\mathbb{r}^*$ by Lemma 1.11, (2).[14]

Let $\mathbb{G}$ be an $(M^*, \mathbb{P}^*)$-generic filter such that $\mathbb{r}^* \in \mathbb{G}$ (there is such $\mathbb{G}$ by Lemma 1.6).

Claim 1.16.1 $M^*[\mathbb{G}] \models \neg\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$.

$\vdash$ Otherwise, there is $\mathbb{s} \in \mathbb{G}$ such that $M^* \models$ "$\mathbb{s} \Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$"" by Theorem 1.12, (2) (which holds in M^*). For $\mathbb{r} \in \mathbb{G}$ with $\mathbb{r} \leq_{\mathbb{P}^*} \mathbb{s}, \mathbb{r}^*$, we have $M^* \models$ "$\mathbb{r} \Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$"" by Lemma 1.11, (1). This is a contradiction to the choice of $\mathbb{r}^*$. $\dashv$ (Claim 1.16.1)

By Claim 1.16.1 and by Theorem 1.12, (2), there is $\mathbb{s} \in \mathbb{G}$ such that $M^* \models$ "$\mathbb{s} \Vdash_{\mathbb{P}^*}$ "$\neg\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$"". Let $\mathbb{q}^* \in \mathbb{G}$ be such that $\mathbb{q}^* \leq_{\mathbb{P}^*} \mathbb{s}, \mathbb{r}^*$. Then $\mathbb{q}^* \leq_{\mathbb{P}^*} \mathbb{p}^*$ and $M^* \models$ "$\mathbb{q}^* \Vdash_{\mathbb{P}^*}$ "$\neg\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$"".

Let $\mathbb{q} \in M$ be such that $i(\mathbb{q}) = \mathbb{q}^*$. Then we have

(1.21) $M \models$ "$\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $q \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$"".

By absoluteness, it follows that $\mathbb{q} \in \mathbb{P}$, $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $q \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0^*, \dots, \underset{\sim}{a}_{n-1}^*)$". ❑ (Lemma 1.16)

The argument with appropriately chosen absolute enough countable M constructed using Lévy-Montague Absoluteness Theorem and Downward Löwenheim-Skolem Theorem and its Mostowski collapse M^* is repeated many times. In the following we shall refer this argument as "*M, M^*-argument*"[15]. The proof of the next Theorem is also an example of such M, M^*-argument.

Theorem 1.17 *The class relations $\cdot \Vdash_{\cdot}$ "$\varphi(\cdots)$", for $\mathcal{L}_\varepsilon$-formulas φ are determined uniquely by the properties in Forcing Lemma 1.11 and the Forcing Theorem 1.12.* P-prel-10

Proof. Suppose that another array of class relations $\cdot \Vdash^*_{\cdot}$ "$\varphi(\cdots)$" for $\mathcal{L}_\varepsilon$-formulas φ satisfies also the properties in Forcing Lemma 1.11 and the Forcing Theorem 1.12 but it is different from the original forcing relation. This means that there are a poset $\mathbb{P}$, $\mathbb{p} \in \mathbb{P}$,

[14] More precisely, there is such $\mathbb{r}^*$ since M and hence M^* satisfies all the axioms of ZFC which are needed to prove Lemma 1.11, (2) and thus $M^* \models$ "Lemma 1.11, (2)".

[15] From some point on, we even drop the mention to the M, M^*-argument and begin to deploy a strictly speaking inconsistent narration (see the explanation on p.14 ∼).

an $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ such that, say $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $\mathbb{p} \nVdash^*_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Since $\cdot \Vdash_{\cdot}$ "$\varphi(\cdots)$" satisfies Lemma 1.11, (1) and $\cdot \Vdash^*_{\cdot}$ "$\varphi(\cdots)$" Lemma 1.16 (which is a consequence of Lemma 1.11 and the Forcing Theorem), there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ such that

(1.22) $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $\mathbb{q} \Vdash^*_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

Let M be an $\in$-model of sufficiently large fragment of ZFC such that all $\mathcal{L}_\varepsilon$-formulas which appear in the following argument are absolute over M and $\mathbb{P}$, $\mathbb{q}$, $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in M$. We have

(1.23) $M \models$ "$\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"" and $M \models$ "$\mathbb{q} \Vdash^*_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"" prel-4-1

by the absoluteness.

Let $i : M \overset{\cong}{\to} M^*$ be the Mostowski collapse of M and let Let $\mathbb{P}^* = i(\mathbb{P})$, $\mathbb{q}^* = i(\mathbb{q})$, $\underset{\sim}{a}^*_0 = i(\underset{\sim}{a}_0), ..., \underset{\sim}{a}^*_{n-1} = i(\underset{\sim}{a}_{n-1})$.

By (1.23) and since i is an isomorphism,

(1.24) $M^* \models$ "$\mathbb{q}^* \Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}^*_0, ..., \underset{\sim}{a}^*_{n-1})$"" and $M^* \models$ "$\mathbb{q}^* \Vdash^*_{\mathbb{P}^*}$ "$\neg\varphi(\underset{\sim}{a}^*_0, ..., \underset{\sim}{a}^*_{n-1})$"". prel-4-2

Let $\mathbb{G}$ be an $(M^*, \mathbb{P}^*)$-generic filter with $\mathbb{q}^* \in \mathbb{G}$. By Forcing Theorem 1.12, (1) for $\cdot \Vdash_{\cdot}$ "$\varphi(\cdots)$" and $\cdot \Vdash^*_{\cdot}$ "$\neg\varphi(\cdots)$", it follows from (1.24) that $M^*[\mathbb{G}] \models \varphi(\underset{\sim}{a}^*_0[\mathbb{G}], ..., \underset{\sim}{a}^*_{n-1}[\mathbb{G}])$ and $M^*[\mathbb{G}] \models \neg\varphi(\underset{\sim}{a}^*_0[\mathbb{G}], ..., \underset{\sim}{a}^*_{n-1}[\mathbb{G}])$. This is a contradiction. ❑ (Theorem 1.17)

Exercise 1.18 *Prove the 2. Forcing Theorem 1.13 from the Forcing Lemma 1.11 and the Forcing Theorem 1.12.* ex-prel-0

Hints. Check that the proof of Theorem 4.7 in [Fuchino 2017] uses only Lemma 1.11, the Forcing Theorem 1.12 (and Lemma 1.16 above which is now proved from Lemma 1.11 and the Forcing Theorem). ❑ (Exercise 1.18)

The following fact about the forcing relation is also used frequently mostly without mention.

Lemma 1.19 (Deduction Lemma for Forcing Relation) *Suppose that $\mathbb{P}$ is an arbitrary poset.* Deduction-l

(1) *For any axiom φ of ZFC (formulated as an $\mathcal{L}_\varepsilon$-sentence) we have $\Vdash_{\mathbb{P}}$ "φ".*

(2) *If an $\mathcal{L}_\varepsilon$-sentence ψ is a theorem in ZFC, then we have $\Vdash_{\mathbb{P}}$ "ψ". In particular, for any tautology ψ, we have $\Vdash_{\mathbb{P}}$ "ψ".*

(3) *For $\mathbb{p} \in \mathbb{P}$, $\mathcal{L}_\varepsilon$-formulas $\varphi = \varphi(x_0, ..., x_{n-1})$, $\psi = \psi(x_0, ..., x_{n-1})$ and $\mathbb{P}$ names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, If $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\forall x_0 \cdots \forall x_{n-1}(\varphi \to \psi)$" and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".*

Proof. (1): Suppose that φ is one of the axioms of ZFC. M^* is a countable transitive model of (sufficiently large fragment of) ZFC which is absolute for the formula in the argument below. For an arbitrary poset $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic filter $\mathbb{G}$, we have $M[\mathbb{G}] \models \varphi$ by Theorem 1.14 (see also the footnote (12)). By Corollary 1.14, it follows that $M \models$ " $\Vdash_{\mathbb{P}}$ " φ "". Thus $M \models$ "$\forall \mathbb{P}$ ("$\mathbb{P}$ is a poset" $\to \Vdash_{\mathbb{P}}$ " φ ")". By the absoluteness, it follows that $\forall \mathbb{P}$ ("$\mathbb{P}$ is a poset" $\to \Vdash_{\mathbb{P}}$ " φ ") holds.

The assertions (2) and (3) can be proved similarly to the proof of Lemma 1.16 given above. Instead, we shall give here the proofs of (2) and (3) which are more related to the alternative proof of Lemma 1.16 given below.

(2): Suppose Γ is a (concretely given) finite fragment of ZFC such that $\Gamma \vdash \psi$. Let M^* be a countable transitive model of a sufficiently large fragment of ZFC such that $M^*[\mathbb{G}] \models \Gamma$ holds for any poset $\mathbb{P}$ and any $(M^*, \mathbb{P})$-generic filter $\mathbb{G}$ and such that the sentence

$$\forall \mathbb{P}\,(\text{“}\mathbb{P} \text{ is a poset”} \to \Vdash_{\mathbb{P}} \text{“}\psi\text{”}) \tag{1.25}$$

is absolute over M^*.

Suppose that $\mathbb{P} \in M$ is a poset and $\mathbb{G}$ is an arbitrary $(M^*, \mathbb{P})$-generic filter. Then we have $M^*[\mathbb{G}] \models \Gamma$. It follows that $M^*[\mathbb{G}] \models \psi$. Since $\mathbb{G}$ was arbitrary, it follows by Corollary 1.14 that $M^* \models$ " $\Vdash_{\mathbb{P}}$ " ψ ""$. Since $\mathbb{P}$ was arbitrary, it follows that

$$M^* \models \text{“}\forall \mathbb{P}\,(\text{“}\mathbb{P} \text{ is a poset”} \to \Vdash_{\mathbb{P}} \text{“}\psi\text{”})\text{”}. \tag{1.26}$$

By the absoluteness it follows that $\forall \mathbb{P}$ ("$\mathbb{P}$ is a poset" $\to \Vdash_{\mathbb{P}}$ " ψ ") holds.

(3): Suppose that $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\psi = \psi(x_0, ..., x_{n-1})$ are $\mathcal{L}_\varepsilon$-formulas. What we have to prove here is that the following universal formula holds (in ZFC):

$$\begin{aligned} \forall \mathbb{P}\, \forall \mathbb{p}\, \forall \underset{\sim}{a}_0 \cdots \forall \underset{\sim}{a}_{n-1}\, ((& \text{“}\mathbb{P} \text{ is a poset”} \wedge \mathbb{p} \in \mathbb{P} \wedge \text{“}\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \text{ are } \mathbb{P}\text{-names”} \\ & \wedge\, \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\forall x_0 \cdots \forall x_{n-1}(\varphi \to \psi)\text{”} \wedge \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}) \\ & \to\ \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}). \end{aligned} \tag{1.27}$$

Let us call this formula η. Let M^* be a countable transitive model of sufficiently large fragment of ZFC such that the sentence η is absolute over M^*.

Let $\mathbb{P} \in M^*$ be an arbitrary poset and suppose that $\mathbb{p} \in \mathbb{P}$ and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ are $\mathbb{P}$-names in M^* such that $M^* \models$ " $\mathbb{p} \Vdash_{\mathbb{P}}$ " $\forall x_0 \cdots \forall x_{n-1}(\varphi \to \psi)$ ""$ and $M^* \models$ " $\mathbb{p} \Vdash_{\mathbb{P}}$ " $\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$ ""$.

Let $\mathbb{G}$ be an arbitrary $(M^*, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$. By Theorem 1.12, (1) we have $M^*[\mathbb{G}] \models$ "$\forall x_0 \cdots \forall x_{n-1}(\varphi \to \psi)$" and $M^*[\mathbb{G}] \models$ "$\varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$". It follows that $M^*[\mathbb{G}] \models$ "$\psi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$". Since $\mathbb{G}$ was arbitrary it follows by Corollary 1.14 that $M^* \models$ " $\mathbb{p} \Vdash_{\mathbb{P}}$ " $\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$ ""$.

This shows that $M^* \models \eta$. By the absoluteness it follows that η really holds.

❑ (Lemma 1.19)

The following Lemma can be proved quite similarly to Lemma 1.16. The following Lemma is going to play one of the crucial roles in the relative consistency proofs using the forcing.

Lemma 1.20 (Consistency of the Forcing Relation) *For any poset $\mathbb{P}$, $\mathbb{p} \in \mathbb{P}$, $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$", Then we have $\mathbb{p} \nVdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".* L-consistency

Proof. Suppose toward a contradiction that

(1.28) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and prel-4-3

(1.29) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". prel-4-4

Let M be a countable $\in$-model of a fragment of ZFC large enough for the following arguments such that $\mathbb{P}$, $\mathbb{p}$, $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in M$ and the forcing relations $\cdot \Vdash_{\cdot}$ "$\varphi(\cdot, ..., \cdot)$" and $\cdot \Vdash_{\cdot}$ "$\neg\varphi(\cdot, ..., \cdot)$" are absolute between M and V. Let M^* be the Mostowski's collapse of M. Let $i : M \overset{\cong}{\to} M^*$ and let $\mathbb{P}^* = i(\mathbb{P})$, $\mathbb{p}^* = i(\mathbb{p})$, $\underset{\sim}{a}^*_0 = i(\underset{\sim}{a}_0), ..., \underset{\sim}{a}^*_{n-1} = i(\underset{\sim}{a}_{n-1})$. By (1.28), (1.29), absoluteness and the isomorphism, we have

(1.30) $M^* \models \mathbb{p}^* \Vdash_{\mathbb{P}^*}$ "$\varphi(\underset{\sim}{a}^*_0, ..., \underset{\sim}{a}^*_{n-1})$" and prel-4-5

(1.31) $M^* \models \mathbb{p}^* \Vdash_{\mathbb{P}^*}$ "$\neg\varphi(\underset{\sim}{a}^*_0, ..., \underset{\sim}{a}^*_{n-1})$". prel-4-6

Also M^* is a model of the same fragment of axioms of ZFC satisfied by M by isomorphism.

Let $\mathbb{G}$ be an $(M^*, \mathbb{P}^*)$-generic filter such that $\mathbb{p}^* \in \mathbb{G}$. Note that such $\mathbb{G}$ exists by Lemma 1.6.

By (1.30) and (1.31), we have $M^*[\mathbb{G}] \models \varphi(\underset{\sim}{a}^*_0[\mathbb{G}], ..., \underset{\sim}{a}^*_{n-1}[\mathbb{G}])$ and $M^*[\mathbb{G}] \models \neg\varphi(\underset{\sim}{a}^*_0[\mathbb{G}], ..., \underset{\sim}{a}^*_{n-1}[\mathbb{G}])$ respectively by Forcing Theorem 1.12, (1). This is a contradiction. ❑ (Lemma 1.20)

Exercise 1.21 *For a poset $\mathbb{P}$, $\mathbb{p} \in \mathbb{P}$, $\mathcal{L}_\varepsilon$-formulas $\varphi = \varphi(x_0, ..., x_{n-1})$, $\psi = \psi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, if $\Vdash_{\mathbb{P}}$ "$(\varphi \to \psi)(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\psi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".* ❑ ex-prel-1

The argument with M and M^* in the proofs of Lemma 1.16, Theorem 1.17 and Lemma 1.19 (as well as in the proofs for the Exercises above) appears repeatedly in may proofs of assertions on the forcing relation. To simplify the presentation in all of these proofs, we often identify the universe V with M^* and assume that, for all $\mathbb{p} \in \mathbb{P}$, there are $(\mathsf{V}, \mathbb{P})$-generic filters $\mathbb{G}$ (outside the universe V!) with $\mathbb{p} \in \mathbb{G}$, $V[\mathbb{G}]$ satisfies assertions corresponding to the assertions of Lemma 1.8 with M there replaced by V and the conclusion of the 2. Forcing Theorem 1.13 holds for V in place of M. In particular, we treat statements of the form "For all $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$, $\mathsf{V}[\mathbb{G}] \models$ ..." as fully interchangeable with the statements of the form "$\mathbb{p} \Vdash_{\mathbb{G}}$ "...""".

The proof of Lemma 1.16 in this "fake" setting would then look like the following:

A simplified but formally "incorrect" proof of Lemma 1.16: fake-statement

Suppose $\mathbb{p} \not\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". By Lemma 1.11, (2), there is $\mathbb{q}^* \leq_{\mathbb{P}} \mathbb{p}$ such that, for all $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}^*$, we have $\mathbb{r} \not\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter with $\mathbb{q}^* \in \mathbb{G}$.

Claim. $V[\mathbb{G}] \models \neg\varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$

$\vdash$ Otherwise $V[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. By Forcing Theorem 1.12, (2), there is $\mathbb{s} \in \mathbb{G}$ such that $\mathbb{s} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Let $\mathbb{r} \in \mathbb{G}$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{s}, \mathbb{q}^*$. Then $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" by Lemma 1.11, (1). This is a contradiction to the choice of $\mathbb{q}^*$.

❑ (Claim)

By the Claim above and by Lemma 1.12, (2), there is $\mathbb{r} \in \mathbb{G}$ such that $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Let $\mathbb{q} \in \mathbb{G}$ be such that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{q}^*$. Then $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" by Lemma 1.11, (1). ❑

Strictly speaking, the "proof" above does not make any sense since V is the class of all sets and hence there can be no "generic" sets outside V. However, when such an "incorrect" forcing proof as above is given, we can systematically translate it into a correct one. We show how this translation works through the example of the "incorrect" proof of Lemma 1.16 as above:

Translation of the "incorrect" proof of Lemma 1.16 above to a correct formal proof:

What we have to do is to find a formal proof in ZFC of the following $\mathcal{L}_\varepsilon$-formula ψ:

$$\begin{aligned}(1.32)\quad \forall\mathbb{P}\forall\mathbb{p}\forall\underset{\sim}{a}_0, ..., \forall\underset{\sim}{a}_{n-1}(&\text{"}\mathbb{P}\text{ is a poset"} \wedge \mathbb{p}\,\varepsilon\,\mathbb{P} \wedge \text{"}\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}\text{ are }\mathbb{P}\text{-names"} \\ &\wedge \mathbb{p} \not\Vdash_{\mathbb{P}} \text{"}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{"} \\ &\rightarrow \exists\mathbb{q}(\mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \wedge \mathbb{q} \Vdash_{\mathbb{P}} \text{"}\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{"})).\end{aligned}$$ prel-6

Let M^* be a countable transitive model which satisfies a sufficiently large fragment of ZFC and such that ψ is absolute over M^*.[16] We can find M^* similarly to the construction of M^* is the (original) proof of Lemma 1.16.

Now insert the "incorrect" proof here with all the appearance of "V" in the proof switched to "M^*". This modified proof is now correct because of Lemma 1.6 and it proves $M^* \models \psi$. By the absoluteness of M^*, it follows that ψ really holds. ❑

In the following, we often give "incorrect" forcing proofs similar to the "simplified but formally "incorrect" proof of Lemma 1.16" as above without further comments. The "incorrect" version of forcing theorems applied in such incorrect proofs is as formulated in the following Theorem 1.22. When the incorrect proofs using these "fake" theorems

[16] That is, $\psi \leftrightarrow$ "$M^* \models \psi$" holds.

are given, we simply assume that the proofs are to be corrected in the way as described in the "Translation of the *incorrect* proof of Lemma 1.16 to a formal proof".

Theorem 1.22 ("Fake" version of the Forcing Theorems) (Cf. Corollary 1.14) *Suppose that $\mathbb{P}$ is a poset. Then, for any $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1} \in \mathsf{V}^{\mathbb{P}}$, we have* P-prel-8'
$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" *if and only if we have* $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$ *for all* $(\mathsf{V}, \mathbb{P})$-*generic filter* $\mathbb{G}$ *with* $\mathbb{p} \in \mathbb{G}$. ❑

Henceforth, we shall simply refer Theorem 1.22 as "Forcing Theorem".

In Section 3.4, we shall consider an axiomatic framework of the set-generic multiverse in which we can interpret the formally "incorrect" forcing proofs with $(V, \mathbb{P})$-generic filters and the Forcing Theorem (Theorem 1.22)more straightforwardly.

1.2 Further properties of forcing relation and some easy applications in relative consistency proofs

The usage of the Axiom of Choice (AC) in the proof of the following theorem is indispensable: it is known that AC is equivalent to the following Lemma over ZF (see [Miller].) fpforcing

Lemma 1.23 (Maximal Principle) *Suppose that $\mathbb{P}$ is a poset, $\mathbb{p} \in \mathbb{P}$, $\varphi = \varphi(x, x_0, ..., x_{n-1})$ an $\mathcal{L}_\varepsilon$-formula and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ are $\mathbb{P}$-names.* L-fpforcing-2

(1) *If $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\exists x \varphi(x, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then there are $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".*

(2) *If $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\exists x \varphi(x, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then there is a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"*

Proof. (1): Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. By Forcing Theorem 1.12, (1), we have $\mathsf{V}[\mathbb{G}] \models \exists x \varphi(x, \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. Let $a \in \mathsf{V}[\mathbb{G}]$ be such that $\mathsf{V}[\mathbb{G}] \models \varphi(a, \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$ and let $\underset{\sim}{a}$ be a $\mathbb{P}$-name such that $\underset{\sim}{a}[\mathbb{G}] = a$. Then $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}[\mathbb{G}], \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. By Forcing Theorem 1.12, (2), there is $\mathbb{r} \in \mathbb{G}$ such that $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Let $\mathbb{q} \in \mathbb{G}$ be such that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{r}$. By Forcing Lemma 1.11, (1), this $\mathbb{q}$ is as desired.

(2): For each $\mathbb{r} \in \mathbb{P}$ choose $\underset{\sim}{a}_{\mathbb{r}}$ such that

$$\underset{\sim}{a}_{\mathbb{r}} = \begin{cases} \text{a } \mathbb{P}\text{-name } \underset{\sim}{a} \text{ such that} & \\ \qquad \mathbb{r} \Vdash_{\mathbb{P}} \text{"}\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{"}, & \text{if there is such } \underset{\sim}{a}; \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1.33}$$

By (1) (and Lemma 1.11, (1),

$$D = \{\mathbb{r} \in \mathbb{P} \downarrow \mathbb{p} : \mathbb{r} \Vdash_{\mathbb{P}} \text{"}\varphi(\underset{\sim}{a}_{\mathbb{r}}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{"}\} \tag{1.34}$$

is dense below $\mathbb{p}$. Let $A \subseteq D$ be a maximal antichain below $\mathbb{p}$.

(1.35) $\quad \underset{\sim}{a} = \{\langle \underset{\sim}{b}, \mathbb{s}\rangle$: there are $\mathbb{r} \in A$ and $\mathbb{t} \in \mathbb{P}$ such that $\langle \underset{\sim}{b}, \mathbb{t}\rangle \in \underset{\sim}{a}_{\mathbb{r}}$ and $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{t}\}$ fpforcing-a-0

We show that this $\underset{\sim}{a}$ is as desired. By 2. Forcing Theorem 1.13, it is enough to show that $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}[\mathbb{G}], \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$ holds for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ with $\mathbb{p} \in \mathbb{G}$, since, by the 2. Forcing Theorem 1.13, this implies $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

We have $\mathbb{G} \cap A \neq \emptyset$ since A is a maximal antichain below $\mathbb{p} \in \mathbb{G}$ (see Exercise 1.10, (2)). Let $\mathbb{r}^* \in \mathbb{G} \cap A$.

Claim 1.23.1 $\underset{\sim}{a}[\mathbb{G}] = \underset{\sim}{a}_{\mathbb{r}^*}[\mathbb{G}]$.

$\vdash$ Suppose $b \in \underset{\sim}{a}[\mathbb{G}]$. Then there are $\mathbb{s} \in \mathbb{G}$ and a $\mathbb{P}$-name $\underset{\sim}{b}$ such that $\langle \underset{\sim}{b}, \mathbb{s}\rangle \in \underset{\sim}{a}$ and $b = \underset{\sim}{b}[\mathbb{G}]$.

By the definition (1.35) of $\underset{\sim}{a}$, there is $\mathbb{r} \in A$ and $\mathbb{t} \in \mathbb{P}$ such that $\langle \underset{\sim}{b}, \mathbb{t}\rangle \in \underset{\sim}{a}_{\mathbb{r}}$ and

(1.36) $\quad \mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{t}.$ fpforcing-a-1

Since A is pairwise incompatible and $\mathbb{r} \in \mathbb{G}$ by (1.36), we have $\mathbb{r} = \mathbb{r}^*$. Since we also have $\mathbb{t} \in \mathbb{G}$, it follows that $\underset{\sim}{b}[\mathbb{G}] \in \underset{\sim}{a}_{\mathbb{r}^*}[\mathbb{G}]$.

Suppose now that $b \in \underset{\sim}{a}_{\mathbb{r}^*}[\mathbb{G}]$. Then $b = \underset{\sim}{b}[\mathbb{G}]$ for some $\langle \underset{\sim}{b}, \mathbb{t}\rangle \in \underset{\sim}{a}_{\mathbb{r}^*}$ such that $\mathbb{t} \in \mathbb{G}$. Let $\mathbb{s} \in \mathbb{G}$ be such that $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}^*, \mathbb{t}$. Then we have $\langle \underset{\sim}{b}, \mathbb{s}\rangle \in \underset{\sim}{a}$ and $b = \underset{\sim}{b}[\mathbb{G}] \in \underset{\sim}{a}[\mathbb{G}]$.

$\dashv$ (Claim 1.23.1)

Since $\mathbb{r}^* \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_{\mathbb{r}^*}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$", it follows by Forcing Theorem 1.12, (1) and Claim 1.23.1 that $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}[\mathbb{G}], \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_[[\mathbb{G}])$. ❑ (Lemma 1.23)

Lemma 1.24 (Semantic Maximal Principle) *Suppose that $\mathbb{P}$ is a poset, $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic filter, $\varphi = \varphi(x_0, ..., x_{m-1}, y_0, ..., y_{n-1})$ an $\mathcal{L}_\varepsilon$-formula and $b_0, ..., b_{n-1} \in \mathsf{V}$. If $\Vdash_{\mathbb{P}}$ "$\exists x_0 \cdots \exists x_{m-1} \varphi(x_0, ..., x_{m-1}, \check{b}_0, ..., \check{b}_{n-1})$" and $a_0, ..., a_{m-1} \in \mathsf{V}[\mathbb{G}]$ are such that $\mathsf{V}[\mathbb{G}] \models (a_0, ..., a_{m-1}, b_0, ..., b_{n-1})$, then there are $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{m-1}$ (in V) such that $\underset{\sim}{a}_0[\mathbb{G}] = a_0, ..., \underset{\sim}{a}_{m-1}[\mathbb{G}] = a_{m-1}$ and $\Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{m-1}, \check{b}_0, ..., \check{b}_{n-1})$".* L-fpforcing-2-0

Proof. ❑ (Lemma 1.24)

The following lemma is also used very often:

Lemma 1.25 *Suppose that $\mathbb{P}$ is a poset, $\mathbb{p} \in \mathbb{P}$, $\varphi = \varphi(x, x_0, ..., x_{n-1})$ an $\mathcal{L}_\varepsilon$-formula, a a set (in V) and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ are $\mathbb{P}$-names. If $\mathbb{p} \Vdash_{\mathbb{P}}$ "$(\exists y \ \varepsilon \ \check{a}_{\mathbb{P}})\, \varphi(y, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$", then there are $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $b \in a$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\check{b}_{\mathbb{P}}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".* P-fpforcing-a

Proof. Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$. By Theorem 1.12, (1), we have $\mathsf{V}[\mathbb{G}] \models (\exists y \in \check{a}_{\mathbb{P}}[\mathbb{G}])\, \varphi(y, \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. Since $\check{a}_{\mathbb{P}}[\mathbb{G}] = a$, there is $b \in a$ such that $\mathsf{V}[\mathbb{G}] \models \varphi(b, \underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. Since $b = \check{b}_{\mathbb{P}}[\mathbb{G}]$, it follows that $\mathsf{V}[\mathbb{G}] \models \varphi(\check{b}_{\mathbb{P}}, \underset{\sim}{a}_0[\mathbb{G}], ...,$

$\underset{\sim}{a}_{n-1}[\mathbb{G}]$). By Theorem 1.12, (2), there is $\mathbb{q} \in \mathbb{G}$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\varphi(\check{b}_{\mathbb{P}}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Without loss of generality, we may assume $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$.[17] ❑ (Lemma 1.25)

For a cardinal κ, a poset $\mathbb{P}$ is sad to satisfy *κ-cc* [18] if any antichain in $\mathbb{P}$ is of size $< \kappa$. For any poset $\mathbb{P}$, if $|\,\mathbb{P}\,| = \kappa$ then $\mathbb{P}$ is κ^+-cc. $\aleph_1$-cc is also called countable chain condition and is abbreviated as *ccc*.

A poset $\mathbb{P}$ is said to *preserve a cardinal λ* if $\Vdash_{\mathbb{P}}$ "$\check{\lambda}_{\mathbb{P}}$ is a cardinal". For a regular cardinal λ, $\mathbb{P}$ preserves cofinality λ if for all ordinal α with $cf(\alpha) = \lambda$ we have $\Vdash_{\mathbb{P}}$ " $cf(\check{\alpha}_{\mathbb{P}}) = \lambda$".

In our approach that we treat V as a representative of a countable transitive model of (a large fragment of) ZFC,

(1.37) $\mathbb{P}$ preserves a cardinal κ if κ remains a cardinal in the generic extension $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$. fpforcing-a-1-0

(1.38) $\mathbb{P}$ preserves cofinality κ if for any ordinal α with $cf(\alpha) = \kappa$, the equation still holds in the generic extension $V[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$. fpforcing-a-1-1

Lemma 1.26 *Suppose that κ is a regular uncountable cardinal, $\mathbb{P}$ a κ-cc poset, $\mathbb{p} \in \mathbb{P}$, A, B are sets and $\underset{\sim}{f}$ a $\mathbb{P}$-name such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{A}_{\mathbb{P}} \to \check{B}_{\mathbb{P}}$".* L-fpforcing-3
Then there is a mapping $F : A \to [B]^{<\kappa}$ such that[19] *$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} \subseteq \check{F}_{\mathbb{P}}$".*

Proof. For each $a \in A$, let

(1.39) $D_a = \{\mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \,:\, \mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\check{a}_{\mathbb{P}}) \equiv \check{b}_{\mathbb{P}}$ for some $b \in B$"$\}$. fpforcing-a-2

D_a is dense below $\mathbb{p}$ by Lemma 1.25. For each $\mathbb{q} \in D_a$, the $b \in a$ as in (1.39) is unique. Let us denote this b by $b^a_{\mathbb{q}}$.

For $a \in A$, let $A_a \subseteq D_a$ be a maximal antichain in $\mathbb{P}$. By the κ-cc of $\mathbb{P}$ we have $|\,A_a\,| < \kappa$ for all $a \in A$. Let $F : A \to [B]^{<\kappa}$ be defined by $F(a) = \{b^a_{\mathbb{q}} \,:\, \mathbb{q} \in A_a\}$ for each $a \in A$. This F is as desired. ❑ (Lemma 1.26)

Corollary 1.27 *Suppose that κ is a regular uncountable cardinal and $\mathbb{P}$ is a κ-cc poset. For any $\lambda \geq \kappa$ with $cf(\lambda) \geq \kappa$, $\mathbb{p} \in \mathbb{P}$, $\mathbb{P}$-name $\underset{\sim}{b}$ and a set A, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \check{A}_{\mathbb{P}}$ and $|\,\underset{\sim}{b}\,| < \lambda$", then there is $B \in [A]^{<\lambda}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \check{B}_{\mathbb{P}}$".* P-fpforcing-a-0

Proof. Since $cf(\lambda) \geq \kappa$ and $\mathbb{P}$ is κ-cc, there is $\lambda' < \lambda$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$|\,\underset{\sim}{b}\,| \leq \lambda'$". Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \lambda' \to \underset{\sim}{b}$ is a surjection". By Lemma 1.26, there

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[17] Since $\mathbb{G}$ is a filter there is $\mathbb{q}' \in \mathbb{G}$ such that $\mathbb{q}' \leq_{\mathbb{P}} \mathbb{p}, \mathbb{q}$. By Lemma 1.11, (1), we have $\mathbb{q}' \Vdash_{\mathbb{P}}$ "$\varphi(\check{b}_{\mathbb{P}}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Thus we may replace $\mathbb{q}$ with $\mathbb{q}'$ ($\leq_{\mathbb{P}} \mathbb{p}$).

[18] "cc" stands for "chain condition". But we also often use the expression as an adjective and say that "$\mathbb{P}$ *is* κ-cc", "a κ-cc poset $\mathbb{P}$,' etc.

[19] For a set X and a cardinal, $[X]^{<\kappa} = \{a \subseteq X \,:\, |\,a\,| < \kappa\}$.

is a mapping $F : \lambda' \to [A]^{<\kappa}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} \subseteq \check{F}_{\mathbb{P}}$". Let $B = \bigcup F''\lambda$. Then $|B| \leq \kappa \cdot \lambda' < \lambda$ and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \check{B}_{\mathbb{P}}$". ❑ (Corollary 1.27)

Lemma 1.28 *For a regular uncountable cardinal κ, if a poset $\mathbb{P}$ is κ-cc then $\mathbb{P}$ preserves all cardinals $\geq \kappa$ and all cofinality $\geq \kappa$.* P-fpforcing-0

Proof. Let $\mathbb{P}$ be a κ-cc poset. We show show that $\mathbb{P}$ preserves all cofinality $\geq \kappa$. Let $\lambda \geq \kappa$ be a regular cardinal and $cf(\alpha) = \lambda$ for an ordinal α. Suppose, toward a contradiction, that $\not\Vdash_{\mathbb{P}}$ "$cf(\check{\alpha}_{\mathbb{P}}) \not\equiv \check{\lambda}_{\mathbb{P}}$". Then, by Lemma 1.16 and Lemma 1.25, there are $\mathbb{p} \in \mathbb{P}$, $\delta < \lambda$ and $\mathbb{P}$-name $\underset{\sim}{f}$ such that

(1.40) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{\delta}_{\mathbb{P}} \to \check{\alpha}_{\mathbb{P}}$ and $\underset{\sim}{f}''\check{\delta}_{\mathbb{P}}$ is cofinal in $\check{\alpha}_{\mathbb{P}}$". fpforcing-0

By Lemma 1.26, there is $F : \delta \to [\alpha]^{<\kappa}$ such that

(1.41) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} \subseteq \check{F}_{\mathbb{P}}$". fpforcing-1

For each $\xi \in \delta$, let $g(\xi) = \sup F(\xi)$. $g(\xi) < \alpha$ since $cf(\alpha) = \lambda \geq \kappa$.

Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{G})$-generic filter with $\mathbb{p} \in \mathbb{G}$. Then we have

(1.42) $\mathsf{V}[\mathbb{G}] \models$ "$\underset{\sim}{f}[\mathbb{G}](\xi) \leq \underbrace{g(\xi)}_{= \check{g}_{\mathbb{P}}[\mathbb{G}](\xi)}$ for all $\xi \in \delta$".

Since $\mathsf{V}[\mathbb{G}] \models$ "$\underset{\sim}{f}[\mathbb{G}]''\delta$ is cofinal in α" by (1.40), we have $\mathsf{V}[\mathbb{G}] \models$ "$g''\delta$ is cofinal in α". Hence $g''\delta$ is cofinal in α. This is a contradiction to $cf(\alpha) = \lambda > \delta$.

From the preservation of cofinalities $\geq \kappa$ it follows immediately that all regular cardinal $\geq \kappa$ are preserved (recall (1.37) and (1.38)). The preservation of singular cardinals follows from this since a limit of cardinals is a cardinal. ❑ (Lemma 1.28)

Corollary 1.29 *If a poset $\mathbb{P}$ is ccc, then $\mathbb{P}$ preserves all cardinals and cofinality.* P-fpforcing-0-0

Proof. If $\mathbb{P}$ is a ccc poset, then all cardinals and cofinality $\geq \omega_1$ are preserved by Lemma 1.28. $\aleph_0$ and cofinality ω are preserved as well because of the absoluteness of ω.

❑ (Corollary 1.29)

The following version of Lemma 1.26 is also used frequently:

Lemma 1.30 *Suppose that $\kappa \leq \lambda$ are regular cardinals, $\mathbb{P}$ a κ-cc poset and $\mathbb{p} \in \mathbb{P}$. If $\underset{\sim}{o}$ is a $\mathbb{P}$-name such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{o} < \lambda$", then there is an ordinal $\alpha < \lambda$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{o} \leq \alpha$".*
In particular, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$|\underset{\sim}{a}| < \lambda$" for a $\mathbb{P}$-name $\underset{\sim}{a}$, then there is $\mu < \kappa$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$|\underset{\sim}{a}| \leq \check{\mu}$". We can also replace "$|\underset{\sim}{a}| < \cdots$" in the statement by any inequality with some other ordinal invariant of the set, e.g. "$rank(\underset{\sim}{a}) < \cdots$". P-fpforcing-0-1

Proof. Let A be a maximal antichain below $\mathbb{p}$ in $\mathbb{P}$ such that, for each $\mathbb{q} \in A$ there is $\alpha_{\mathbb{q}} < \lambda$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{o} = \alpha_{\mathbb{q}}$". Let $\alpha = \sup\{\alpha_{\mathbb{q}} : \mathbb{q} \in A\}$. Since $|A| < \kappa$ and κ is regular we have $\alpha < \kappa$. Clearly we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{o} \leq \mu$". ❑ (Lemma 1.30)

The next lemma is going to be an important tool in connection with iteration of forcing:

Lemma 1.31 *Suppose that κ is a regular uncountable cardinal and $\mathbb{P}$ a κ-cc poset with $\mathbb{P} \subseteq \mathcal{H}(\kappa)$. Then, for any $\mathbb{P}$-name $\underset{\sim}{a}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \varepsilon \mathcal{H}(\kappa)$" there is a $\mathbb{P}$-name $\underset{\sim}{a}^* \in \mathcal{H}(\kappa)$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{a}^*$".* P-fpforcing-0-2

Proof. We prove the following statement by induction on $\alpha < \kappa$:

(1.43)$_\alpha$ For any $\mathbb{P}$-name $\underset{\sim}{a}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \mathcal{H}(\kappa) \ \wedge \ rank(\underset{\sim}{a}) \leq \check{\alpha}$" there is a $\mathbb{P}$-name $\underset{\sim}{a}^* \in \mathcal{H}(\kappa)$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{a}^*$". fpforcing-1-0

By Lemma 1.30 (and since $\mathcal{H}(\kappa) \subseteq V_\kappa$ (see Lemma 2.9 in [Fuchino 2017]), (1.43)$_\alpha$ implies the Lemma.

Suppose that (1.43)$_\beta$ holds for all $\beta < \alpha$ and $\underset{\sim}{a}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \mathcal{H}(\kappa) \ \wedge \ rank(\underset{\sim}{a}) \leq \check{\alpha}$".

By Lemma 1.30, there is $\lambda < \kappa$ such that

(1.44) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \emptyset \ \vee \ \exists f\,(f : \check{\lambda} \to \underset{\sim}{a} \ \wedge \ f$ is a surjection)". fpforcing-1-1

By Maximal Principle (Lemma 1.23), there is a $\mathbb{P}$-name $\underset{\sim}{f}$ such that

(1.45) $\Vdash_{\mathbb{P}}$ "$(\underset{\sim}{a} \not\equiv \emptyset \ \to \ (\underset{\sim}{f} : \check{\lambda} \to \underset{\sim}{a} \ \wedge \ \underset{\sim}{f}$ is a surjection)) $\wedge \ \forall \xi \leq \check{\lambda}\,(rank(\underset{\sim}{f}(\xi)) < \alpha)$" fpforcing-1-2

By the induction hypothesis, for each $\xi < \lambda$, there is a $\mathbb{P}$-name $\underset{\sim}{a}^*_\xi$ such that $\underset{\sim}{a}^*_\xi \in \mathcal{H}(\kappa)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}^*_\xi \equiv \underset{\sim}{f}(\check{\xi})$".

Let $A \subseteq \mathbb{P}$ be a maximal pairwise incompatible set such that each $\mathbb{r} \in A$ decides "$\underset{\sim}{a} \equiv \emptyset$". $|A| < \kappa$ by the κ-cc of $\mathbb{P}$.

Let

(1.46) $\underset{\sim}{a}^* = \{\langle \underset{\sim}{a}_\xi, \mathbb{r}\rangle : \xi < \lambda, \mathbb{r} \in A, \mathbb{r} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \not\equiv \emptyset$"$\}$. fpforcing-1-3

Then $\underset{\sim}{a}^* \in \mathcal{H}(\kappa)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{a}^*$" as desired. ❑ (Lemma 1.31)

The following Lemma was proved in [Fuchino 2017] using the method of elementary submodels. Recall that a family of sets A is said to be a Δ-system with the root r if $r = a \cap a'$ for all distinct $a, a' \in A$.

Lemma 1.32 (Generalized Δ-System Lemma, see Theorem 2.29 in [Fuchino 2017]) *Suppose that κ is an infinite cardinal and λ with $\kappa < \lambda$ a regular cardinal such that,* P-fpforcing-1

(1.47) *for all $\alpha < \lambda$, we have $|[\alpha]^{<\kappa}| < \lambda$.*[20] fpforcing-2

If $\langle a_\alpha : \alpha < \lambda\rangle$ is a sequence of sets of size $< \kappa$, then there is $I \in [\lambda]^\lambda$ and r such that $\{a_\alpha : \alpha \in I\}$ is a Δ-system with the root r. ❑

Since $\kappa = \aleph_0$ and arbitrary uncountable λ satisfy (1.47), we obtain the following

Corollary 1.33 (Δ-System Lemma) *Suppose that λ is a regular uncountable cardinal and $\langle a_\alpha : \alpha < \lambda \rangle$ a sequence of finite sets. Then there is $I \in [\lambda]^\lambda$ such that $\{a_\alpha : \alpha \in I\}$ is a Δ-system (with a root r).* ❑ P-fpforcing-2

For sets I, J and cardinal κ with $|I| \leq \kappa$, let $\mathrm{Fn}(I, J, <\kappa)$ be the poset $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ defined by

(1.48) $\mathbb{P} = \{\mathbb{p} : \mathbb{p} : I_0 \to J \text{ for some } I_0 \in [I]^{<\kappa}\}$, fpforcing-3

(1.49) $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \;\Leftrightarrow\; \mathbb{p} \subseteq \mathbb{q}$ fpforcing-4

for $\mathbb{p}$, $\mathbb{q} \in \mathrm{Fn}(I, J, <\kappa)$ and

(1.50) $\mathbb{1}_{\mathbb{P}} = \emptyset$. fpforcing-5

Lemma 1.34 *For an infinite cardinal κ, $\mathbb{P} = \mathrm{Fn}(I, J, <\kappa)$ has the $(|J|^{<\kappa})^+$-cc.* P-fpforcing-3

Proof. Note that κ and $\lambda = (|J|^{<\kappa})^+$ satisfy (1.47). Suppose that $A \in [\mathbb{P}]^\lambda$. We show that A is not an antichain.

Let $\langle \mathbb{p}_\xi : \xi < \lambda \rangle$ be a 1-1 enumeration of A. By Lemma 1.32, there is $I \in [\lambda]^\lambda$ such that $\{\mathrm{dom}(\mathbb{p}_\xi) : \xi \in I\}$ is a Δ-system with the root $d \in [I]^{<\kappa}$. Since $\mathbb{p}_\xi \restriction d$, $\xi \in I$ have at most $|J|^{<\kappa} < \lambda$ many possibilities, there is $I' \in [I]^\lambda$ such that $\mathbb{p}_\xi \restriction d$, $\xi \in I'$ are all the same, by the pigeonhole principle. Clearly, for all ξ, $\xi' \in I'$, $\mathbb{p}_\xi$ and $\mathbb{p}_{\xi'}$ are compatible.

❑ (Lemma 1.34)

$\mathrm{Fn}(I, J, <\aleph_0)$ is also denoted as $\mathrm{Fn}(I, J)$.

Corollary 1.35 *For any set I, $\mathrm{Fn}(I, 2)$ has the ccc.* P-fpforcing-3-0

Lemma 1.36 *Suppose that I and J are sets and κ a cardinal such that $\kappa \leq |I|$. Let $\mathbb{P} = \mathrm{Fn}(I, J, <\kappa)$ and $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic ultrafilter.* P-fpforcing-4

(1) *$\bigcup \mathbb{G} : I \to J$ and $\bigcup \mathbb{G}$ is a surjection onto J.*

(2) *Suppose further that $I = I_0 \times \kappa$. Let $g_i : \kappa \to J$ be defined by $g_i(\alpha) = \bigcup \mathbb{G}(\langle i, \alpha \rangle)$, for all $i \in I_0$ and $\alpha \in \kappa$. For any distinct i, $i' \in I_0$, we have $g_i \neq g_{i'}$.*

Proof. (1): Since, $\mathbb{G}$ is a filter, $\bigcup \mathbb{G}$ is a mapping from a subset of I to J. To show that $\mathrm{dom}(\bigcup \mathbb{G}) = I$, let $i \in I$. In V, let $D_i = \{\mathbb{p} \in \mathbb{P} : i \in \mathrm{dom}(\mathbb{p})\}$. Since D_i is dense in $\mathbb{P}$, there is $\mathbb{p} \in \mathbb{G} \cap D_i$. Thus $i \in \mathrm{dom}(\mathbb{p}) \subseteq \mathrm{dom}(\bigcup \mathbb{G})$.

To show that $\bigcup \mathbb{G}$ is a surjection, let $j \in J$. In V, let $D_j = \{\mathbb{p} \in \mathbb{P} : j \in \mathbb{p}''\mathrm{dom}(\mathbb{p})\}$. Since D_j is dense in $\mathbb{P}$, there is $\mathbb{p} \in \mathbb{G} \cap D_j$. Thus $j \in \mathbb{p}''\mathrm{dom}(\mathbb{p}) \subseteq \bigcup \mathbb{G}''I$.

(2): Suppose i, $i' \in I_0$ with $i \neq i'$. Then

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(20) λ is said to be $<\kappa$-inaccessible if it satisfies (1.47).

(1.51) $\quad D_{i,i'} = \{\mathbb{p} \in \mathbb{P} \,:\, \text{there is } \xi \in \kappa \text{ such that } \mathbb{p}(\langle i, \xi\rangle) \neq \mathbb{p}(\langle i', \xi\rangle)\}$

is dense in $\mathbb{P}$. Since $D_{i,i'} \in \mathsf{V}$, there is $\mathbb{p} \in \mathbb{G} \cap D_{i,i'}$. Let $\xi \in \kappa$ be as in the definition of $D_{i,i'}$ for this $\mathbb{p}$. Then

(1.52) $\quad g_i(\xi) = \bigcup \mathbb{G}(\langle i, \xi\rangle) = \mathbb{p}(\langle i, \xi\rangle) \neq \mathbb{p}(\langle i', \xi\rangle) = \bigcup \mathbb{G}(\langle i', \xi\rangle) = g_{i'}(\xi).$

Note that the condition $\kappa \leq |\, I \,|$ is needed to show the denseness of D_i, D_j and $D_{i,i'}$.

❑ (Lemma 1.36)

Recall that the Continuum Hypothesis (CH) is the statement $2^{\aleph_0} = \aleph_1$. Since ZFC proves $2^{\aleph_0} \geq \aleph_1$, the negation $\neg\mathsf{CH}$ of CH is equivalent to $2^{\aleph_0} > \aleph_1$ under ZFC.

Proposition 1.37 *Suppose that κ is a cardinal and let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Then $\mathbb{P}$ preserves all cardinals and $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} \geq \kappa$". In particular, if $\kappa \geq \aleph_2$, then $\Vdash_{\mathbb{P}}$ "$\neg\mathsf{CH}$".* *P-fpforcing-5*

Proof. We may assume that κ is an infinite cardinal. $\mathbb{P}$ preserves all cardinals by Corollary 1.35 and Corollary 1.29. $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} \geq \kappa$" follows from Lemma 1.36 and $\mathrm{Fn}(\kappa, 2) \cong \mathrm{Fn}(\kappa \times \omega, 2)$. ❑ (Proposition 1.37)

Note that the proof of Proposition 1.37 can be recast into a "correct" proof without mention of "generic filters over V" by deploying the argument with M and M^*, like in the proof of Lemma 1.16 (see also the explanation on p.14 ∼). This remark applies also to the proofs of Proposition 1.43 and other statements about forcing relations in V.

Proposition 1.37 implies the relative consistency of $\neg\mathsf{CH}$ by the following general fact:

Theorem 1.38 *Suppose that T is a (concretely given) theory in $\mathcal{L}_\varepsilon$ extending ZFC and let φ be a sentence in $\mathcal{L}_\varepsilon$. Suppose* *P-fpforcing-5-0*

(1.53) $\quad T \vdash$ *"there is a poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "φ".* *fpforcing-5-a*

Then we can conclude that, if T is consistent then $\mathsf{ZFC} + \varphi$ is also consistent.[21]

Proof. Suppose that $\mathsf{ZFC} + \varphi$ is not consistent. We show that T is then not consistent. By the assumption, we have $\mathsf{ZFC} \vdash \neg\varphi$. By the Deduction Lemma for Forcing Relation (Lemma 1.19) and since $\Vdash_{\mathbb{P}}$ "ψ" holds for all poset $\mathbb{P}$ for each axiom ψ of ZFC (Theorem 1.15), it follows that

(1.54) $\quad \mathsf{ZFC} \vdash$ "for all poset $\mathbb{P}$, $\Vdash_{\mathbb{P}}$ "$\neg\varphi$" holds". fpforcing-5-a-0

By (1.53) and (1.54), if follows that

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[21] This theorem is a meta meta-theorem since it mentions about general pattern of certain meta mathematical proofs.

In most of the applications of this theorem in the following, T is simply ZFC itself. In a consistency proof which stats from the assumption that there is a certain large cardinal, T is the theory ZFC + the corresponding large cardinal axiom.

(1.55) $T \vdash$ "there is a poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\varphi \wedge \neg\varphi$" ". fpforcing-5-a-1

But then, by the Consistency of the Forcing Relation (Lemma 1.20), it follows that $T \vdash 0 \equiv 1$. ❑ (Theorem 1.38)

Corollary 1.39 (P. Cohen) *If* ZFC *is consistent then so is* ZFC + $\neg$CH. *P-fpforcing-6*

Proof. By Proposition 1.37 and Theorem 1.38. ❑ (Corollary 1.39)

For a cardinal κ, a poset $\mathbb{P}$ is said to be $<\kappa$-closed if, for any decreasing sequence $\langle \mathbb{p}_\xi : \xi < \delta \rangle$ in $\mathbb{P}$ (with respect to $\leq_{\mathbb{P}}$) for any $\delta < \kappa$, there is $\mathbb{p} \in \mathbb{P}$ such that $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_\xi$ for all $\xi < \delta$.

Lemma 1.40 *For any regular cardinal κ, $\mathbb{P} = \mathrm{Fn}(I, J, <\kappa)$ is $<\kappa$-closed.* *P-fpforcing-7*

Proof. Suppose that $\langle \mathbb{p}_\xi : \xi < \delta \rangle$ for a $\delta < \kappa$ is decreasing sequence in $\mathbb{P}$. This means that $\langle \mathbb{p}_\xi : \xi < \delta \rangle$ is an increasing chain of partial functions in $\mathbb{P}$. $\mathbb{p} = \bigcup_{\xi<\delta} \mathbb{p}_\xi$ is then a partial function from I to J and $|\operatorname{dom}(\mathbb{p})| \leq \sum_{\xi<\delta} |\operatorname{dom}(\mathbb{p}_\xi)| < \kappa$ by the regularity of κ. Thus $\mathbb{p} \in \mathbb{P}$ and $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_\xi$ for all $\xi < \delta$. ❑ (Lemma 1.40)

Lemma 1.41 *For a cardinal κ, if a poset $\mathbb{P}$ is $<\kappa$-closed, A, B are sets with $|A| < \kappa$, then $\mathbb{P}$ does not add any new function from A to B.* *P-fpforcing-8*

Remark. The statement of the Lemma formulated in terms of generic extension is:

(1.56) for any $<\kappa$-closed poset $\mathbb{P}$ and sets A, B as above and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, if $f \in \mathsf{V}[\mathbb{G}]$ is such that $\mathsf{V}[\mathbb{G}] \models$ "$g : A \to B$", then $f \in V$. fpforcing-5-0

The (correct) statement of the Lemma in terms of forcing relations is the following:

(1.57) for any $<\kappa$-closed poset $\mathbb{P}$ and sets A, B as above, for any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{f}$ with $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{A}_{\mathbb{P}} \to \check{B}_{\mathbb{P}}$" there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $f : A \to B$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} \equiv \check{f}_{\mathbb{P}}$".[22] fpforcing-5-1

Proof. Let $\mathbb{P}$, A and B be as above. Suppose that $\mathbb{p} \in \mathbb{P}$ and $\underset{\sim}{f}$ is a $\mathbb{P}$-name such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{A}_{\mathbb{P}} \to \check{B}_{\mathbb{P}}$". Let $A = \{a_\xi : \xi < \delta\}$ for some $\delta < \kappa$.

Let $\langle \mathbb{p}_\xi : \xi < \delta \rangle$ be a decreasing sequence of elements of $\mathbb{P}$ such that

(1.58) $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}$; fpforcing-6

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[22] That (1.56) follows from (1.57) can be seen as follows: Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter and $f = \underset{\sim}{f}[\mathbb{G}]$ is such that $\mathsf{V}[\mathbb{G}] \models$ "$f : A \to B$". Without loss of generality, we may assume that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \check{A}_{\mathbb{P}} \to \check{B}_{\mathbb{P}}$".

In V let

$$D = \{\mathbb{q} \in \mathbb{P} : \text{there is } f_{\mathbb{q}} : A \to B \text{ such that } \mathbb{q} \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{f} \equiv \check{f}_{\mathbb{P}}(f_{\mathbb{q}})\text{"}\}.$$

D is dense in $\mathbb{P}$ by (1.57). Hence there is $\mathbb{q} \in \mathbb{G} \cap D$. By Forcing Theorem 1.12, (1), $\underset{\sim}{f}[G] = f_{\mathbb{q}}$.

(1.59) $\mathbb{p}_\xi \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\surd_{\mathbb{P}}(a_\xi)) \equiv \surd_{\mathbb{P}}(b_\xi)$" for some $b_\xi \in B$. fpforcing-7

The condition (1.59) can be satisfied by Lemma 1.25 and the construction at a limit step is possible by the $<\kappa$-closedness of $\mathbb{P}$. By the $<\kappa$-closedness of $\mathbb{P}$ there is $\mathbb{q} \in \mathbb{P}$ such that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_\xi$ for all $\xi < \delta$. $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ by (1.58). Let $f : A \to B$ be defined by $f(a_\xi) = b_\xi$ for all $\xi < \delta$. Then we have $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} \equiv \check{f}_{\mathbb{P}}$" (Exercise). ❑ (Lemma 1.41)

Corollary 1.42 *If $\mathbb{P}$ is a $<\kappa$-closed poset, then $\mathbb{P}$ preserves all cardinals and cofinalities $\leq \kappa$.* ❑ P-fpforcing-9

Proposition 1.43 *Let $\kappa = 2^{\aleph_0}$ and let $\mathbb{P} = \mathrm{Fn}(\omega_1, \kappa, <\aleph_1)$. Then we have $\Vdash_{\mathbb{P}}$ "CH".* P-fpforcing-10

Proof. Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{G})$-generic filter. By 2. Forcing Theorem 1.13, it is enough to show that $\mathsf{V}[\mathbb{G}] \models \mathsf{CH}$. By Lemma 1.41, ${}^{\omega}2)^{\mathsf{V}[\mathbb{G}]} = ({}^{\omega}2)^{\mathsf{V}}$. thus $\left(2^{\aleph_0}\right)^{\mathsf{V}[\mathbb{G}]} = \left(2^{\aleph_0}\right)^{\mathsf{V}} = \kappa$.

On the other hand, (in $\mathsf{V}[\mathbb{G}]$) $\bigcup \mathbb{G} : \omega_1 \to \kappa$ is a surjection by Lemma 1.36, (1). Thus $\mathsf{V}[\mathbb{G}] \models$ "$2^{\aleph_0} \equiv \aleph_1$". ❑ (Lemma 1.43)

Corollary 1.44 (P. Cohen) *If ZFC is consistent then so is $\mathsf{ZFC} + \mathsf{CH}$.* P-fpforcing-11

Proof. By Proposition 1.43 and Theorem 1.38. ❑ (Corollary 1.44)

For a poset $\mathbb{P}$ and a $\mathbb{P}$-name $\underset{\sim}{a}$, a $\mathbb{P}$-name $\underset{\sim}{c}$ is a *nice $\mathbb{P}$-name of a subset of $\underset{\sim}{a}$* if $\underset{\sim}{c}$ is of the form

(1.60) $\underset{\sim}{c} = \{\langle \underset{\sim}{b}, \mathbb{r} \rangle : \underset{\sim}{b} \in \mathrm{dom}(\underset{\sim}{a}), \mathbb{r} \in A_{\underset{\sim}{b}}\}$ fpforcing-9

where $A_{\underset{\sim}{b}}$ is an antichain (possibly $\emptyset$) for each $\underset{\sim}{b} \in \mathrm{dom}(\underset{\sim}{a})$. Note that in spite of the naming such $\underset{\sim}{c}$ does not necessarily satisfy $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{c} \subseteq \underset{\sim}{a}$".

Lemma 1.45 *For any poset $\mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{a}$, $\underset{\sim}{b}$, there is a nice $\mathbb{P}$-name $\underset{\sim}{c}$ of a subset of $\underset{\sim}{a}$ such that* P-fpforcing-12

(1.61) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \underset{\sim}{a} \to \underset{\sim}{b} \equiv \underset{\sim}{c}$". fpforcing-9-0

In particular, for any $\mathbb{p} \in \mathbb{P}$, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \underset{\sim}{a}$" then we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{c}$" (see Exercise 1.21).

Proof. For each $\underset{\sim}{s} \in \mathrm{dom}(\underset{\sim}{a})$, let $A_{\underset{\sim}{s}}$ be an antichain in $\mathbb{P}$ such that

(1.62) for all $\mathbb{r} \in A_{\underset{\sim}{s}}$ we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{s} \varepsilon \underset{\sim}{b}$"; and fpforcing-10

(1.63) $A_{\underset{\sim}{s}}$ is maximal among antichains in $\mathbb{P}$ with the property (1.62). fpforcing-11

Let

(1.64) $\underset{\sim}{c} = \{\langle \underset{\sim}{b}, \mathbb{r} \rangle : \underset{\sim}{b} \in \mathrm{dom}(\underset{\sim}{a}), \mathbb{r} \in A_{\underset{\sim}{b}}\}$. fpforcing-12

We show that this $\underset{\sim}{c}$ is as desired. By the 2. Forcing Theorem 1.13 it is enough to show that the implication in (1.61) holds in an arbitrary generic extension.

Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter. If $\underset{\sim}{b}[\mathbb{G}] \not\subseteq \underset{\sim}{a}[\mathbb{G}]$ then we have $\mathsf{V}[\mathbb{G}] \models \underset{\sim}{b}[\mathbb{G}] \subseteq \underset{\sim}{a}[\mathbb{G}] \to \underset{\sim}{b}[\mathbb{G}] \equiv \underset{\sim}{c}[\mathbb{G}]$.

So assume $\underset{\sim}{b}[\mathbb{G}] \subseteq \underset{\sim}{a}[\mathbb{G}]$. Then there is a $\mathbb{p} \in \mathbb{G}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \underset{\sim}{a}$". We have to show that $\underset{\sim}{b}[\mathbb{G}] = \underset{\sim}{c}[\mathbb{G}]$.

Suppose that $d \in \underset{\sim}{b}[\mathbb{G}]$. Then $d \in \underset{\sim}{a}[\mathbb{G}]$. Thus there is $\langle \underset{\sim}{s}, \mathbb{r} \rangle \in \underset{\sim}{a}$ such that $\mathbb{r} \in \mathbb{G}$ and $\underset{\sim}{s}[\mathbb{G}] = d$. By Forcing Theorem 1.12, (2),

(1.65) there is $\mathbb{q} \in \mathbb{G}$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{s} \varepsilon \underset{\sim}{b}$". fpforcing-13

Without loss of generality, we may assume that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$. Thus we have $A_{\underset{\sim}{s}} \neq \emptyset$.

Let $\hat{A}$ be a maximal antichain in $\mathbb{P}$ extending $A_{\underset{\sim}{s}}$. There is $\mathbb{r} \in \mathbb{G} \cap \hat{A}$ by the genericity of $\mathbb{G}$. But it must be $\mathbb{r} \in A_{\underset{\sim}{s}}$ by (1.65). Thus $\langle \underset{\sim}{s}, \mathbb{r} \rangle \in \underset{\sim}{c}$ and $d = \underset{\sim}{s}[\mathbb{G}] \in \underset{\sim}{c}[\mathbb{G}]$.

Suppose now that $d \in \underset{\sim}{c}[\mathbb{G}]$. Then there is $\langle \underset{\sim}{s}, \mathbb{r} \rangle \in \underset{\sim}{c}$ such that $\mathbb{r} \in \mathbb{G}$ and $d = \underset{\sim}{s}[\mathbb{G}]$. By the definition (1.64) of $\underset{\sim}{c}$, we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{s} \varepsilon \underset{\sim}{b}$". By Forcing Theorem 1.12, (1), it follows that $d = \underset{\sim}{s}[\mathbb{G}] \in \underset{\sim}{b}[\mathbb{G}]$. ❑ (Lemma 1.45)

Lemma 1.46 *Suppose that $\mathbb{P}$ is a μ-cc poset with $|\mathbb{P}| = \kappa \geq \aleph_0$. For a cardinal λ, and $\theta = (\kappa^{<\mu})^{\lambda}$, we have $\Vdash_{\mathbb{P}}$ "$2^{\lambda} \leq \theta$".* P-fpforcing-13

Proof. Let $A = \{\underset{\sim}{c} : \underset{\sim}{c}$ is a nice $\mathbb{P}$-name of a subset of $\check{\lambda}_{\mathbb{P}}\}$. By μ-cc of $\mathbb{P}$, there are at most $\kappa^{<\mu}$ many antichains in $\mathbb{P}$. Since $|\check{\lambda}_{\mathbb{P}}| = \lambda$, it follows that $|A| \leq (\kappa^{<\mu})^{\lambda} = \theta$. By Lemma 1.45, $\mathcal{P}(\lambda)^{\mathsf{V}[\mathbb{G}]} \subseteq \{\underset{\sim}{c}[\mathbb{G}] : \underset{\sim}{c} \in A\}$ for an arbitrary $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$. It follows that $(2^{\lambda})^{\mathsf{V}[\mathbb{G}]} \leq \theta$. To prove that $\mathsf{V}[\mathbb{G}]$ sees this inequality, we show that (in V) there is a $\mathbb{P}$-name of a surjection from θ to $\mathcal{P}(\lambda)^{\mathsf{V}[\mathbb{G}]}$.

In V, let $\langle \underset{\sim}{c}_{\xi} : \xi < \theta \rangle$ be an enumeration of all nice $\mathbb{P}$-names of subsets of $\check{\lambda}_{\mathbb{P}}$. Let

(1.66) $\underset{\sim}{f} = \{\langle odp(\check{\xi}_{\mathbb{P}}, \underset{\sim}{c}_{\xi}), \mathbb{1}_{\mathbb{P}} \rangle : \xi < \theta\}$.[23] fpforcing-14

Then for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\underset{\sim}{f}[\mathbb{G}]$ is a surjection from θ to $\mathcal{P}(\lambda)$ in $\mathsf{V}[\mathbb{G}]$. Thus $\mathsf{V}[\mathbb{G}] \models 2^{\lambda} \leq \theta$. By 2. Forcing Theorem 1.13, this implies that $\Vdash_{\mathbb{P}}$ "$2^{\lambda} \leq \theta$". ❑ (Lemma 1.46)

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[23] For a poset $\mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{a}$, $\underset{\sim}{b}$, let $uop(\underset{\sim}{a}, \underset{\sim}{b}) = \{\langle \underset{\sim}{a}, \mathbb{1}_{\mathbb{P}} \rangle, \langle \underset{\sim}{b}, \mathbb{1}_{\mathbb{P}} \rangle\}$. $uop(\underset{\sim}{a}, \underset{\sim}{b})$ is a standard $\mathbb{P}$-name for unordered pair of (the interpretations of) $\underset{\sim}{a}$ and $\underset{\sim}{b}$: $uop(\underset{\sim}{a}, \underset{\sim}{b})$ is a $\mathbb{P}$-name and, for any filter $\mathbb{G}$ on $\mathbb{P}$, we have $uop(\underset{\sim}{a}, \underset{\sim}{b})[\mathbb{G}] = \{\underset{\sim}{a}[\mathbb{G}], \underset{\sim}{b}[\mathbb{G}]\}$.

For $\mathbb{P}$-names $\underset{\sim}{a}$, $\underset{\sim}{b}$, let $odp(\underset{\sim}{a}, \underset{\sim}{b}) = \{\langle uop(\underset{\sim}{a}, \underset{\sim}{a}), \mathbb{1}_{\mathbb{P}} \rangle, \langle uop(\underset{\sim}{a}, \underset{\sim}{b}), \mathbb{1}_{\mathbb{P}} \rangle\}$. $odp(\underset{\sim}{a}, \underset{\sim}{b})$ is a is a standard $\mathbb{P}$-name for ordered pair of (the interpretations of) $\underset{\sim}{a}$ and $\underset{\sim}{b}$: $odp(\underset{\sim}{a}, \underset{\sim}{b})$ is a $\mathbb{P}$-name and, for any filter $\mathbb{G}$ on $\mathbb{P}$, we have $odp(\underset{\sim}{a}, \underset{\sim}{b})[\mathbb{G}] = \langle \underset{\sim}{a}[\mathbb{G}], \underset{\sim}{b}[\mathbb{G}] \rangle$.

Similarly we can introduce the standard $\mathbb{P}$-name of the ordered triplet $otr(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c})$, e.g., by $otr(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = odp(\underset{\sim}{a}, odp(\underset{\sim}{b}, \underset{\sim}{c}))$.

The following is one of the applications of Lemma 1.46:

Proposition 1.47 *If* ZFC *is consistent then so is* ZFC + $2^{\aleph_0} = \aleph_2$. P-fpforcing-14

Proof. In V, let $\mathbb{P}_0 = \mathrm{Fn}(\aleph_1, 2^{\aleph_0}, <\aleph_1)$. Then

(1.67) $\Vdash_{\mathbb{P}_0}$ "CH" fpforcing-14-0

by Lemma 1.43.

Now let $\underset{\sim}{\mathbb{P}}_1$ be a $\mathbb{P}_0$-name of the poset $\mathrm{Fn}(\omega_2 \times \omega, 2)$. Then we have

(1.68) $\Vdash_{\mathbb{P}_0}$ " $\Vdash_{\underset{\sim}{\mathbb{P}}_1}$ " $2^{\aleph_0} \equiv \aleph_2$ " " fpforcing-15

by Proposition 1.37, (1.67), Lemma 1.46 and Deduction Lemma 1.19[(24)].

Now assume, toward a contradiction, that ZFC + $2^{\aleph_0} \equiv \aleph_2$ is inconsistent. Then there is a proof $\mathcal{P}$ of $2^{\aleph_0} \not\equiv \aleph_2$ from ZFC. Thus we have

(1.69) $\Vdash_{\mathbb{P}_0}$ " $\Vdash_{\underset{\sim}{\mathbb{P}}_1}$ " $2^{\aleph_0} \not\equiv \aleph_2$ " " fpforcing-16

by Deduction Lemma 1.19[(25)]. This is a contradiction to Lemma 1.20 (which is forced by $\mathbb{P}_0$ again by the Deduction Lemma 1.19). ❑ (Proposition 1.47)

Exercise 1.48 *If* ZFC *is consistent, then so is* ZFC+ $2^{\aleph_0} = \aleph_{2018}$. ❑

2 Regular subordering and projection

2.1 Dense embeddings and forcing equivalence reg

Posets $\mathbb{P}$ and $\mathbb{Q}$ are said to be *semantically forcing equivalent* and denoted by $\mathbb{P} \sim \mathbb{Q}$, if forcing-eq

(2.1) If $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, then there is a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$; and forcing-eq-a-0

(2.2) If $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter, then there is a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ such that $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$. forcing-eq-a-1

For posets $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, $\mathbb{Q} = (\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$, a mapping $i : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* of $\mathbb{P}$ in $\mathbb{Q}$, if

(2.3) $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$; forcing-eq-a-2

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(24) Note that CH implies that $(2^{<\aleph_1})^{\aleph_2} = \aleph_2$. We shall later introduce the notion of two-step iteration with which we can define a poset $\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}_1$ and show $\Vdash_{\mathbb{P}_0 * \underset{\sim}{\mathbb{P}}_1}$ " $2^{\aleph_0} \equiv \aleph_2$ ". The relative consistency of ZFC+ $2^{\aleph_0} = \aleph_2$ follows then directly from this and Theorem 1.43.

(25) More precisely, the instance of Lemma 1.19 needed to establish (1.69) which is forced by $\mathbb{P}_0$ by the Deduction Lemma 1.19 applied to $\mathbb{P}_0$.

(2.4) i is order and incompatibility preserving[26] ; and, forcing-eq-a-3

(2.5) $i''\mathbb{P}$ is dense in $\mathbb{P}$. forcing-eq-a-4

Note that we do not demand that a dense embedding i should be a 1-1 mapping. If there is a dense embedding of $\mathbb{P}$ into $\mathbb{Q}$ we shall also say that $\mathbb{P}$ is *densely embedded* into $\mathbb{Q}$. If the dense embedding is 1-1 then we say that $\mathbb{P}$ is *injectively and densely embedded* into $\mathbb{Q}$.

Exercise 2.1 *Suppose that $\mathbb{P}$ is a poset and $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$ with $\mathbb{1}_{\mathbb{P}} \in D$. Let $\mathbb{D} = \langle \mathbb{D}, \leq_{\mathbb{D}}, \mathbb{1}_{\mathbb{P}} \rangle$ be the poset defined by $\mathbb{D} = D$, $\leq_{\mathbb{D}} = \leq_{\mathbb{P}} \cap D^2$ and $\mathbb{1}_{\mathbb{D}} = \mathbb{1}_{\mathbb{P}}$. Then the identity mapping $id : \mathbb{D} \to \mathbb{P}$ is a dense embedding.* ❑ ex-forcing-eq-0

Lemma 2.2 *For posets $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$, if $i : \mathbb{P} \to \mathbb{Q}$ and $j : \mathbb{Q} \to \mathbb{R}$ are dense embeddings, then $j \circ i$ is also a dense embedding.* P-forcing-eq-a-0

Proof. Easy. ❑ (Lemma 2.2)

Lemma 2.3 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets and $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding. Then $\mathbb{P}$ and $\mathbb{Q}$ are semantically forcing equivalent. More specifically:* P-forcing-eq-0

(1) *If $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter then $i''\mathbb{G}$ generates a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ and $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$.*

(2) *If $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter, then $i^{-1}{}''\mathbb{H}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter and $\mathsf{V}[\mathbb{H}] = \mathsf{V}[i^{-1}{}''\mathbb{H}]$.*

Proof. Suppose that $\mathbb{P}$, $\mathbb{Q}$ and $i : \mathbb{P} \to \mathbb{Q}$ are as above.

(1): We first show that $i''\mathbb{G}$ generates a filter. Let $\mathbb{q}_0$, $\mathbb{q}_1 \in i''\mathbb{G}$, say $\mathbb{q}_0 = i(\mathbb{p}_0)$ and $\mathbb{q}_1 = i(\mathbb{p}_1)$ for $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{G}$. Let $\mathbb{p}_2 \in \mathbb{G}$ be such that $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_0$, $\mathbb{p}_1$. Then $i(\mathbb{p}_2) \in i''\mathbb{G}$ and $i(\mathbb{p}_2) \leq_{\mathbb{P}} \mathbb{q}_0, \mathbb{q}_1$ since i is order preserving. Thus

(2.6) $\mathbb{H} = \{\mathbb{q} \in \mathbb{Q} : \text{there is } \mathbb{q}' \in i''\mathbb{G} \text{ with } \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}\}$ forcing-eq-0

is a filter on $\mathbb{Q}$.

Suppose that $D \subseteq \mathbb{Q}$ is open dense. By Lemma 1.5, it is enough to show that $\mathbb{H} \cap D \neq \emptyset$.

Claim 2.3.1 *$i^{-1}{}''D$ is a dense subset of $\mathbb{P}$.*

$\vdash$ For $\mathbb{p} \in \mathbb{P}$, let $\mathbb{d} \in D$ be such that $\mathbb{d} \leq_{\mathbb{Q}} i(\mathbb{p})$. Since $i''\mathbb{P}$ is dense in $\mathbb{Q}$, there is $\mathbb{p}' \in \mathbb{P}$ such that $i(\mathbb{p}') \leq_{\mathbb{Q}} \mathbb{d}$. Since $i(\mathbb{p})$ and $i(\mathbb{p}')$ are compatible in $\mathbb{Q}$, $\mathbb{p}$ and $\mathbb{p}'$ are compatible

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[26] $i : \mathbb{P} \to \mathbb{Q}$ is *order preserving* if $\mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0$ implies $i(\mathbb{p}_1) \leq_{\mathbb{Q}} i(\mathbb{p}_0)$ for all $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$. i is *strictly order preserving* if $\mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0$ is equivalent to $i(\mathbb{p}_1) \leq_{\mathbb{Q}} i(\mathbb{p}_0)$ for all $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$. i is *incompatibility preserving* if $\mathbb{p}_1 \perp_{\mathbb{P}} \mathbb{p}_0$ implies $i(\mathbb{p}_1) \perp_{\mathbb{Q}} i(\mathbb{p}_0)$ for all $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$.

Note that, if $i : \mathbb{P} \to \mathbb{Q}$ is order preserving, then $i(\mathbb{p}_0) \perp_{\mathbb{Q}} i(\mathbb{p}_1)$ implies $\mathbb{p}_0 \perp_{\mathbb{P}} \mathbb{p}_1$, for any $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$.

in $\mathbb{P}$ as i is incompatibility preserving. Let $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$, $\mathbb{p}'$. Then we have $i(\mathbb{r}) \leq_{\mathbb{Q}} i(\mathbb{p}')$. Hence $i(\mathbb{r}) \leq_{\mathbb{Q}} \mathbb{d}$. It follows that $i(\mathbb{r}) \in D$ since D is open dense. Thus $\mathbb{r} \in i^{-1}{}''D$ (and $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$). ⊣ (Claim 2.3.1)

By genericity of $\mathbb{G}$, there is $\mathbb{p}^* \in \mathbb{G} \cap (i^{-1}{}''D)$. It follows that $i(\mathbb{p}^*) \in i''\mathbb{G} \cap D \subseteq \mathbb{H} \cap D$. This shows that $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter. $\mathbb{H}$ is definable in $V[\mathbb{G}]$ by (2.6). Hence $\mathsf{V}[\mathbb{H}] \subseteq \mathsf{V}[\mathbb{G}]$.

Claim 2.3.2 $\mathbb{G} = i^{-1}{}''\mathbb{H}$. *cl-forcing-eq-0*

$\vdash$ $i^{-1}{}''\mathbb{H}$ is a filter on $\mathbb{P}$. This will be shown in the proof of (2) below. Clearly $\mathbb{G} \subseteq i^{-1}{}''\mathbb{H}$. Since $\mathbb{G}$ is $(\mathsf{V}, \mathbb{P})$-generic filter and hence a maximal filter by Lemma 1.1, it follows that $\mathbb{G} = i^{-1}{}''\mathbb{H}$. ⊣ (Claim 2.3.2)

Thus $\mathbb{G}$ is definable in $\mathsf{V}[\mathbb{H}]$ and $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$.

(2): Suppose that $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter and $\mathbb{G} = i^{-1}\mathbb{H}$. Suppose that $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{G}$. Since $i(\mathbb{p}_0)$, $i(\mathbb{p}_1) \in \mathbb{H}$ and $\mathbb{H}$ is a filter, $i(\mathbb{p}_0)$ and $i(\mathbb{p}_1)$ are compatible. Since i is incompatibility preserving, it follows that $\mathbb{p}_0$ and $\mathbb{p}_1$ are compatible. If $\mathbb{p}_1 \in \mathbb{G}$ and $\mathbb{p}_0 \geq_{\mathbb{P}} \mathbb{p}_0$ then $i(\mathbb{p}_0) \in \mathbb{H}$ and $i(\mathbb{p}_0) \geq_{\mathbb{P}} i(\mathbb{p}_1)$. Since $\mathbb{H}$ is a filter it follows that $i(\mathbb{p}_0) \in \mathbb{H}$. Thus $\mathbb{p}_0 \in \mathbb{G}$.

If $D \subseteq \mathbb{P}$ is dense then $i''D$ is dense in $\mathbb{Q}$. By genericity of $\mathbb{H}$, there is $\mathbb{q} \in i''D \cap \mathbb{H}$. Let $\mathbb{p} \in D$ be such that $i(\mathbb{p}) = \mathbb{q}$. Then we have $\mathbb{p} \in \mathbb{G} \cap D$. This shows together with the fact shown above that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter.

Since $\mathbb{G}$ is definable in $\mathsf{V}[\mathbb{H}]$ as $i^{-1}{}''\mathbb{H}$ and $\mathbb{H}$ in $\mathsf{V}[\mathbb{G}]$ as $i''\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$.

❑ (Lemma 2.3)

A poset $\mathbb{P}$ is *separative* if

(2.7) for all $\mathbb{p}$, $\mathbb{q} \in \mathbb{P}$ we have *forcing-eq-1*

$$\mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \;\Leftrightarrow\; \text{there is no } \mathbb{r} \leq_{\mathbb{P}} \mathbb{q} \text{ such that } \mathbb{r} \perp_{\mathbb{P}} \mathbb{p}.$$

Note that the implication "⇒" in (2.7) holds always. Thus

Lemma 2.4 *A poset $\mathbb{P}$ is separative if* *P-forcing-eq-0-0*

(2.8) *for all $\mathbb{p}$, $\mathbb{q} \in \mathbb{P}$ we have* *forcing-eq-1-0*

$$\mathbb{q} \not\leq_{\mathbb{P}} \mathbb{p} \;\Rightarrow\; \textit{there is an } \mathbb{r} \leq_{\mathbb{P}} \mathbb{q} \textit{ such that } \mathbb{r} \perp_{\mathbb{P}} \mathbb{p}.$$

❑

Lemma 2.5 (1) *For a poset $\mathbb{P}$ and the standard $\mathbb{P}$-name $\underset{\sim}{\mathbb{G}}$ of a $\mathbb{P}$-generic filter, we have* P-forcing-eq-1

(2.9) $\mathbb{q} \nVdash_{\mathbb{P}}$ "$\mathbb{p} \varepsilon \underset{\sim}{\mathbb{G}}$" $\Leftrightarrow$ *there is* $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}$ *such that* $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$ forcing-eq-2

for all $\mathbb{p}, \mathbb{q} \in \mathbb{P}$.

(2) $\mathbb{P}$ *is separative if and only if*

(2.10) $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\mathbb{p} \varepsilon \underset{\sim}{\mathbb{G}}$" $\Leftrightarrow$ $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ forcing-eq-3

holds for all $\mathbb{p}, \mathbb{q} \in \mathbb{P}$.

Proof. (1): "$\Rightarrow$": If $\mathbb{q} \nVdash_{\mathbb{P}}$ "$\mathbb{p} \varepsilon \underset{\sim}{\mathbb{G}}$", then there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}$ such that

(2.11) $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\mathbb{p} \not\varepsilon \underset{\sim}{\mathbb{G}}$" forcing-eq-3-0

by Lemma 1.16 $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$: Otherwise, there is $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{p}$. We have $\mathbb{s} \Vdash_{\mathbb{P}}$ "$\mathbb{p} \varepsilon \underset{\sim}{\mathbb{G}}$" by Lemma 1.11, (1). A contradiction to (2.11).

"$\Leftarrow$": Suppose that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}$ is such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$. Then, since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{G}}$ is a filter" and $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\mathbb{r} \varepsilon \underset{\sim}{\mathbb{G}}$" by the definition (1.15) of $\underset{\sim}{\mathbb{G}}$ of the generic filter and 2. Forcing Theorem 1.13, we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\mathbb{p} \not\varepsilon \underset{\sim}{\mathbb{G}}$". Thus $\mathbb{q}$ cannot force "$\mathbb{p} \in \underset{\sim}{\mathbb{G}}$".

(2): By (1), the equivalence (2.10) is equivalent to

(2.12) there is no $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$ $\Leftrightarrow$ $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$. forcing-eq-4

However, this is the equivalence (2.7) in the definition of separativeness. ❑ (Lemma 2.5)

For a poset $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, let $\leq^s_{\mathbb{P}}$ be the binary relation[27] on (the underlying set of) $\mathbb{P}$ defined by

(2.13) $\mathbb{q} \leq^s_{\mathbb{P}} \mathbb{p}$ $\Leftrightarrow$ for all $\mathbb{r} \in \mathbb{P}$, $\mathbb{r} \top_{\mathbb{P}} \mathbb{q}$ implies $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}$ forcing-eq-5

for $\mathbb{p}, \mathbb{q} \in \mathbb{P}$.

Let $\mathbb{P}^s = (\mathbb{P}, \leq^s_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$.

Lemma 2.6 (1) $\leq^s_{\mathbb{P}}$ *is a transitive relation on* $\mathbb{P}$. P-forcing-eq-2

(2) $\leq_{\mathbb{P}} \subseteq \leq^s_{\mathbb{P}}$.

(3) *For* $\mathbb{p}, \mathbb{q} \in \mathbb{P}$, *if* $\mathbb{q} \leq^s_{\mathbb{P}} \mathbb{p}$ *then* $\mathbb{q} \top_{\mathbb{P}} \mathbb{p}$.

(4) *For all* $\mathbb{p}, \mathbb{q} \in \mathbb{P}$, *we have* $\mathbb{p} \perp_{\mathbb{P}} \mathbb{q}$ $\Leftrightarrow$ $\mathbb{p} \perp_{\mathbb{P}^s} \mathbb{q}$.

(5) $\mathbb{P}^s$ *is a separative poset.*

(6) $id : \mathbb{P} \to \mathbb{P}$ *is a dense embedding of* $\mathbb{P}$ *into* $\mathbb{P}^s$.

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[27] 's' in '$\leq^s_{\mathbb{P}}$' stands for "separative". See Lemma 2.6 below.

Proof. (1): Suppose that $\mathbb{p}_1 \leq^s_{\mathbb{P}} \mathbb{p}_0$ and $\mathbb{p}_2 \leq^s_{\mathbb{P}} \mathbb{p}_1$. By definition, this means "for all $\mathbb{r} \in \mathbb{P}$, $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_1$ implies $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_0$" and "for all $\mathbb{r} \in \mathbb{P}$, $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_2$ implies $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_1$". It follows "for all $\mathbb{r} \in \mathbb{P}$, $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_2$ implies $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}_0$" which means $\mathbb{p}_2 \leq^s_{\mathbb{P}} \mathbb{p}_0$.

(2): Suppose that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$. If $\mathbb{r} \top_{\mathbb{P}} \mathbb{q}$, let $\mathbb{r}_1 \in \mathbb{P}$ be such that $\mathbb{r}_1 \leq_{\mathbb{P}} \mathbb{r}, \mathbb{q}$. It follows that $\mathbb{r}_1 \leq_{\mathbb{P}} \mathbb{p}$. Thus $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}$. This shows $\mathbb{q} \leq^s_{\mathbb{P}} \mathbb{p}$.

(3): Consider the right side of the equivalence of (2.13) with $\mathbb{r} = \mathbb{q}$.

(4): If $\mathbb{p} \top_{\mathbb{P}} \mathbb{q}$ then we have $\mathbb{p} \top_{\mathbb{P}^s} \mathbb{q}$ by (2).

Suppose now that $\mathbb{p} \top_{\mathbb{P}^s} \mathbb{q}$ and let $\mathbb{r} \in \mathbb{P}$ be such that $\mathbb{r} \leq_{\mathbb{P}^s} \mathbb{p}, \mathbb{q}$. Then there is $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{p}$ by (3). Since $\mathbb{s} \top_{\mathbb{P}} \mathbb{r}$, we have $\mathbb{s} \top_{\mathbb{P}} \mathbb{q}$ and hence there is $\mathbb{t} \leq_{\mathbb{P}} \mathbb{s}, \mathbb{q}$. Since $\mathbb{t} \leq_{\mathbb{P}} \mathbb{s} \leq_{\mathbb{P}} \mathbb{p}$, this $\mathbb{t}$ witnesses $\mathbb{p} \top_{\mathbb{P}} \mathbb{q}$.

(5): Note that, by Lemma 2.4, it is enough to show (2.8). Suppose that $\mathbb{q} \not\leq^s_{\mathbb{P}} \mathbb{p}$. This means that there is an $\mathbb{r} \in \mathbb{P}$ such that $\mathbb{r} \top_{\mathbb{P}} \mathbb{q}$ and $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$. For such $\mathbb{r}$, let $\mathbb{s} \in \mathbb{P}$ be such that $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{q}$. We have $\mathbb{s} \leq^s_{\mathbb{P}} \mathbb{q}$ by (2).

Since $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$ and $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}$, we have $\mathbb{s} \perp_{\mathbb{P}} \mathbb{p}$. By (4), it follows that $\mathbb{s} \perp_{\mathbb{P}^s} \mathbb{p}$.

(6): $id(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{P}} = \mathbb{1}_{\mathbb{P}^s}$. id is order preserving by (2) and incompatibility preserving by (4). Since id is a surjection, $id''\mathbb{P}$ is dense in $\mathbb{P}^s$. ❑ (Lemma 2.6)

For a poset $\mathbb{P}$, let $\sim_{\mathbb{P}}$ be the equivalence relation on (the underlying set of) $\mathbb{P}$ defined by

(2.14) $\quad \mathbb{q} \sim_{\mathbb{P}} \mathbb{p} \;\Leftrightarrow\; \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p} \leq_{\mathbb{P}} \mathbb{q}$ forcing-eq-5-a

for $\mathbb{p}, \mathbb{q} \in \mathbb{P}$.

$\sim_{\mathbb{P}}$ is an equivalence relation since $\leq_{\mathbb{P}}$ is transitive and reflective. For $\mathbb{p} \in \mathbb{P}$ let us denote with $[\mathbb{p}]$ the equivalence class of $\mathbb{p}$ with respect to $\sim_{\mathbb{P}}$ and $\mathbb{P}/\sim_{\mathbb{P}} = \{[\mathbb{p}] : \mathbb{p} \in \mathbb{P}\}$.

Let $\leq_{\mathbb{P}/\sim_{\mathbb{P}}}$ be the binary relation defined by

(2.15) $\quad [\mathbb{q}] \leq_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{p}] \;\Leftrightarrow\; \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ forcing-eq-5-a-0

for $\mathbb{p}, \mathbb{q} \in \mathbb{P}$. The definition of $[\mathbb{q}] \leq_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{p}]$ does not depend on the representatives $\mathbb{p}$ and $\mathbb{q}$ since if $\mathbb{q}' \sim_{\mathbb{P}} \mathbb{q}$, $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p} \sim_{\mathbb{P}} \mathbb{p}'$ then we have $\mathbb{q}' \leq_{\mathbb{P}} \mathbb{q}$ and $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}'$, and hence, by transitivity of $\leq_{\mathbb{P}}$, we have $\mathbb{q}' \leq_{\mathbb{P}} \mathbb{p}$.

Let $\mathbb{1}_{\mathbb{P}/\sim_{\mathbb{P}}} = [\mathbb{1}_{\mathbb{P}}]$ and $\mathbb{P}/\sim_{\mathbb{P}} = (\mathbb{P}/\sim_{\mathbb{P}}, \leq_{\mathbb{P}/\sim_{\mathbb{P}}}, \mathbb{1}_{\mathbb{P}/\sim_{\mathbb{P}}})$. $\mathbb{P}/\sim_{\mathbb{P}}$ is a poset.

Lemma 2.7 (1) *$\leq_{\mathbb{P}/\sim_{\mathbb{P}}}$ is a partial ordering on $\mathbb{P}/\sim_{\mathbb{P}}$.* *P-forcing-eq-2-0*

(2) *If $\mathbb{P}$ is separative then $\mathbb{P}/\sim_{\mathbb{P}}$ is also separative.*

(3) *$i : \mathbb{P} \to \mathbb{P}/\sim_{\mathbb{P}}$; $\mathbb{p} \mapsto [\mathbb{p}]$ is a dense embedding.*

Proof. (1): We have to show that $\leq_{\mathbb{P}/\sim_{\mathbb{P}}}$ is antisymmetric but this is clear from the definition of $\sim_{\mathbb{P}}$ and $\leq_{\mathbb{P}/\sim_{\mathbb{P}}}$.

(2) Suppose that $\mathbb{p}, \mathbb{q} \in \mathbb{P}$ and $[\mathbb{q}] \not\leq_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{p}]$. Then we have $\mathbb{q} \not\leq_{\mathbb{P}} \mathbb{p}$. By separativity of $\leq_{\mathbb{P}}$, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{q}$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}$. It follows that $[\mathbb{r}] \leq_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{q}]$ and $[\mathbb{r}] \perp_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{p}]$. This shows by Lemma 2.4 that $\mathbb{P}/\sim_{\mathbb{P}}$ is separative.

(3): i is clearly order preserving.

If $i(\mathbb{p}) \top_{\mathbb{P}/\sim_{\mathbb{P}}} i(\mathbb{q})$ for $\mathbb{p}$, $\mathbb{q} \in \mathbb{P}$ (i.e., if $[\mathbb{p}] \top_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{q}]$) then there is $\mathbb{r} \in \mathbb{P}$ such that $[\mathbb{r}] \leq_{\mathbb{P}/\sim_{\mathbb{P}}} [\mathbb{p}], [\mathbb{q}]$. But this means that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$, $\mathbb{q}$ and hence $\mathbb{p} \top_{\mathbb{P}} \mathbb{q}$.

Since i is a surjection, $i''\mathbb{P}$ is dense in $\mathbb{P}/\sim_{\mathbb{P}}$. ❑ (Lemma 2.7)

Let us call a poset $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ a *sub-Boolean* poset if $\mathbb{P}$ is separative and $\leq_{\mathbb{P}}$ is a partial ordering on $\mathbb{P}$.

Lemma 2.8 *For any poset $\mathbb{P}$ there is a sub-Boolean poset $\mathbb{Q}$ and a dense embedding $i : \mathbb{P} \to \mathbb{Q}$.* P-forcing-eq-2-1

Proof. By Lemma 2.6, (5), (6), Lemma 2.7 and Lemma 2.2. ❑ (Lemma 2.8)

For a poset $\mathbb{P}$, we shall call the sub-Boolean poset constructed from $\mathbb{P}$ as in the proof of Lemma 2.8 the *sub-Booleanization* of $\mathbb{P}$. The naming "sub-Boolean" is justified by Corollary 2.11 of the next Lemma.

In the proof of Lemma 2.8, a sub-Boolean poset $\mathbb{Q}$ in which $\mathbb{P}$ is densely embedded is obtained from $\mathbb{P}$ by successive application of the operations $\mathbb{P} \mapsto \mathbb{P}_s$ and $\mathbb{P} \mapsto \mathbb{P}/\sim_{\mathbb{P}}$. We shall call the poset obtained form $\mathbb{P}$ in this way a *sub-Booleanization* of $\mathbb{P}$.

Lemma 2.9 *For any poset $\mathbb{P}$, there is a complete Boolean algebra $\mathbb{B}$ such that $\mathbb{P}$ is densely embedded in $\mathbb{B}^+$. Furthermore if $\mathbb{P}$ is sub-Boolean poset, then the embedding can be chosen to be 1-1.* P-forcing-eq-2-2

Proof. For a poset $\mathbb{P}$, let

$$\mathcal{B} := \{\bigcap\{\mathbb{P} \downarrow \mathbb{p} \,:\, \mathbb{p} \in u\} \,:\, u \in [\mathbb{P}]^{<\aleph_0}\} \tag{2.16}$$ forcing-eq-5-a-1

where $\mathbb{P} \downarrow \mathbb{p} = \{\mathbb{q} \in \mathbb{P} \,:\, \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}\}$. Let $\mathcal{O}$ be the topology on the set $\mathbb{P}$ generated by $\mathcal{B}$.

Let

$$RO(\mathbb{P}) := \{O \subseteq \mathbb{P} \,:\, O \text{ is regular open in the topological space } \langle \mathbb{P}, \mathcal{O}\rangle\}. \tag{2.17}$$ forcing-eq-5-a-2

Note that $RO(\mathbb{P})$ (with the ordering $\subseteq$) is a complete Boolean algebra[28].

Note that, for $\mathbb{p} \in \mathbb{P}$, $\mathbb{P} \downarrow \mathbb{p}$ is the minimal open set containing $\mathbb{p}$ (if $\mathbb{p} \in \bigcap\{\mathbb{P} \downarrow \mathbb{q} \,:\, \mathbb{q} \in u\}$ for some $u \in [\mathbb{P}]^{<\aleph_0}$, $\mathbb{P} \downarrow \mathbb{p} \subseteq \mathbb{P} \downarrow \mathbb{q}$ for all $\mathbb{q}$. Thus $\mathbb{P} \downarrow \mathbb{p} \subseteq \bigcap\{\mathbb{P} \downarrow \mathbb{q} \,:\, \mathbb{q} \in u\}$).

Claim 2.9.1 *If $\mathbb{P}$ is separative then for any $\mathbb{p} \in \mathbb{P}$, $\mathbb{P} \downarrow \mathbb{p}$ is a regular open subset of $\mathbb{P}$ with respect to $\mathcal{O}$.*

$\vdash$ For $\mathbb{q} \in \mathbb{P}$, we have

[28] Recall that a set $U \subseteq X$ for a topological space $X = (X, \mathcal{O})$ is said to be regular open if $(U^-)^\circ = U$ where $\cdot^-$ and $\cdot^\circ$ respectively denote the closure and interior operators of the topology.

It is easy to check that, for any topological space X, the set algebra $RO(X)$ is a complete Boolean algebra (though the infinite sum in the complete Boolean algebra may differ from the set-operation of infinite union).

(2.18) $\quad \mathbb{q} \notin (\mathbb{P} \downarrow \mathbb{p})^- \;\Leftrightarrow\; (\mathbb{P} \downarrow \mathbb{q}) \cap (\mathbb{P} \downarrow \mathbb{p}) = \emptyset \;\Leftrightarrow\; \mathbb{q} \perp_{\mathbb{P}} \mathbb{p}.$

Thus, for $\mathbb{r} \in \mathbb{P}$,

$$\begin{aligned} (2.19) \qquad \mathbb{r} \in ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ \;&\Leftrightarrow\; \mathbb{P} \downarrow \mathbb{r} \subseteq (\mathbb{P} \downarrow \mathbb{p})^- \\ &\Leftrightarrow\; (\forall \mathbb{r}' \in \mathbb{P} \downarrow \mathbb{r})\ (\mathbb{r}' \in (\mathbb{P} \downarrow \mathbb{p})^-) \\ &\Leftrightarrow\; (\forall \mathbb{r}' \leq_{\mathbb{P}} \mathbb{r})\ (\mathbb{r}' \top_{\mathbb{P}} \mathbb{p}) \end{aligned}$$

If $\mathbb{P}$ is separative, the last condition of (2.19) is equivalent to $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$.

Thus $((\mathbb{P} \downarrow \mathbb{p})^-)^\circ = \mathbb{P} \downarrow \mathbb{p}$ as desired. ⊣ (Claim 2.9.1)

Let $i : \mathbb{P} \to \mathbb{B}^+;\ \mathbb{p} \mapsto ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ$. This is well defined since $i(\mathbb{p}) \supseteq \mathbb{P} \downarrow \mathbb{p} \neq \emptyset$.

We show that the embedding i of $\mathbb{P}$ into $\mathbb{B}^+$ is as desired in the following Claim:

Claim 2.9.2 (i) *if $\mathbb{P}$ is sub-Boolean then i is injective.*

(ii) *i is order preserving.*

(iii) *i is incompatibility preserving.*

(iv) *$i''\mathbb{P}$ is dense in $RO(\mathbb{P})^+$.*

⊢ (i): Suppose that $\mathbb{p}, \mathbb{q} \in \mathbb{P}$ with $\mathbb{p} \neq \mathbb{q}$. Since $\leq_{\mathbb{P}}$ is a partial ordering on $\mathbb{P}$, we have $\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{q}$ or $\mathbb{q} \not\leq_{\mathbb{P}} \mathbb{p}$. Suppose without loss of generality that $\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{q}$ holds. Then, since $\mathbb{P}$ is separative, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{q}$. It follows that $\mathbb{r} \in \mathbb{P} \downarrow \mathbb{p} = i(\mathbb{p})$ and $\mathbb{r} \notin \mathbb{P} \downarrow \mathbb{q} = i(\mathbb{q})$. Thus $i(\mathbb{p}) \neq i(\mathbb{q})$.

(ii): If $\mathbb{p}, \mathbb{q} \in \mathbb{P}$ and $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$, then $\mathbb{P} \downarrow \mathbb{q} \subseteq \mathbb{P} \downarrow \mathbb{p}$. Hence $i(\mathbb{q}) = ((\mathbb{P} \downarrow \mathbb{q})^-)^\circ \subseteq ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ = i(\mathbb{p})$.

(iii): If $\mathbb{p} \perp_{\mathbb{P}} \mathbb{q}$, then $(\mathbb{P} \downarrow \mathbb{p}) \cap (\mathbb{P} \downarrow \mathbb{q}) = \emptyset$. Since $(\mathbb{P} \downarrow \mathbb{p})^- = \{\mathbb{r} \in \mathbb{P} : (\mathbb{P} \downarrow \mathbb{r}) \cap (\mathbb{P} \downarrow \mathbb{p}) \neq \emptyset\}$. It follows that $(\mathbb{P} \downarrow \mathbb{p})^- \cap (\mathbb{P} \downarrow \mathbb{q})^- = \emptyset$. Thus $i(\mathbb{p}) \cap i(\mathbb{q}) = ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ \cap ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ = \emptyset$.

(iv): Suppose that $\emptyset \neq O \in RO(\mathbb{P})$. Let $\mathbb{p} \in O$. Then $i(\mathbb{p}) = ((\mathbb{P} \downarrow \mathbb{p})^-)^\circ \subseteq O$.

⊣ (Claim 2.9.2)

❑ (Lemma 2.9)

In the construction of the complete Boolean algebra $\mathbb{B}$ in Lemma 2.9, the dense embedding of $\mathbb{P}$ into $\mathbb{B}^+$ constructed there was injective if $\mathbb{P}$ is separative. Actually a more general statement holds (see Lemma 2.19 , (1)). In the following we just prove a special case of Lemma 2.19 , (1):

Lemma 2.10 *If $\mathbb{P}$ is sub-Boolean, and $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding of $\mathbb{P}$ in a poset $\mathbb{Q}$, then i is injective.* *P-forcing-eq-3-a-0*

Proof. Suppose that $\mathbb{p}, \mathbb{p}' \in \mathbb{P}$ are distinct. Since $\leq_{\mathbb{P}}$ is a partial ordering, we have either $\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{p}'$ or $\mathbb{p}' \not\leq_{\mathbb{P}} \mathbb{p}$. Suppose, without loss of generality, that $\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{p}'$ holds. Since

$\mathbb{P}$ is separative, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}'$. Since i is order and incompatibility preserving. We have $i(\mathbb{r}) \leq_{\mathbb{Q}} i(\mathbb{p})$ and $i(\mathbb{r}) \perp_{\mathbb{Q}} i(\mathbb{p}')$. Thus $i(\mathbb{p}) \neq i(\mathbb{p}')$. ❑ (Lemma 2.10)

For a poset $\mathbb{P}$, since $\mathbb{P}$ is densely embedded into $(RO(\mathbb{P}))^+$, $\mathbb{P}$ and $(RO(\mathbb{P}))^+$ are semantically forcing equivalent by Lemma 2.3. We shall call $RO(\mathbb{P})$ the *Boolean completion* of $\mathbb{P}$. $RO(\mathbb{P})$ is uniquely characterized up to isomorphism over $\mathbb{P}$ as the complete Boolean algebra $\mathbb{B}$ in to such that $\mathbb{P}$ can be densely embedded.

By Lemma 2.3 it follows further that if posets $\mathbb{P}$ and $\mathbb{Q}$ have isomorphic Boolean completions then $\mathbb{P}$ and $\mathbb{Q}$ are semantically forcing equivalent. We shall call $\mathbb{P}$ and $\mathbb{Q}$ with $RO(\mathbb{P}) \cong RO(\mathbb{Q})$ *forcing equivalent* and denote this by $\mathbb{P} \approx \mathbb{Q}$. Thus, if $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent then they are semantically forcing equivalent. In general, semantic forcing equivalence does not imply forcing equivalence, e.g. the pair of any non trivial poset $\mathbb{P}$ and the lottery sum $\mathbb{Q}$ of many copies of $\mathbb{P}$ is a counterexample[(29)]. In Corollary 3.24 we shall show that this counterexample is a very symptomatic one and that the two notions coincide for locally homogeneous posets.

Corollary 2.11 *A poset $\mathbb{P}$ is sub-Boolean if and only if $\mathbb{P}$ is a dense subordering of positive elements of a Boolean algebra.* ❑ *P-forcing-eq-3-0*

We shall call a poset $\mathbb{P}$ a *Boolean poset*, if $\mathbb{P} = \mathbb{B}^+$ for a Boolean algebra $\mathbb{B}$ (with $\mathbb{1}_{\mathbb{P}} = \mathbb{1}_{\mathbb{B}}$ and $\leq_{\mathbb{P}} = \leq_{\mathbb{B}} \cap (\mathbb{B}^+)^2$). If $\mathbb{P}$ is a Boolean poset, the Boolean algebra corresponding to $\mathbb{P}$ is denoted by $\mathbb{B}_{\mathbb{P}}$. It is $\mathbb{B}_{\mathbb{P}} = \mathbb{P} \cup \{\mathbb{1}_{\mathbb{B}_{\mathbb{P}}}\}$.

A Boolean poset $\mathbb{P}$ a *cBa poset* if $\mathbb{B}_{\mathbb{P}}$ is a complete Boolean algebra.

In the proof of Lemma 2.9, if $\mathbb{P}$ is Boolean, then we can check that the mapping i defined there is a complete Boolean monomorphism. This is an instance of more general fact:

Lemma 2.12 (1) *For posets $\mathbb{P}$ and $\mathbb{Q}$, if $\mathbb{P}$ is sub-Boolean, then, any mapping $i : \mathbb{P} \to \mathbb{Q}$ satisfying (2.4) is 1-1 and strictly order preserving, that is, we have:* *P-forcing-eq-2-3*

(2.20) *For all $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}$, $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p} \Leftrightarrow i(\mathbb{p}') \leq_{\mathbb{Q}} i(\mathbb{p})$.* *forcing-eq-5-1-0*

(2) *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are Boolean posets. If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\tilde{i} = i \cup \{\langle \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}, \mathbb{0}_{\mathbb{B}_{\mathbb{Q}}} \rangle\}$ is a dense complete Boolean embedding of $\mathbb{B}_{\mathbb{P}}$ into $\mathbb{B}_{\mathbb{Q}}$.*

(3) *Suppose that $\mathbb{P}$ is a cBa poset and $\mathbb{Q}$ is a Boolean poset. If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\mathbb{B}_{\mathbb{Q}}$ is also a complete Boolean algebra and $\tilde{i} = i \cup \{\langle \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}, \mathbb{0}_{\mathbb{B}_{\mathbb{Q}}} \rangle\}$ is a Boolean isomorphism from $\mathbb{B}_{\mathbb{P}}$ to $\mathbb{B}_{\mathbb{Q}}$.*

Proof. (1): Suppose that $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}$ are distinct. Since $\leq_{\mathbb{P}}$ is a partial ordering on $\mathbb{P}$, we have either $\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{p}'$ or $\mathbb{p}' \not\leq_{\mathbb{P}} \mathbb{p}$. Suppose without loss of generality that

[(29)] A lottery sum $\mathbb{Q}$ of $\mathbb{P}$ is such a poset that there is a maximal pairwise incompatible elements $\mathbb{q}_\alpha$, $\alpha < \kappa$ of $\mathbb{P}$ such that $\mathbb{Q} \restriction \mathbb{q}_\alpha \cong \mathbb{P}$ for all $\alpha < \kappa$. If $\kappa > |\mathbb{P}|$, we cannot have $\mathbb{P} \approx \mathbb{Q}$. On the other hand, $\mathbb{P}$ and $\mathbb{Q}$ are semantically forcing equivalent since both of the posets simply add a $(\mathsf{V}, \mathbb{P})$-generic set.

$\mathbb{p} \not\leq_{\mathbb{P}} \mathbb{p}'$ holds. Since $\mathbb{P}$ is separative, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}'$. Since i is order and incompatibility preserving, it follows that $i(\mathbb{r}) \leq_{\mathbb{P}} i(\mathbb{p})$ and $i(\mathbb{r}) \perp_{\mathbb{Q}} i(\mathbb{p}')$. Thus $i(\mathbb{p}) \neq i(\mathbb{p}')$. This argument also shows "$\Leftarrow$" of (2.20). "$\Rightarrow$" of (2.20) is a part of (2.4).

(2): This follow from Lemma 2.17, (1) and Lemma 2.19 below.

(3): By (2), $\tilde{i}$ is a Boolean monomorphism. Since $i''\mathbb{B}_{\mathbb{P}}$ ($\cong \mathbb{B}_{\mathbb{P}}$) is a dense complete subalgebra of $\mathbb{B}_{\mathbb{Q}}$, it follows that $i''\mathbb{B}_{\mathbb{P}} = \mathbb{B}_{\mathbb{Q}}$. ❑ (Lemma 2.12)

For a Boolean poset $\mathbb{P}$ and $S \subseteq \mathbb{P}$, we shall also write $\prod^{\mathbb{P}} S$ and $\sum^{\mathbb{P}} S$ in place of $\prod^{\mathbb{B}_{\mathbb{P}}} S$ and $\sum^{\mathbb{B}_{\mathbb{P}}} S$ respectively. Note that if $\mathbb{P}$ is not a cBa poset, then it may happen that $\prod^{\mathbb{P}} S$ or $\sum^{\mathbb{P}} S$ does not simply exist. Even if they exist it is possible that we have $\prod^{\mathbb{P}} S = \mathbb{0}_{\mathbb{B}_{\mathbb{P}}} \notin \mathbb{P}$ or $\sum^{\mathbb{P}} S = \mathbb{0}_{\mathbb{B}_{\mathbb{P}}} \notin \mathbb{P}$.[30].

2.2 Boolean valued models

In Section 2.1, we have shown that any poset can be densely embedded into a cBa poset and hence is forcing equivalent to the cBa poset. If $\mathbb{P}$ is a cBa poset, we can see $\mathsf{V}^{\mathbb{P}}$ as a Boolean valued model. bvm

For a cBa poset $\mathbb{P}$, $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, let

$$(2.21) \quad [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = \textstyle\sum^{\mathbb{B}_{\mathbb{P}}} \{\mathbb{p} \in \mathbb{P} : \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}\}$$ forcing-eq-5-2

where we let $\sum^{\mathbb{B}_{\mathbb{P}}} \emptyset = \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}$.

Lemma 2.13 *Suppose that $\mathbb{P}$ is a cBa poset , $\mathbb{p} \in \mathbb{P}$, $\varphi = \varphi(x_0, ..., x_{n-1})$ an $\mathcal{L}_\varepsilon$-formula and $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ $\mathbb{P}$-names.* forcing-eq-5-3

(1) *If $[\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} \neq \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}$ then we have*

$$(2.22) \quad [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”}.$$

(2) *For a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, we have*

$$(2.23) \quad \mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}]) \;\Leftrightarrow\; [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} \in \mathbb{G}.$$

Proof. (1): By Lemma 1.11, (2).

(2): By (1) and Forcing Theorem 1.12. ❑ (Lemma 2.13)

By Lemma 2.13, (1), $[\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}}$ is the maximal condition $\mathbb{p} \in \mathbb{P}$ which forces $\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$.

For a poset $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{a}$ we define the hereditary domain $hd(\underset{\sim}{a})$ of $\underset{\sim}{a}$ by the following: Let

$$(2.24) \quad d(\underset{\sim}{a}) = \mathrm{dom}(\underset{\sim}{a})$$ bvalue-0

[30] $\sum^{\mathbb{P}} S = \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}$ does happen if $S = \emptyset$.

and let

(2.25) $hd_0(\underset{\sim}{a}) = \{\underset{\sim}{a}\}$ and bvalue-1

(2.26) $hd_{n+1}(\underset{\sim}{a}) = d''hd_n(\underset{\sim}{a})$ bvalue-2

for $n \in \omega$. Let

(2.27) $hd(\underset{\sim}{a}) = \bigcup_{n\in\omega} hd_n(\underset{\sim}{a})$. bvalue-3

A $\mathbb{P}$-name $\underset{\sim}{a}$ is said to be a *very nice $\mathbb{P}$-name*, if, for any $\mathbb{P}$-name $\underset{\sim}{b}$, there is at most one $\mathbb{p} \in \mathbb{P}$ such that $\langle \underset{\sim}{b}, \mathbb{p}\rangle \in \underset{\sim}{a}$. Since each singleton $\subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$, very nice $\mathbb{P}$-name is a nice $\mathbb{P}$-name. If $\underset{\sim}{a}$ is a very nice $\mathbb{P}$-name then $\underset{\sim}{a} : \mathrm{dom}(\underset{\sim}{a}) \to \mathbb{P}$.

A $\mathbb{P}$-name is a *hereditarily very nice $\mathbb{P}$-name* if all elements of $hd(\underset{\sim}{a})$ are very nice $\mathbb{P}$-names.

In the theory of Boolean valued models the Boolean valued universe $\mathsf{V}^{\mathbb{B}}$ for a given complete Boolean algebra $\mathbb{B}$ is constructed as follows: we first define $\mathsf{V}^{\mathbb{B}}_\alpha$ for $\alpha \in \mathsf{On}$ recursively by

(2.28) $\mathsf{V}^{\mathbb{B}}_0 = \emptyset$; bvalue-4

(2.29) $\mathsf{V}^{\mathbb{B}}_{\alpha+1} = \{f \,:\, f : x \to \mathbb{B}^+ \text{ for some } x \subseteq \mathsf{V}^{\mathbb{B}}_\alpha\}$; and bvalue-5

(2.30) $\mathsf{V}^{\mathbb{B}}_\gamma = \bigcup_{\alpha<\gamma} \mathsf{V}^{\mathbb{B}}_\alpha$ for a limit ordinal γ. bvalue-6

Let $\mathsf{V}^{\mathbb{B}} = \bigcup_{\alpha\in\mathsf{On}} \mathsf{V}^{\mathbb{B}}_\alpha$.

Lemma 2.14 *For any complete Boolean algebra $\mathbb{B}$ and the poset $\mathbb{P} = \mathbb{B}^+$, we have $\mathsf{V}^{\mathbb{B}} \subseteq \mathsf{V}^{\mathbb{P}}$.* P-bvalue-0

Proof. We prove that $\mathsf{V}^{\mathbb{B}}_\alpha \subseteq V^{\mathbb{P}}$ holds for all $\alpha \in \mathsf{On}$ by induction on α.

For $\alpha = 0$ this is clear by (2.28). Suppose that $\mathsf{V}^{\mathbb{B}}_\alpha \subseteq V^{\mathbb{P}}$ holds. Then, by (2.29)

(2.31)
$$\begin{aligned}\mathsf{V}^{\mathbb{B}}_{\alpha+1} &= \{f \,:\, f : x \to \mathbb{B}^+ \text{ for some } x \subseteq \mathsf{V}^{\mathbb{B}}_\alpha\} \\ &\subseteq \{f \,:\, f : x \to \mathbb{B}^+ \text{ for some set } x \text{ of } \mathbb{P}\text{-names}\} \subseteq \mathsf{V}^{\mathbb{P}}.\end{aligned}$$

If γ is a limit ordinal and $\mathsf{V}^{\mathbb{B}}_\alpha \subseteq V^{\mathbb{P}}$ holds for all $\alpha < \gamma$, then $\mathsf{V}^{\mathbb{B}}_\gamma \subseteq V^{\mathbb{P}}$ also holds by (2.30). ❑ (Lemma 2.14)

Lemma 2.15 *Suppose that $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{P} = \mathbb{B}^+$.* P-bvalue-1

(1) *$V^{\mathbb{B}}$ is the class of all hereditarily very nice $\mathbb{P}$-names.*

(2) *For any $\mathbb{P}$-name $\underset{\sim}{a}$ there is a hereditarily very nice $\mathbb{P}$-name $\underset{\sim}{c}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{c}$".*

(3) *For any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, we have $\{\underset{\sim}{a}[\mathbb{G}] \,:\, \underset{\sim}{a} \in \mathsf{V}^{\mathbb{B}}\} = \mathsf{V}[\mathbb{G}]$.*

Proof. (1): Let $\mathcal{C} = \{\underset{\sim}{a} \,:\, \underset{\sim}{a}$ is a hereditarily very nice $\mathbb{P}$-name$\}$.

We first prove that all $f \in \mathsf{V}^{\mathbb{B}}$ are elements of $\mathcal{C}$ by induction on $rank(f)$. Suppose that for all $g \in \mathsf{V}^{\mathbb{B}}$ with $rank(g) < rank(f)$ we have $g \in \mathcal{C}$. Then, by (2.29), we have $f : \mathrm{dom}(f) \to \mathbb{P}$ and $\mathrm{dom}(f) \subseteq \mathcal{C}$. Thus it follows that $f \in \mathcal{C}$.

Similar induction also proves that all $\underset{\sim}{a} \in \mathcal{C}$ is an element of $\mathsf{V}^{\mathbb{B}}$.

(2): We prove the assertion by induction on $rank(\underset{\sim}{a})$. Assume that for all $\mathbb{P}$-name $\underset{\sim}{b}$ with $rank(\underset{\sim}{b}) < rank(\underset{\sim}{a})$ there is an hereditarily very nice $\mathbb{P}$-name $\underset{\sim}{d}_{\underset{\sim}{b}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{d}_{\underset{\sim}{b}}$". Let $\underset{\sim}{a}_0 = \{\langle \underset{\sim}{d}_{\underset{\sim}{b}}, \mathbb{p}\rangle : \langle \underset{\sim}{b}, \mathbb{p}\rangle \in \underset{\sim}{a}\}$. Then we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{a}_0$". Let $\underset{\sim}{c} = \{\langle \underset{\sim}{d}, \sum^{\mathbb{B}}\{\mathbb{p} : \langle \underset{\sim}{d}, \mathbb{p}\rangle \in \underset{\sim}{a}_0\}\rangle : \underset{\sim}{d} \in \mathrm{dom}(\underset{\sim}{a}_0)\}$. Then $\underset{\sim}{c}$ is a hereditarily very nice $\mathbb{P}$-name and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{c}$".

(3): By Lemma 2.14 and (2). ❑ (Lemma 2.15)

Lemma 2.16 *Suppose that $\mathbb{P}$ is a cBa poset.* P-bvalue-2

(1) *For $\mathcal{L}_\varepsilon$-formulas $\varphi = \varphi(x_0, ..., x_{n-1})$, $\psi = \psi(x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$,*

(a) $[\![(\varphi \wedge \psi)(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} \wedge^{\mathbb{B}_{\mathbb{P}}} [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}}$,

(b) $[\![(\varphi \vee \psi)(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} \vee^{\mathbb{B}_{\mathbb{P}}} [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}}$,

(c) $[\![\neg\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = \neg^{\mathbb{B}_{\mathbb{P}}} [\![\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}}$.

(2) *For $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x, x_0, ..., x_{n-1})$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$,*

(a) $[\![\exists x\varphi(x, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = \sum^{\mathbb{B}_{\mathbb{P}}}\{[\![\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} : \underset{\sim}{a}$ *is a $\mathbb{P}$-name*$\}$,

(b) $[\![\forall x\varphi(x, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} = \prod^{\mathbb{B}_{\mathbb{P}}}\{[\![\varphi(\underset{\sim}{a}, \underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})]\!]^{\mathbb{B}_{\mathbb{P}}} : \underset{\sim}{a}$ *is a $\mathbb{P}$-name*$\}$,

Proof. ❑ (Lemma 2.16)

2.3 Complete embedding and projection

For posets $\mathbb{P}$ and $\mathbb{Q}$, a mapping $i : \mathbb{P} \to \mathbb{Q}$ is said to be a *complete embedding* if proj

(2.32) i satisfies the conditions (2.3) and (2.4); and forcing-eq-6

(2.33) for any predense subset D of $\mathbb{P}$, $i''D$ is predense in $\mathbb{Q}$. forcing-eq-8

Similarly to dense embedding, we do not assume in general that a complete embedding is a 1-1 mapping.

If $\mathbb{P}$ is a subposet of $\mathbb{Q}$ and the identity function from $\mathbb{P}$ to $\mathbb{Q}$ is a complete embedding, we say that $\mathbb{P}$ is a regular subposet of $\mathbb{Q}$ and denote this by $\mathbb{P} \lessdot \mathbb{Q}$. If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding we denote this by $i : \mathbb{P} \overset{\lessdot}{\to} \mathbb{Q}$.

Lemma 2.17 (1) *If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding for posets $\mathbb{P}$ and $\mathbb{Q}$, then i is a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$.* P-forcing-eq-4

(2) *Suppose $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ are posets. If $i : \mathbb{P} \to \mathbb{Q}$, $j : \mathbb{Q} \to \mathbb{R}$ are complete embeddings, then $j \circ i : \mathbb{P} \to \mathbb{R}$ is also a complete embedding.*

Proof. (1): Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding and $D \subseteq \mathbb{P}$ is a predense subset of $\mathbb{P}$. We have to show that $i''D$ is a predense subset of $\mathbb{Q}$. We check that $i''D$ satisfies the condition (1.5) in Lemma 1.4, (1). For $\mathbb{q} \in \mathbb{Q}$, there is $\mathbb{p} \in \mathbb{P}$ such that $i(\mathbb{p}) \leq_{\mathbb{Q}} \mathbb{q}$ since $i''\mathbb{P}$ is dense in $\mathbb{Q}$. Since D is predense in $\mathbb{P}$, there is a $\mathbb{d} \in D$ such that $\mathbb{p} \top_{\mathbb{P}} \mathbb{d}$ by Lemma 1.4, (1). Since i is order preserving it follows that $i(\mathbb{p}) \top_{\mathbb{Q}} i(\mathbb{d})$. Thus $\mathbb{q} \top_{\mathbb{Q}} i(\mathbb{d})$.

(2): Easy to check. ❑ (Lemma 2.17)

Lemma 2.18 *For posets $\mathbb{P}$ and $\mathbb{Q}$, the following are equivalent:* *P-forcing-eq-5*

(a) *$i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding.*

(b) *$i : \mathbb{P} \to \mathbb{Q}$ satisfies (2.3), (2.4) and,*

(2.34) *for any $\mathbb{q} \in \mathbb{Q}$, there is $\mathbb{p} \in \mathbb{P}$ such that, for any $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$, $i(\mathbb{r}) \top_{\mathbb{Q}} \mathbb{q}$.* *forcing-eq-11*

(c) *$i : \mathbb{P} \to \mathbb{Q}$ satisfies (2.3), (2.4) and,*

(2.35) *for every maximal antichain A in $\mathbb{P}$, $i''A$ is a maximal antichain in $\mathbb{Q}$.* *forcing-eq-12*

Proof. (b) $\Rightarrow$ (a): Suppose that $i : \mathbb{P} \to \mathbb{Q}$ satisfies (2.3), (2.4), and (2.34). We have to show that i satisfies (2.33). If not, there is a predense $D \subseteq \mathbb{P}$ such that $i''D$ is not predense in $\mathbb{P}$. Thus there is $\mathbb{q}^* \in \mathbb{Q}$ such that,

(2.36) for all $\mathbb{d} \in D$, we have $i(\mathbb{d}) \perp_{\mathbb{Q}} \mathbb{q}^*$ forcing-eq-13

by Lemma 1.4, (1).

Claim 2.18.1 *There is no $\mathbb{p} \in \mathbb{P}$ such that $i(\mathbb{r}) \top_{\mathbb{Q}} \mathbb{q}^*$ for all $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$.*

$\vdash$ For $\mathbb{p} \in \mathbb{P}$, since D is predense, there is $\mathbb{d} \in D$ such that $\mathbb{p} \top_{\mathbb{P}} \mathbb{d}$. Let $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}, \mathbb{d}$. Then $i(\mathbb{r}) \leq_{\mathbb{Q}} i(\mathbb{d})$ since i is order preserving. We have $\mathbb{q}^* \perp_{\mathbb{Q}} i(\mathbb{d})$ by (2.36). Thus we have $\mathbb{q}^* \perp_{\mathbb{Q}} i(\mathbb{r})$. $\dashv$ (Claim 2.18.1)

This is a contradiction to (2.34).

(a) $\Rightarrow$ (c): Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and $A \subseteq \mathbb{P}$ is a maximal antichain. By Lemma 1.4, (3), this means that $A \subseteq \mathbb{P}$ is an antichain in $\mathbb{P}$ and it is predense in $\mathbb{P}$. $i''A$ is an antichain by (2.4) and $i''A$ is predense in $\mathbb{Q}$ by (2.33). Thus, again by Lemma 1.4, (3), $i''A$ is a maximal antichain in $\mathbb{Q}$.

(c) $\Rightarrow$ (b): Suppose that i satisfies (2.3) and (2.4) but not (2.34). Then there is $q^* \in \mathbb{Q}$ such that,

(2.37) for any $\mathbb{p} \in \mathbb{P}$, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ such that $i(\mathbb{r}) \perp_{\mathbb{Q}} \mathbb{q}^*$. forcing-eq-14

Let

(2.38) $D = \{\mathbb{r} \in \mathbb{P} : i(\mathbb{r}) \perp_{\mathbb{Q}} \mathbb{q}^*\}$. forcing-eq-15

D is dense in $\mathbb{P}$ by (2.37). Let $A \subseteq D$ be a maximal antichain in $\mathbb{P}$ (there is such A by Lemma 1.4, (4)). $i''A$ is not a maximal antichain in $\mathbb{Q}$ since $i''A \cup \{\mathbb{q}^*\}$ is an antichain by (2.38). ❑ (Lemma 2.18)

A condition $\mathbb{p} \in \mathbb{P}$ for the condition $\mathbb{q} \in \mathbb{Q}$ in (2.34) is called a reduction of $\mathbb{q}$ (for i). Note that a projection of $\mathbb{q}$ is not unique in general: if $\mathbb{p}$ is a projection of $\mathbb{q}$, any $\mathbb{p}' \in \mathbb{P}$ with $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ is also a projection of $\mathbb{q}$.

Later we introduce a notion about a mapping which is also called projection (see (2.42) $\sim$ (2.43)). This notion is connected with but different from the projection of a condition introduced here.

If $\mathbb{P}$ is assumed to be sub-Boolean and we add the condition $i(\mathbb{p}) \leq_{\mathbb{Q}} \mathbb{q}$ to (2.34), then the condition $\mathbb{q}$ is unique. If $\mathbb{Q}$ is also sub-Boolean then the mapping $\mathbb{p} \mapsto \mathbb{q}$ is a projection in the sense of (2.42) $\sim$ (2.44) (Exercise).

Lemma 2.19 (1) *For posets $\mathbb{P}$ and $\mathbb{Q}$, if $\mathbb{P}$ is sub-Boolean, then, any complete embedding $i : \mathbb{P} \to \mathbb{Q}$ is 1-1 and strictly order preserving, that is, for all $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}$, $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p} \Leftrightarrow i(\mathbb{p}') \leq_{\mathbb{Q}} i(\mathbb{p})$.* *P-proj-a-0*

(2) *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are Boolean posets. If $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, then $\tilde{i} = i \cup \{\langle \mathbb{0}_{\mathbb{A}}, \mathbb{0}_{\mathbb{B}} \rangle\}$ is a complete Boolean monomorphism from $\mathbb{B}_{\mathbb{P}}$ to $\mathbb{B}_{\mathbb{Q}}$.*

(3) *Suppose that $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{P}'$, $\mathbb{Q}'$ are posets, $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, $i_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}'$, $i_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}'$ are dense embeddings and $\mathbb{Q}'$ is a cBa poset with a complete Boolean algebra $\mathbb{A}$ such that $\mathbb{Q}' = \mathbb{A}^+$. Then there is the unique complete embedding $i^* : \mathbb{P}' \to \mathbb{Q}'$ which makes the diagram below commutative:*

$$\begin{tikzcd} \mathbb{P}' \arrow[r, "i^*"] & \mathbb{Q}' \\ \mathbb{P} \arrow[u, "i_{\mathbb{P}}"] \arrow[r, "i"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{Q} \arrow[u, "i_{\mathbb{Q}}"'] \end{tikzcd} \tag{2.39}$$ *forcing-eq-16*

Proof. (1): This follows from Lemma 2.12, (1).

(2): It is enough to prove that i preserves $\neg$ and $\wedge$.

Suppose that $\mathbb{p} \in \mathbb{P}$ and $\mathbb{p}' = \neg^{\mathbb{A}}\mathbb{p}$. If $\mathbb{p}' = \mathbb{0}_{\mathbb{A}}$ then $\mathbb{p} = \mathbb{1}_{\mathbb{A}}$ and we have $\tilde{i}(\mathbb{1}_{\mathbb{A}}) = \mathbb{1}_{\mathbb{B}}$ and $\tilde{i}(\mathbb{0}_{\mathbb{A}}) = \mathbb{0}_{\mathbb{B}}$. Otherwise $D = \{\mathbb{p}, \mathbb{p}'\}$ is a predense subset of $\mathbb{A}^+$. Thus $i''D = \{i(\mathbb{p}), i(\mathbb{p}')\}$ is a predense subset of $\mathbb{B}^+$. This means that $i(\mathbb{p}) \vee_{\mathbb{B}} i(\mathbb{p}') = \mathbb{1}_{\mathbb{B}}$. Since $i(\mathbb{p}) \wedge^{\mathbb{B}} i(\mathbb{p}') = \mathbb{0}_{\mathbb{B}}$ by incompatibility preservingness of i, we have $i(\mathbb{p}') = \neg^{\mathbb{B}} i(\mathbb{p})$.

Suppose now that $\mathbb{p}$, $\mathbb{p}' \in \mathbb{A}^+$ and let $\mathbb{p}'' = \mathbb{p} \wedge^{\mathbb{A}} \mathbb{p}'$. If $\mathbb{p}'' = \mathbb{0}_{\mathbb{A}}$, we have $\mathbb{p} \perp_{\mathbb{A}^+} \mathbb{p}'$. By incompatibility preserving of i, it follows that $i(\mathbb{p}) \perp_{\mathbb{B}^+} i(\mathbb{p}')$. Thus $\tilde{i}(\mathbb{p}) \wedge^{\mathbb{B}} \tilde{i}(\mathbb{p}') = \mathbb{0}_{\mathbb{B}} = \tilde{i}(\mathbb{p} \wedge^{\mathbb{A}} \mathbb{p}')$.

Otherwise $\mathbb{p}'' \in \mathbb{A}^+$. By order preservingness of i, we have $i(\mathbb{p}'') \leq_{\mathbb{B}^+} i(\mathbb{p})$, $i(\mathbb{p}')$ and hence $i(\mathbb{p}'') \leq_{\mathbb{B}^+} i(\mathbb{p}) \wedge^{\mathbb{B}} i(\mathbb{p}')$. Assume, toward a contradiction, that $i(\mathbb{p}'') \lneq_{\mathbb{B}^+} i(\mathbb{p}) \wedge^{\mathbb{B}} i(\mathbb{p}')$. Then there is $\mathbb{r} \leq_{\mathbb{B}^+} i(\mathbb{p}) \wedge^{\mathbb{B}} i(\mathbb{p}')$ such that $i(\mathbb{p}'') \perp_{\mathbb{B}^+} \mathbb{r}$. Let

(2.40) $D = \{\mathbb{d} \in \mathbb{A}^+ : \mathbb{d} \leq_{\mathbb{A}^+} \mathbb{p}'' \text{ or } \mathbb{d} \perp_{\mathbb{A}^+} \mathbb{p} \text{ or } \mathbb{d} \perp_{\mathbb{A}^+} \mathbb{p}'\}$.

D is dense in $\mathbb{A}^+$ and hence predense but $\mathbb{r} \perp_{\mathbb{B}^+} \mathbb{q}$ for all $\mathbb{q} \in i''D$. This is a contradiction to the assumption that i is a complete embedding[(31)].

(3): For $\mathbb{p}' \in \mathbb{P}'$, let

(2.41) $i^*(\mathbb{p}') = \sum^{\mathbb{A}}\{i_{\mathbb{Q}} \circ i(\mathbb{p}) : \mathbb{p} \in \mathbb{P},\ i_{\mathbb{P}}(\mathbb{p}) \leq_{\mathbb{P}'} \mathbb{p}'\}$. forcing-eq-17

This mapping $i^* : \mathbb{P}' \to \mathbb{Q}'$ is as desired. [The rets will be written later.] ❑ (Lemma 2.19)

For posets $\mathbb{P}$ and $\mathbb{Q}$, a mapping $p : \mathbb{Q} \to \mathbb{P}$ is said to be a *projection*, if Scan_2023-06-21–6.40 iterated-forcing-katowice-2018 - annotated p.1〜加筆すべきものあり

(2.42) $p(\mathbb{1}_{\mathbb{Q}}) = \mathbb{1}_{\mathbb{P}}$; proj-0

(2.43) p is order preserving; and, proj-2

(2.44) for any $\mathbb{q} \in \mathbb{Q}$, if $\mathbb{p} \in \mathbb{P}$ is such that $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$, then there is $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ such that $p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{p}$. proj-3

Projections are often defined to be surjective but we do not assume this here. The following replaces the condition:

Lemma 2.20 *If $p : \mathbb{Q} \to \mathbb{P}$ is a projection then $p''\mathbb{Q}$ is dense in $\mathbb{P}$.* *P-proj-a-1*

Proof. Suppose $\mathbb{p} \in \mathbb{P}$. Then $\mathbb{p} \leq_{\mathbb{P}} \mathbb{1}_{\mathbb{P}} \overbrace{=}^{\text{by (2.42)}} p(\mathbb{1}_{\mathbb{Q}})$. By (2.44), there is $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{1}_{\mathbb{Q}}$ such that $p(\mathbb{q}') \leq_{\mathbb{Q}} \mathbb{p}$. ❑ (Lemma 2.20)

Lemma 2.21 *For complete Boolean algebras $\mathbb{A}$ and $\mathbb{B}$, there is a complete embedding $i : \mathbb{A}^+ \to \mathbb{B}^+$ if and only if there is a projection $p : \mathbb{B}^+ \to \mathbb{A}^+$* *P-proj-0*

Proof. Suppose first that $i : \mathbb{A}^+ \to \mathbb{B}^+$ is a complete embedding. Let $p : \mathbb{B}^+ \to \mathbb{A}$ be defined by

(2.45) $p(\mathbb{b}) = \prod^{\mathbb{A}}\{\mathbb{a} \in \mathbb{A}^+ : i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}\}$ proj-4

for $\mathbb{b} \in \mathbb{B}^+$.

Claim 2.21.1 *$\tilde{i}(p(\mathbb{b})) \geq_{\mathbb{B}^+} \mathbb{b}$ holds for all $\mathbb{b} \in \mathbb{B}^+$. In particular, $p : \mathbb{B}^+ \to \mathbb{A}^+$ and $p(\mathbb{b})$ is the minimal element $\mathbb{a}$ in $\mathbb{A}^+$ with $i(\mathbb{a}) \geq_{\mathbb{B}} \mathbb{b}$.* *Cl-proj-0*

$\vdash$ Otherwise there would be a $\mathbb{b}^* \in \mathbb{B}^+$ such that $\tilde{i}(p(\mathbb{b}^*)) \not\geq_{\mathbb{B}} \mathbb{b}^*$. Note that $\mathbb{b}^* \wedge \neg\tilde{i}(p(\mathbb{b}^*)) \neq \mathbb{0}_{\mathbb{B}}$.

Let

(31) In the terminology of Lemma 3.11, the argument here proves the following: *Suppose $\mathbb{P}$ is a conjunctive sub-Boolean poset, $\mathbb{Q}$ a conjunctive poset and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Then i preserves $\wedge$.*

(2.46) $D = \{\mathbb{d} \in \mathbb{A}^+ : i(\mathbb{d}) \leq_{\mathbb{B}^+} \tilde{i}(p(\mathbb{b}^*))$ or $\mathbb{d} \perp_{\mathbb{A}^+} \mathbb{a}$ for some $\mathbb{a} \in \mathbb{A}^+$ with $i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}^*\}$. proj-5

Subclaim 2.21.1.1 *D is dense in $\mathbb{A}^+$.*

$\vdash$ Suppose $\mathbb{c} \in \mathbb{A}^+$. If $\mathbb{c} \top_{\mathbb{A}^+} p(\mathbb{b}^*)$ then, letting $\mathbb{d} = \mathbb{c} \wedge p(\mathbb{b}^*)$, $\mathbb{d} \leq_{\mathbb{A}^+} \mathbb{c}$ and $\mathbb{d} \in D$.

If $\mathbb{c} \perp_{\mathbb{A}^+} p(\mathbb{b}^*)$, $\mathbb{c} \not\leq_{\mathbb{A}^+} \mathbb{a}^*$ for some $\mathbb{a}^* \in \mathbb{A}^+$ with $i(\mathbb{a}^*) \geq_{\mathbb{B}^+} \mathbb{b}^*$ (otherwise we would have $\mathbb{c} \leq_{\mathbb{A}^+} p(\mathbb{b}^*)$: a contradiction to the assumption).

Let $\mathbb{d} = \mathbb{c} \wedge \neg\mathbb{a}^*$. We have $\mathbb{d} \perp_{\mathbb{A}^+} \mathbb{a}^*$ hence $\mathbb{d} \in D$ and $\mathbb{d} \leq_{\mathbb{A}^+} \mathbb{c}$. $\dashv$ (Subclaim 2.21.1.1)

If $\mathbb{d} \in D$, then $i(\mathbb{d}) \leq_{\mathbb{B}^+} \tilde{i}(p(\mathbb{b}^*))$ or $i(\mathbb{d}) \perp_{\mathbb{B}^+} i(\mathbb{a})$ for some $\mathbb{a} \in \mathbb{A}^+$ with $i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}^*$.

Thus $\mathbb{b}^* \wedge \neg\tilde{i}(p(\mathbb{b}^*))$ is incompatible to all element of $i''D$ in $\mathbb{B}^+$. This is a contradiction to the assumption that i is a complete embedding.

The minimality of $p(\mathbb{b})$ among elements $\mathbb{a}$ of A^+ with $i(\mathbb{a}) \geq_{\mathbb{B}^+} b$ follows from this by the definition of $p(\mathbb{b})$. $\dashv$ (Claim 2.21.1)

We show that p is as desired in the following Claim:

Claim 2.21.2 (i) $p(\mathbb{1}_{\mathbb{B}}) = \mathbb{1}_{\mathbb{A}}$.

(ii) $p(i(\mathbb{a})) = \mathbb{a}$ *for all* $\mathbb{a} \in A^+$. *In particular p is a surjection.*

(iii) *p is order preserving.*

(iv) *p satisfies* (2.44).

$\vdash$ (i) and (ii) are clear by the definition (2.45) of p.

(iii): For $\mathbb{b}$, $\mathbb{b}' \in \mathbb{B}^+$ with $\mathbb{b}^+ \leq_{\mathbb{B}^+} \mathbb{b}$, we have $\{\mathbb{a} \in \mathbb{A}^+ : i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}'\} \supseteq \{\mathbb{a} \in \mathbb{A}^+ : i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}\}$. Hence $p(\mathbb{b}') = \prod^{\mathbb{A}}\{\mathbb{a} \in \mathbb{A}^+ : i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}'\} \leq_{\mathbb{A}^+} \prod^{\mathbb{A}}\{\mathbb{a} \in \mathbb{A}^+ : i(\mathbb{a}) \geq_{\mathbb{B}^+} \mathbb{b}\} = p(\mathbb{b})$.

(iv): Suppose that $\mathbb{p} \in \mathbb{A}^+$ and $\mathbb{q} \in \mathbb{B}^+$ are such that $\mathbb{p} \leq_{\mathbb{A}^+} p(\mathbb{q})$. We have to show that there is $\mathbb{q}' \leq_{\mathbb{B}^+} \mathbb{q}$ such that $p(\mathbb{q}') \leq_{\mathbb{A}^+} \mathbb{p}$.

Let $\mathbb{q}' = i(\mathbb{p}) \wedge^{\mathbb{B}} \mathbb{q}$.

Subclaim 2.21.2.1 $\mathbb{q}' \neq \mathbb{0}_{\mathbb{B}}$.

$\vdash$ Suppose otherwise. Then $\neg i(\mathbb{p}) \geq_{\mathbb{B}^+} \mathbb{q}$. By Lemma 2.19, (2) and Claim 2.21.1, $i(p(\mathbb{q}) \wedge \neg\mathbb{p}) = i(p(\mathbb{q})) \wedge \neg i(\mathbb{p}) \geq_{\mathbb{B}^+} \mathbb{q}$. Since $p(\mathbb{q}) \wedge \neg\mathbb{p} \lneq_{\mathbb{A}^+} p(\mathbb{q})$, this is a contradiction to the minimality of $p(\mathbb{q})$ in the sense of Claim 2.21.1. $\dashv$ (Subclaim 2.21.2.1)

$\mathbb{q}' \leq_{\mathbb{B}^+} \mathbb{q}$ and $p(\mathbb{q}') \leq_{\mathbb{A}^+} p(i(\mathbb{p})) = \mathbb{p}$ (by (iii)). Thus this $\mathbb{q}'$ is as desired. $\dashv$ (Claim 2.21.2)

Suppose now that $p : \mathbb{B}^+ \to \mathbb{A}^+$ is a projection. Let $i : \mathbb{A}^+ \to \mathbb{B}^+$ be defined by

(2.47) $i(\mathbb{a}) = \sum^{\mathbb{B}}\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}\}$. proj-6

We show that this i is a complete embedding of $\mathbb{A}^+$ into $\mathbb{B}^+$:

Claim 2.21.3 (i) $i(\mathbb{1}_{\mathbb{A}}) = \mathbb{1}_{\mathbb{B}}$.

(ii) *i is well-defined.*

(iii) *i is order preserving.*

(iv) *i is incompatibility preserving.*

(v) *$i(p(\mathbb{d})) \geq_{\mathbb{B}^+} \mathbb{d}$ for all $\mathbb{d} \in \mathbb{B}^+$.*

(vi) *i satisfies the condition (2.34).*

$\vdash$ (i) is clear by the definition (2.47) of i.

(ii): We have to show that $i(\mathbb{a}) \neq \mathbb{0}_{\mathbb{B}}$ for all $\mathbb{a} \in \mathbb{A}^+$. For $\mathbb{a} \in \mathbb{A}^+$, since p is a surjection by the definition of a projection, there is $\mathbb{b} \in \mathbb{B}^+$ such that $p(\mathbb{b}) = \mathbb{a}$. Hence $\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}\} \neq \emptyset$ and this implies that $i(\mathbb{a}) \neq \mathbb{0}_{\mathbb{B}}$.

(iii): Suppose $\mathbb{a}' \leq_{\mathbb{A}^+} \mathbb{a}$ for $\mathbb{a}$, $\mathbb{a}' \in \mathbb{A}^+$. then $\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}'\} \subseteq \{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}\}$. Thus $i(\mathbb{a}') = \sum^{\mathbb{B}}\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}'\} \leq_{\mathbb{B}^+} \sum^{\mathbb{B}}\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}\} = i(\mathbb{a})$.

(iv): Suppose that $\mathbb{a}$, $\mathbb{a}' \in \mathbb{A}^+$ and $i(\mathbb{a}) \top_{\mathbb{B}^+} i(\mathbb{a}')$. Then there are $\mathbb{b}$, $\mathbb{b}' \in \mathbb{B}^+$ such that $p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}$, $p(\mathbb{b}') \leq_{\mathbb{A}^+} \mathbb{a}'$ and $\mathbb{b}_0 = \mathbb{b} \wedge \mathbb{b}' \in \mathbb{B}^+$. Let $\mathbb{a}_0 = p(\mathbb{b}_0)$. Them $\mathbb{a}_0 = p(\mathbb{b}_0) \leq_{\mathbb{A}^+} p(\mathbb{b}) \leq_{\mathbb{A}^+} \mathbb{a}$ and $\mathbb{a}_0 = p(\mathbb{b}_0) \leq_{\mathbb{A}^+} p(\mathbb{b}') \leq_{\mathbb{A}^+} \mathbb{a}'$. Thus $\mathbb{a} \top_{\mathbb{A}^+} \mathbb{a}'$.

(v): By the definition (2.47) of i, we have

$$i(p(\mathbb{d})) = \sum^{\mathbb{B}} \underbrace{\{\mathbb{b} \in \mathbb{B}^+ : p(\mathbb{b}) \leq_{\mathbb{A}^+} p(\mathbb{d})\}}_{\substack{\cup \\ \mathbb{d}}} \geq_{\mathbb{B}} \mathbb{d}. \tag{2.48}$$

(vi): For $\mathbb{q} \in \mathbb{B}^+$, let $\mathbb{p} = p(\mathbb{q})$. We show that $\mathbb{p}$ satisfies (2.34). Suppose $\mathbb{r} \leq_{\mathbb{A}^+} \mathbb{p}$. We have to show that $i(\mathbb{r}) \top_{\mathbb{B}^+} \mathbb{q}$.

Since $\mathbb{r} \leq_{\mathbb{A}^+} \mathbb{p} = p(\mathbb{q})$ and since p satisfies (2.44), it follows that there is $\mathbb{q}' \leq_{\mathbb{B}^+} \mathbb{q}$ such that $p(\mathbb{q}') \leq_{\mathbb{A}^+} \mathbb{r}$. By (iii) and (v), it follows that $\mathbb{q}' \leq i(p(\mathbb{q}')) \leq_{\mathbb{B}^+} i(\mathbb{r})$. Thus $i(\mathbb{r}) \top_{\mathbb{B}^+} \mathbb{q}$. $\dashv$ (Claim 2.21.3)

$\square$ (Lemma 2.21)

Lemma 2.22 *Suppose that $\mathbb{P}$, $\mathbb{Q}$ are posets and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding.* *P-complete-0*

(1) *If $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter, then $\mathbb{G} = i^{-1}{}''\mathbb{H}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter and we have $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ and $\mathsf{V}[\mathbb{G}]$ is transitive in $\mathsf{V}[\mathbb{H}]$.*

(2) *If $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, then there is a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $i''\mathbb{G} \subseteq \mathbb{H}$. In this case we also have $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ and $\mathsf{V}[\mathbb{G}]$ is transitive in $\mathsf{V}[\mathbb{H}]$.*

Proof. (1): Let $\mathbb{G} = i^{-1}{}''\mathbb{H}$. As in the proof of Lemma 2.3, (2), $\mathbb{G}$ is upward closed and pairwise compatible. By Lemma 1.5, it is enough to show that $\mathbb{G} \cap D \neq \emptyset$ for all predense $D \subseteq \mathbb{P}$ (in V). bbd Lemma1.5

Suppose that $D \subseteq \mathbb{P}$ is predense. Then $i''D$ is predense in $\mathbb{Q}$ by (2.33). Thus $\mathbb{H} \cap i''D \neq \emptyset$. Let $\mathbb{q} \in \mathbb{H} \cap i''D$ and let $\mathbb{p} \in D$ be such that $i(\mathbb{p}) = \mathbb{q}$. Then $\mathbb{p} \in \mathbb{G} \cap D$ by the definition of $\mathbb{G}$. Since $\mathbb{G} \in \mathsf{V}[\mathbb{H}]$, we have $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ by Theorem 1.9.

The transitivity of $\mathsf{V}[\mathbb{G}]$ in $\mathsf{V}[\mathbb{H}]$ follows from Lemma 1.8, (1).

(2): The proof of the first part of the assertions is postponed to 3.1. The same argument as that of the proof of Claim 2.3.2 shows that $\mathbb{G} = i^{-1}{}''\mathbb{H}$. Thus $\mathbb{G} \in \mathsf{V}[\mathbb{H}]$ and $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ by Theorem 1.9.

Further working in $\mathsf{V}[\mathbb{G}]$, let $\mathbb{P}$ be a ccc poset of cardinality $< \kappa$ and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \aleph_0$. By Maximal Principle, ❑ (Lemma 2.22)

Lemma 2.23 *Suppose that $p : \mathbb{Q} \to \mathbb{P}$ is a projection.* *Lemma 2.23 bbd*

(1) *If $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter, then $p''\mathbb{H}$ generates a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ and we have $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$.*

(2) *If $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, then there is a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $p''\mathbb{H} = \mathbb{G}$ and hence $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$.*

Proof. (1): $p''\mathbb{H}$ is pairwise compatibles by (2.43) and since $\mathcal{H}$ is pairwise compatible. Suppose $D \subseteq \mathbb{P}$ is open dense.

Claim 2.23.1 *$p^{-1}{}''D$ is dense in $\mathbb{Q}$.* *Cl-complete-0*

$\vdash$ If $\mathbb{q} \in \mathbb{Q}$, then there is $\mathbb{p} \in D$ such that $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$. By (2.44), there is $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ such that $p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{p}$ since D is open $p(\mathbb{q}') \in D$. Thus $\mathbb{q}' \in p^{-1}{}''D$. ⊣ (Claim 2.23.1)

Let $\mathbb{q} \in p^{-1}{}''D \cap \mathbb{H}$ and let $\mathbb{p} = p(\mathbb{p})$. Then $\mathbb{p} \in \mathbb{G} \cap D$.

Since $\mathbb{G}$ is definable from $i \in \mathsf{V}$ and $\mathbb{H}$, $\mathbb{G} \in \mathsf{V}[\mathbb{H}]$ and $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ by the minimality of $\mathsf{V}[\mathbb{G}]$.

(2): Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter. In $\mathsf{V}[\mathbb{G}]$, let

$$\mathbb{Q}^* = \{\mathbb{q} \in \mathbb{Q} : p(\mathbb{q}) \in \mathbb{G}\}. \tag{2.49}$$ complete-a-0

Then $\mathbb{Q}^*$ is an upward closed subset of $\mathbb{Q}$ and in particular $\mathbb{1}_{\mathbb{Q}} \in \mathbb{Q}^*$. Consider the poset $\mathbb{Q}^* = \langle \mathbb{Q}^*, \leq_{\mathbb{Q}} \cap (\mathbb{Q}^*)^2, \mathbb{1}_{\mathbb{Q}} \rangle$ and let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}], \mathbb{Q}^*)$-generic filter.

We show that $\mathbb{H}$ is as desired.

Claim 2.23.2 *$\mathbb{H}$ is a $(V, \mathbb{Q})$-generic filter.* *Cl-complete-1*

$\vdash$ $\mathbb{H}$ is upward closed in $\mathbb{Q}$ since $\mathbb{Q}^*$ is upward closed in $\mathbb{Q}$. Thus $\mathbb{H}$ is a filter on $\mathbb{Q}$.

To prove the genericity of $\mathbb{H}$, suppose that D is a dense open subset of $\mathbb{Q}$. We have to show that $\mathbb{H} \cap D \neq \emptyset$.

Subclaim 2.23.2.1 *In $\mathsf{V}[\mathbb{G}]$, $D \cap \mathbb{Q}^*$ is dense in $\mathbb{Q}^*$.* *SubCl-complete-1*

$\vdash$ Suppose that $\mathbb{q} \in \mathbb{Q}^*$. By the definition (2.49) of $\mathbb{Q}^*$, $\mathbb{p} = p(\mathbb{q}) \in \mathbb{G}$. Let

(2.50) $E = \{\mathbb{r} \in \mathbb{P} : \mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ there is $\mathbb{s} \leq_{\mathbb{P}} \mathbb{p}$ such that $\mathbb{s} \in D$ and $p(\mathbb{s}) = \mathbb{r}\}$. complete-a-1

E is dense in below $\mathbb{p}$ in $\mathbb{P}$: Suppose $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$. By (2.44), there is $\mathbb{q}_0 \in \mathbb{Q}$ such that $\mathbb{q}_0 \leq_{\mathbb{P}} \mathbb{q}$ and $p(\mathbb{q}_0) \leq \mathbb{p}'$. Let $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}_0$ be such that $\mathbb{q}' \in D$. Then $\mathbb{r} = p(\mathbb{q}') \in D$ and $\mathbb{r} \leq_{\mathbb{P}} p(\mathbb{q}_0) \leq \mathbb{p}' \leq \mathbb{p}$. Thus $\mathbb{r} \in E$. ⊣ (Subclaim 2.23.2.1)

Let $\mathbb{r} \in \mathbb{G} \cap E$. Then there is $\mathbb{q}' \in D$ such that $p(\mathbb{q}') = \mathbb{r}$. Thus $\mathbb{q}' \in \mathbb{Q}^*$. ⊣ (Claim 2.23.2)

By definition of $\mathbb{Q}^*$, $\mathbb{G} \supseteq p''\mathbb{H}$. By (1), $p''\mathbb{H}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter. By Lemma 1.1, it follows that $\mathbb{G} = p''\mathbb{H}$. By (1), it follows that $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$. ❑ (Lemma 2.23)

The inclusion $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ in Lemma 2.22 can be described more precisely.

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets and $i : \mathbb{P} \to \mathbb{Q}$ a complete embedding. We define the class mapping $\tilde{i} : \mathsf{V}^{\mathbb{P}} \to \mathsf{V}^{\mathbb{Q}}$ associated with i by recursion on $rank(\underset{\sim}{a})$ for a $\mathbb{P}$-name $\underset{\sim}{a}$ by:

(2.51) $\tilde{i}(\underset{\sim}{a}) = \{\langle \tilde{i}(\underset{\sim}{b}), i(\mathbb{p})\rangle : \langle \underset{\sim}{b}, \mathbb{p}\rangle \in \underset{\sim}{a}\}$. complete-0

Lemma 2.24 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets, and $i : \mathbb{P} \to \mathbb{Q}$ a complete embedding.* P-complete-2

(1) $\tilde{i} : \mathsf{V}^{\mathbb{P}} \to \mathsf{V}^{\mathbb{Q}}$ *is well-defined.*

(2) *If $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter and $\mathbb{H}$ a $(\mathsf{V}, \mathbb{P})$-generic filter with $i''\mathbb{G} \subseteq \mathbb{H}$, then we have $\underset{\sim}{a}[\mathbb{G}] = \tilde{i}(\underset{\sim}{a})[\mathbb{H}]$ for all $\mathbb{P}$-name $\underset{\sim}{a}$.*

$$\begin{tikzcd}
\mathsf{V}[\mathbb{G}] \arrow[r, "\subseteq"] & \mathsf{V}[\mathbb{H}] \\
\mathsf{V}^{\mathbb{P}} \arrow[u] \arrow[r, "\tilde{i}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathsf{V}^{\mathbb{Q}} \arrow[u] \\
\mathbb{P} \arrow[r, "i"'] & \mathbb{Q}
\end{tikzcd}$$

(3) *Suppose that $\varphi = \varphi(x_0, ..., x_{n-1})$ is an $\mathcal{L}_\varepsilon$-formula which is absolute between transitive models of ZFC. For any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" if and only if $i(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))$".*

(4) *Suppose that $\varphi = \varphi(x_0, ..., x_{n-1})$ is an $\mathcal{L}_\varepsilon$-formula which is upward absolute between transitive models of ZFC. For any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$" then $i(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))$".*

(5) *Suppose that $\varphi = \varphi(x_0, ..., x_{n-1})$ is an $\mathcal{L}_\varepsilon$-formula which is downward absolute between transitive models of ZFC. For any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, if $i(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))$" then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".*

(6) *Suppose that i is a dense embedding. Then, for any $\mathbb{Q}$-name $\underset{\sim}{b}$, there is a $\mathbb{P}$-name $\underset{\sim}{a}$ such that $\Vdash_{\mathbb{Q}}$ "$\tilde{i}(\underset{\sim}{a}) \equiv \underset{\sim}{b}$".*

(7) *Suppose again that i is a dense embedding. For any $\mathbb{p} \in \mathbb{P}$, $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$, and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$, we have*

$$\mathbb{p} \Vdash_{\mathbb{P}} \text{“}\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})\text{”} \quad \Leftrightarrow \quad i(\mathbb{p}) \Vdash_{\mathbb{Q}} \text{“}\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))\text{”}. \tag{2.52}$$ complete-1

Proof. (1): We have to show that $\tilde{i}(\underset{\sim}{a})$ is a $\mathbb{Q}$-name for all $\mathbb{P}$-name $\underset{\sim}{a}$. This can be done by induction on $rank(\underset{\sim}{a})$.

(2): The assertion can be shown by induction on $rank(\underset{\sim}{a})$ for a $\mathbb{P}$-name $\underset{\sim}{a}$.

(3): Suppose that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$". Let $\mathbb{H}$ be an arbitrary $(\mathsf{V}, \mathbb{Q})$-generic filter such that $i(\mathbb{p}) \in \mathbb{H}$. Then $\mathbb{G} = i^{-1}{}''\mathbb{H}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter by Lemma 2.22, (1) and $\mathbb{p} \in \mathbb{G}$. Also by Lemma 2.22, (1) we have $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$.

By Forcing Theorem 1.12, (1), we have $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$. By the absoluteness of φ and (2), it follows that $\mathsf{V}[\mathbb{H}] \models \varphi(\tilde{i}(\underset{\sim}{a}_0)[\mathbb{H}], ..., \tilde{i}(\underset{\sim}{a}_{n-1})[\mathbb{H}])$.

Since $\mathbb{H}$ was arbitrary $(\mathsf{V}, \mathbb{Q})$-generic filter with $i(\mathbb{p}) \in \mathbb{H}$, it follows by the 2. Forcing Theorem 1.13 that $i(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))$"

Assume now that $i(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(\tilde{i}(\underset{\sim}{a}_0), ..., \tilde{i}(\underset{\sim}{a}_{n-1}))$" and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$. By Lemma 2.22, (2), there is a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $i''\mathbb{G} \subseteq \mathbb{H}$. We have $i(\mathbb{p}) \in \mathbb{H}$ and $V[\mathbb{G}] \subseteq V[\mathbb{H}]$ again by Lemma 2.22, (2). By Forcing Theorem 1.12, (1) and by the assumption above, we have $\mathsf{V}[\mathbb{H}] \models \varphi(\tilde{i}(\underset{\sim}{a}_0)[\mathbb{H}], ..., \tilde{i}(\underset{\sim}{a}_{n-1})[\mathbb{H}])$. By the absoluteness of φ and (2), it follows that $\mathsf{V}[\mathbb{G}] \models \varphi(\underset{\sim}{a}_0[\mathbb{G}], ..., \underset{\sim}{a}_{n-1}[\mathbb{G}])$.

Since $\mathbb{G}$ was an arbitrary $(\mathsf{V}, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$, it follows by the 2. Forcing Theorem 1.13 that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$".

(4), (5): The proofs of the two directions of (3) also prove (4) and (5).

(6): By induction on $rank(\underset{\sim}{b})$. Suppose that, for all $\mathbb{Q}$-names $\underset{\sim}{d}$ with $rank(\underset{\sim}{d}) < rank(\underset{\sim}{b})$, there is a $\mathbb{P}$-name $\underset{\sim}{a}_{\underset{\sim}{d}}$ such that $\Vdash_{\mathbb{Q}}$ "$\tilde{i}(\underset{\sim}{a}_{\underset{\sim}{d}}) \equiv \underset{\sim}{d}$".

For each $\langle \underset{\sim}{d}, \mathbb{q} \rangle \in \underset{\sim}{b}$, let

$$\underset{\sim}{a}_{\langle \underset{\sim}{d}, \mathbb{q} \rangle} = \{\langle \underset{\sim}{a}_{\underset{\sim}{d}}, \mathbb{p} \rangle : \mathbb{p} \in \mathbb{P},\ \underset{\sim}{i}(\mathbb{p}) \leq_{\mathbb{Q}} \mathbb{q}\}. \tag{2.53}$$ complete-1-0

Let

$$\underset{\sim}{a} = \bigcup \{\underset{\sim}{a}_{\langle \underset{\sim}{d}, \mathbb{q} \rangle} : \langle \underset{\sim}{d}, \mathbb{q} \rangle \in \underset{\sim}{b}\}. \tag{2.54}$$ complete-1-1

Then $\underset{\sim}{a}$ is a $\mathbb{P}$-name and we can use the Forcing Theorem (Theorem 1.22) that $\Vdash_{\mathbb{Q}}$ "$\tilde{i}(\underset{\sim}{a}) \equiv \underset{\sim}{b}$" holds.

(7): follows from (3) and (6). ❑ (Lemma 2.24)

3 Two step iteration

3.1 Two step iteration and complete embedding

two

Lemma 3.1 *For any poset $\mathbb{P}$, $\mathbb{p} \in \mathbb{P}$, and $\mathbb{P}$-name $\underset{\sim}{a}$, there is a $\mathbb{P}$-name $\underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$ such that* two-complete *P-two-a-0*

(3.1) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$", *and* *two-a-a-0*

(3.2) *for all $\langle \underset{\sim}{d}, \mathbb{q} \rangle \in \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$, we have $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$.* *two-a-a-1*

Proof. Fore each $\langle \underset{\sim}{d}, \mathbb{r} \rangle \in \underset{\sim}{a}$ with $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}$, let $A_{\langle \underset{\sim}{d}, \mathbb{r} \rangle}$ be a maximal antichain below $\mathbb{p}$ and $\mathbb{r}$ [32].

Let

$$\underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}} = \bigcup \{\{\langle \underset{\sim}{d}, \mathbb{q} \rangle : \mathbb{q} \in A_{\langle \underset{\sim}{d}, \mathbb{r} \rangle}\} : \langle \underset{\sim}{d}, \mathbb{r} \rangle \in \underset{\sim}{a},\ \mathbb{r} \top_{\mathbb{P}} \mathbb{p}\}. \tag{3.3}$$ two-a-a-2

We show that $\underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$ is as desired. Since $\underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$ satisfies (3.2) by definition, It is enough to show that it satisfies (3.1). We show this via the Forcing Theorem (Theorem 1.22). Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter with $\mathbb{p} \in \mathbb{G}$. We have to show that $\underset{\sim}{a}[\mathbb{G}] = \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}[\mathbb{G}]$.

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[32] This means that $A_{\langle \underset{\sim}{d}, \mathbb{r} \rangle}$ is an antichain in $\mathbb{P}$ which is maximal among antichains $A \subseteq \{\mathbb{q} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{r}$ and $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}\}$.

If $b \in \underset{\sim}{a}[\mathbb{G}]$, then there is a $\langle \underset{\sim}{d}, \mathbb{r} \rangle \in \underset{\sim}{a}$ such that $\mathbb{r} \in \mathbb{G}$ and $\underset{\sim}{d}[\mathbb{G}] = b$. Since $\mathbb{p} \in \mathbb{G}$, we have $\mathbb{r} \top_{\mathbb{P}} \mathbb{p}$. Since $A_{\langle \underset{\sim}{d}, \mathbb{r} \rangle}$ is a maximal antichain below $\mathbb{p}$ and $\mathbb{r}$, there is $\mathbb{q} \in A_{\langle \underset{\sim}{d}, \mathbb{r} \rangle} \cap \mathbb{G}$. Since $\langle \underset{\sim}{d}, \mathbb{q} \rangle \in \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}$, it follows that $b = \underset{\sim}{d}[\mathbb{G}] \in \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}[\mathbb{G}]$.

Conversely, if $b \in \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}}[\mathbb{G}]$, there is $\langle \underset{\sim}{d}, \mathbb{r} \rangle \in \underset{\sim}{a}$ and $\mathbb{q} \in A_{\underset{\sim}{d}, \mathbb{r}}$ such that $\mathbb{q} \in \mathbb{G}$ and $\underset{\sim}{d}[\mathbb{G}] = b$. Since $\mathbb{q} \leq_{\mathbb{P}} r$, we have $\mathbb{r} \in \mathbb{G}$ and hence $b = \underset{\sim}{d}[\mathbb{G}] \in \underset{\sim}{a}[\mathbb{G}]$. ❑ (Lemma 3.1)

Lemma 3.2 *Suppose that $\mathbb{P}$ is a poset, $\mathbb{p} \in \mathbb{P}$ and $\underset{\sim}{a}$, $\underset{\sim}{b}$ are $\mathbb{P}$-names such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \varepsilon \underset{\sim}{b}$". Then there is a $\mathbb{P}$-name $\underset{\sim}{a}'$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}' \equiv \underset{\sim}{a}$" and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \not\equiv \emptyset \to \underset{\sim}{a}' \varepsilon \underset{\sim}{b}$". In particular, if $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \not\equiv \emptyset$", then we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}' \equiv \underset{\sim}{a}$" and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a}' \varepsilon \underset{\sim}{b}$".* P-two-a-1

Proof. Let A be an antichain in $\mathbb{P}$ maximal among antichains which are $\subseteq \{\mathbb{q} \in \mathbb{P} : \mathbb{q} \perp_{\mathbb{P}} \mathbb{p}$ and there is a $\mathbb{P}$-name $\underset{\sim}{d}$ such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{d} \varepsilon \underset{\sim}{b}$"$\}$ For each $\mathbb{q} \in A$, let $\underset{\sim}{d}_{\mathbb{q}}$ be a $\mathbb{P}$-name such that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{d}_{\mathbb{q}} \varepsilon \underset{\sim}{b}$".

Then

$$\underset{\sim}{a}' = \underset{\sim}{c}_{\mathbb{p}, \underset{\sim}{a}} \cup \bigcup \{\underset{\sim}{c}_{\mathbb{q}, \underset{\sim}{d}_{\mathbb{q}}} : \mathbb{q} \in A\} \tag{3.4}$$ P-two-a-2

is as desired. ❑ (Lemma 3.2)

In the following, (the underlying set of) a two step iteration $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is defined by (3.7), Lemma 3.2 implies that the variation

$$\mathbb{P} * \underset{\sim}{\mathbb{Q}} = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} \times \mathsf{V}^{\mathbb{P}} : \underset{\sim}{\mathbb{q}} \text{ is a canonical } \mathbb{P}\text{-name and } \mathbb{p} \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{\mathbb{q}} \in \underset{\sim}{\mathbb{Q}}\text{"}\} \tag{3.5}$$ two-a-a-3

would introduce a poset which is forcing equivalent to our $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$. (For the definition of canonical $\mathbb{P}$-name see below).

Note that, for a regular cardinal θ and a poset $\mathbb{P}$, if $\mathbb{P} \in \mathcal{H}(\theta)$, then $\mathbb{P}$ has the θ-cc. Also note that if $\mathbb{P}$ has the θ-cc, then $\Vdash_{\mathbb{P}}$ "θ is a regular cardinal".

Lemma 3.3 *Suppose that θ is a regular cardinal, and $\mathbb{P} \subseteq \mathcal{H}(\theta)$ a θ-cc poset. If $\underset{\sim}{x}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \in \mathcal{H}(\theta)$", then there is a $\mathbb{P}$-name $\underset{\sim}{x}' \in \mathcal{H}(\theta)$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \equiv \underset{\sim}{x}'$".* P-two-a

Proof. By induction on the rank of $\underset{\sim}{x}$. Suppose that $\underset{\sim}{x}$ is a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \in \mathcal{H}(\check{\theta})$" and the Lemma holds for all $\mathbb{P}$-names of rank $< rank(\underset{\sim}{x})$.

Since $\mathbb{P} \in \mathcal{H}(\theta)$ and hence $\mathbb{P}$ has the θ-cc, there are $\mu < \theta$ and $\mathbb{P}$-name $\underset{\sim}{f}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \mu \to \underset{\sim}{x}$ is a surjection".

For each $\alpha < \mu$, let A_α be a maximal antichain $\subseteq \{\mathbb{p} \in \mathbb{P} : \mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) \equiv \underset{\sim}{c}_u$" for some $u \in \underset{\sim}{c}$ with $u = \langle \underset{\sim}{c}_u, \mathbb{p}_u \rangle\}$. For each $\mathbb{p} \in A_\alpha$ let $\underset{\sim}{c}_{\alpha, \mathbb{p}}$ be the name $\underset{\sim}{c}_u$ as above. Since $\underset{\sim}{c}_{\alpha, \mathbb{p}} \in \cdots \in \underset{\sim}{x}$, we have $\underset{\sim}{c}_{\alpha, \mathbb{p}} \in \mathcal{H}(\theta)$ and $rank(\underset{\sim}{c}_{\alpha, \mathbb{p}}) < rank(\underset{\sim}{x})$. By the assumption of the induction, the Lemma holds for $\underset{\sim}{c}_{\alpha, \mathbb{p}}$'s. For $\alpha < \mu$ and $\mathbb{p} \in A_\alpha$, let $\underset{\sim}{c}'_{\alpha, \mathbb{p}}$ be such that $\underset{\sim}{c}'_{\alpha, \mathbb{p}} \in \mathcal{H}(\theta)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{c}'_{\alpha, \mathbb{p}} \equiv \underset{\sim}{c}_{\alpha, \mathbb{p}}$".

Let

(3.6) $\quad \underset{\sim}{x}' = \bigcup\{\{\langle \underset{\sim}{c}'_{\alpha,\mathbb{p}}, \mathbb{p}\rangle : \mathbb{p} \in A_\alpha\} : \alpha < \mu\}.$ two-a-0

Then $\underset{\sim}{x}' \in \mathcal{H}(\theta)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \equiv \underset{\sim}{x}'$". ❑ (Lemma 3.3)

Lemma 3.4 *For a poset $\mathbb{P}$ and a $\mathbb{P}$-name $\underset{\sim}{a}$, there is $\alpha \in \mathsf{On}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \subseteq \mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$". It follows that there is a cardinal θ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \mathcal{H}(\check{\theta})$". Thus, in particular, we also have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \subseteq \mathcal{H}(\check{\theta})$".* P-two-0

Proof. Let $\alpha = rank(\underset{\sim}{a})$. Then, by Lemma 3.4, (3) in [Fuchino 2017], we have $V[\mathbb{G}] \models \underset{\sim}{a} \subseteq \underbrace{V_{\check{\alpha}_{\mathbb{P}}[\mathbb{G}]}}_{=\alpha}$ for any $(V, \mathbb{P})$-generic $\mathbb{G}$. By the 2. Forcing Theorem 1.13, it follows that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \subseteq \mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$".

By Lemma 2.9, (4) in [Fuchino 2017], we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \in \mathcal{H}(\check{\theta})$" if θ is chosen to be $\Vdash_{\mathbb{P}}$ "$|V_{\check{\alpha}}| < \check{\theta}$". ❑ (Lemma 3.4)

Lemma 3.5 *For a poset $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{a}$, there is a set A such that, for any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{b}$, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \varepsilon \underset{\sim}{a}$", then there is a $\mathbb{P}$-name $\underset{\sim}{c} \in A$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{c}$".* P-two-1

Proof. By Lemma 3.4, there is $\alpha \in \mathsf{On}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \subseteq \mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$". Let $\underset{\sim}{v}$ be a $\mathbb{P}$-name for $\mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$.[33]

Let $A = \{\underset{\sim}{c} : \underset{\sim}{c}$ is a nice $\mathbb{P}$-name for a subset of $\underset{\sim}{v}\}$. Note that A is a set by the definition (1.60) of nice $\mathbb{P}$-name of a subset of $\underset{\sim}{a}$.

This A is as desired: if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \varepsilon \underset{\sim}{a}$", then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \subseteq \mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$". Hence, by Lemma 1.45, there is a nice $\mathbb{P}$-name $\underset{\sim}{c}$ of a subset of $\underset{\sim}{v}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{c}$". ❑ (Lemma 3.5) bbd Lemma5.6

For a poset $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{a}$, let $\mu_{\mathbb{P}}(\underset{\sim}{a})$ (or simply $\mu(\underset{\sim}{a})$ if it is clear which $\mathbb{P}$ is meant) be the minimal cardinal μ such that $\mathbb{P} \subseteq \mathcal{H}(\mu)$, $\underset{\sim}{a} \in \mathcal{H}(\mu)$ and, for all $\mathbb{p} \in \mathbb{P}$ and and $\mathbb{P}$-name $\underset{\sim}{b}$, if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \varepsilon \underset{\sim}{a}$" then there is a $\mathbb{P}$-name $\underset{\sim}{c} \in \mathcal{H}(\mu)$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{b} \equiv \underset{\sim}{c}$".

Adopting the terminology of [Cummings 2009], we say that a $\mathbb{P}$-name $\underset{\sim}{a}$ is a *canonical $\mathbb{P}$-name*, if for any further $\mathbb{P}$-name $\underset{\sim}{b}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{b}$", we have $|\,trcl(\underset{\sim}{a})\,| \leq |\,trcl(\underset{\sim}{b})\,|$. Thus, if $\underset{\sim}{x}$ is a canonical $\mathbb{P}$-name and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \varepsilon \underset{\sim}{a}$", then $\underset{\sim}{x} \in \mathcal{H}(\mu(\underset{\sim}{a}))$.

The following Lemma will be often used in connection with iterated forcing discussed in Section 4:

Lemma 3.6 *Suppose that κ is a regular cardinal. If $\mathbb{P} \subseteq \mathcal{H}(\kappa)$ is a poset with the κ-cc and $\underset{\sim}{a} \in \mathcal{H}(\kappa)$ a $\mathbb{P}$-name, any nice $\mathbb{P}$-name of a subset of $\underset{\sim}{a}$ is an element of $\mathcal{H}(\kappa)$. Hence we have $\mu(\underset{\sim}{a}) \leq \kappa$.* P-two-1-0

Proof. Clear by the definition (1.60) of nice $\mathbb{P}$-names of subsets of $\underset{\sim}{a}$. ❑ (Lemma 3.6)

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[33] This means that $\underset{\sim}{v}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{v} \equiv \mathsf{V}_{\check{\alpha}_{\mathbb{P}}}$". Note that there is such a $\mathbb{P}$-name by Maximal Principle (Lemma 1.23).

For a poset $\mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{\mathbb{Q}}$, $\leq_{\underset{\sim}{\mathbb{Q}}}$, $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}}$ such that $\Vdash_{\mathbb{P}}$ " $otr(\underset{\sim}{\mathbb{Q}}, \leq_{\underset{\sim}{\mathbb{Q}}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}})$ is a poset ", let two-2 定義に変更あり．要チェック．
$\mathbb{P} * \underset{\sim}{\mathbb{Q}} = \langle \mathbb{P} * \underset{\sim}{\mathbb{Q}}, \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}, \mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \rangle$ be the poset defined by

$$\mathbb{P} * \underset{\sim}{\mathbb{Q}} = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} \times \mathsf{V}^{\mathbb{P}} : \underset{\sim}{\mathbb{q}} \text{ is a canonical } \mathbb{P}\text{-name and } \Vdash_{\mathbb{P}} \text{"} \underset{\sim}{\mathbb{q}} \in \underset{\sim}{\mathbb{Q}} \text{"}\}, \tag{3.7}$$ two-2

$$\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \;\Leftrightarrow\; \mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0 \text{ and } \mathbb{p}_1 \Vdash_{\mathbb{P}} \text{"} \underset{\sim}{\mathbb{q}}_1 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0 \text{"} \tag{3.8}$$ two-3

for all $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$, and

$$\mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} = \langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle. \tag{3.9}$$ two-4

Here we identify $\underset{\sim}{\mathbb{Q}}$ with $otr(\underset{\sim}{\mathbb{Q}}, \leq_{\underset{\sim}{\mathbb{Q}}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}})$. In particular, with $\mu(\underset{\sim}{\mathbb{Q}})$ we actually mean $\mu(otr(\underset{\sim}{\mathbb{Q}}, \leq_{\underset{\sim}{\mathbb{Q}}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}}))$. Note that, by this convention, we always have $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \in \mathcal{H}(\mu(\underset{\sim}{\mathbb{Q}}))$.

In the following we always assume that $\mathbb{P}$ is a poset and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name of a poset.

Lemma 3.7 (1) $\mathbb{P} * \underset{\sim}{\mathbb{Q}} = \langle \mathbb{P} * \underset{\sim}{\mathbb{Q}}, \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}, \mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \rangle$ *is a poset.* *P-two-2*

(2) *If* $\mathbb{P}$ *is separative and* $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}}$ *is separative*", *then* $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ *is also separative.*

Proof. (1): We check that $\leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ satisfies transitivity. The rest is (also) easy (Exercise).

Suppose that $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$ and $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$. By (3.8), we have $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_1$ and $\mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0$. Since $\leq_{\mathbb{P}}$ is transitive it follows that $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_0$. Also by (3.8), we have $\mathbb{p}_2 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{q}}_2 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_1$ " and $\mathbb{p}_1 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{q}}_1 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0$ ". By Forcing Lemma 1.11, (1) it follows that $\mathbb{p}_2 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{q}}_1 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0$ ". By $\Vdash_{\mathbb{P}}$ " $\leq_{\underset{\sim}{\mathbb{Q}}}$ is transitive " and by Deduction Lemma 1.19, it follows that $\mathbb{p}_2 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{q}}_2 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0$ ". Thus $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$.

(2): Suppose that $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \not\leq_{\mathbb{P}} \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$. This means that

$$\text{either } \mathbb{p}_0 \not\leq_{\mathbb{P}} \mathbb{p}_1 \text{ or } (\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}_1 \text{ but } \mathbb{p}_0 \Vdash_{\mathbb{P}} \text{"} \underset{\sim}{\mathbb{q}}_0 \not\leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_1 \text{"}). \tag{3.10}$$ two-4-a-0

If $\mathbb{p}_0 \not\leq_{\mathbb{P}} \mathbb{p}_1$, then, since $\mathbb{P}$ is separative, there is $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}_0$ such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}_1$. Then $\langle \mathbb{r}, \mathbb{q}_0 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$ but $\langle \mathbb{r}, \underset{\sim}{\mathbb{q}}_0 \rangle \perp_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$.

If $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}_1$, then, by (3.10), we have $\mathbb{p}_0 \Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{q}}_0 \not\leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_1$ ". By the Maximal Principle (Lemma 1.23), and since $\underset{\sim}{\mathbb{Q}}$ is forced to be a separative poset, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{r}}$ such that

$$\mathbb{p}_0 \Vdash_{\mathbb{P}} \text{"} \underset{\sim}{\mathbb{r}} \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0 \text{ but } \underset{\sim}{\mathbb{r}} \perp_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_1 \text{"}. \tag{3.11}$$ two-4-a-1

By Lemma 3.2, we may also assume that $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{Q}}$ ". Further, we may also assume that $\underset{\sim}{\mathbb{r}}$ is a canonical $\mathbb{P}$-name.

Then we have $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{r}} \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$ and $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{r}} \rangle \perp_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$. ❑ (Lemma 3.7)

Note that, in general, $\leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ is not a partial ordering but merely a preorder even if $\mathbb{P}$ is a partial order and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is a partial order".

Lemma 3.8 (1) *$i : \mathbb{P} \to \mathbb{P} * \underset{\sim}{\mathbb{Q}}$; $\mathbb{p} \mapsto \langle \mathbb{p}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle$ is a complete embedding.* *P-two-3*

(2) *$p : \mathbb{P} * \underset{\sim}{\mathbb{Q}} \to \mathbb{P}$; $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \mapsto \mathbb{p}$ is a projection.*

(3) *$p \circ i = \mathrm{id}_{\mathbb{P}}$.*

(4) *If $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}_1$ and $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$, then $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}} \rangle = i(\mathbb{p}_0) \wedge \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}} \rangle$.*

Proof. (1): $i(\mathbb{1}_{\mathbb{P}}) = \langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle = \mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$.

i is order preserving: Suppose $\mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0$. Then $i(\mathbb{p}_1) = \langle \mathbb{p}_1, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle = i(\mathbb{p}_0)$.

i is incompatibility preserving: Suppose that $i(\mathbb{p}_0) = \langle \mathbb{p}_0, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle$ and $i(\mathbb{p}_1) = \langle \mathbb{p}_1, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle$ are compatible, say, $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}} \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle, \langle \mathbb{p}_1, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle$. Then $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_0, \mathbb{p}_1$. Thus $\mathbb{p}_0$ and $\mathbb{p}_1$ are compatible.

To prove that i satisfies (2.34), let $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. we show that $\mathbb{p}$ is a projection of $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$ for i. Suppose $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$. Then $\langle \mathbb{r}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and $\langle \mathbb{r}, \underset{\sim}{\mathbb{q}} \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{r}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle = i(\mathbb{r}), \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$. Thus $i(\mathbb{r})$ and $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$ are compatible.

(2): $p(\mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}) = p(\langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle) = \mathbb{1}_{\mathbb{P}}$.

p is a surjection since, for each $\mathbb{p} \in \mathbb{P}$, we have $p(\mathbb{p}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}}) = \mathbb{p}$.

p is order preserving since, if $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, then $p(\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle) = \mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}_0 = \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$ by (3.8).

To prove that p satisfies (2.44), suppose that $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ and $\mathbb{p} \leq_{\mathbb{P}} p(\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle) = \mathbb{p}_0$. Then we have $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$ and $p(\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle) = \mathbb{p} \leq_{\mathbb{P}} \mathbb{p}$.

(3): Clear by definition.

(4): 要証明!

❑ (Lemma 3.8)

Lemma 3.9 (1) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter and* $\mathbb{H}$ *is a* $(\mathsf{V}[\mathbb{G}], \underset{\sim}{\mathbb{Q}}[\mathbb{G}])$*-generic filter. Then* P-two-4

(3.12) $\mathbb{G} * \mathbb{H} = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}} \ : \ \mathbb{p} \in \mathbb{G},\ \underset{\sim}{\mathbb{q}}[\mathbb{G}] \in \mathbb{H}\}$ two-5

is a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$*-generic filter and* $\mathsf{V}[\mathbb{G} * \mathbb{H}] = (\mathsf{V}[\mathbb{G}])[\mathbb{H}]$.[34]

(2) *Suppose that* $\mathbb{K}$ *is a* $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$*-generic filter,*

(3.13) $\mathbb{G} = \{\mathbb{p} \in \mathbb{P} \ : \ \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{K}$ *for some* $\underset{\sim}{\mathbb{q}}\}$ *and*[35] two-5-0

(3.14) $\mathbb{H} = \{\underset{\sim}{\mathbb{q}}[\mathbb{G}] \ : \ \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{K}$ *for some* $\mathbb{p} \in \mathbb{P}\}$. two-6

Then $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter,* $\mathbb{H}$ *is a* $(\mathsf{V}[\mathbb{G}], \underset{\sim}{\mathbb{Q}}[\mathbb{G}])$*-generic filter and* $\mathbb{G} * \mathbb{H} = \mathbb{K}$*. Hence we also have* $\mathsf{V}[\mathbb{K}] = \mathsf{V}[\mathbb{G}][\mathbb{H}]$.

Proof. (1): Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter and $\mathbb{H}$ a $(\mathsf{V}[\mathbb{G}], \underset{\sim}{\mathbb{Q}}[\mathbb{G}])$-generic filter. We show first that $\mathbb{G} * \mathbb{H}$ defined as in (3.12) is a filter.

$\mathbb{1}_{\mathbb{P}} \in \mathbb{G}$ since $\mathbb{G}$ is a filter. $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \in \mathbb{H}$ since[36] $\mathbb{H}$ is a filter on $\underset{\sim}{\mathbb{Q}}[\mathbb{G}]$. Thus $\langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}} \rangle \in \mathbb{G} * \mathbb{H}$.

$\mathbb{G} * \mathbb{H}$ is upward closed: Suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{G} * \mathbb{H}$, that is,

(3.15) $\mathbb{p} \in \mathbb{G}$ and two-6-0

(3.16) $\underset{\sim}{\mathbb{q}}[\mathbb{G}] \in \mathbb{H}$, two-6-1

and $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, that is,

(3.17) $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_0$ and two-7

(3.18) $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \leq_{\underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{\mathbb{q}}_0$. two-8

We want to show $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{G} * \mathbb{H}$: $\mathbb{p}_0 \in \mathbb{G}$ by (3.15), (3.17) and since $\mathbb{G}$ is a filter. $\underset{\sim}{\mathbb{q}}_0[\mathbb{G}] \in \mathbb{H}$ by (3.15), (3.16), (3.18), Forcing Theorem 1.12, (1) and since $\mathbb{H}$ is a filter (in $V[\mathbb{G}]$).

Suppose that $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{G} * \mathbb{H}$. Then we have $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{G}$. Let $\mathbb{r} \in \mathbb{G}$ be such that

(3.19) $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}_0,\ \mathbb{p}_1$. two-8-0

Since $\underset{\sim}{\mathbb{q}}_0[\mathbb{G}]$, $\underset{\sim}{\mathbb{q}}_1[\mathbb{G}] \in \mathbb{H}$, there is a $\mathbb{q}_2 \in \mathbb{H}$ such that

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[34] In the following we often write $\mathsf{V}[\mathbb{G}][\mathbb{H}]$ for $(\mathsf{V}[\mathbb{G}])[\mathbb{H}]$.

[35] Note that $\mathbb{G} = i^{-1}{}''\mathbb{K}$ for i as in Lemma 3.8, (1).

[36] In the following, $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]}$ and $\leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]}$ denote the interpretation of $\mathbb{P}$-names $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}}$ and $\leq_{\underset{\sim}{\mathbb{Q}}}$ by $\mathbb{G}$ respectively.

(3.20) $\mathbb{q}_2 \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \underset{\sim}{\mathbb{q}}_0[\mathbb{G}], \underset{\sim}{\mathbb{q}}_1[\mathbb{G}]$. two-9

Let $\underset{\sim}{\mathbb{q}}_2 \in (\mu(\underset{\sim}{\mathbb{Q}}))$ be a $\mathbb{P}$-name such that

(3.21) $\underset{\sim}{\mathbb{q}}_2[\mathbb{G}] = \mathbb{q}_2$. two-10

Thus we have $\underset{\sim}{\mathbb{q}}_2[\mathbb{G}] \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \underset{\sim}{\mathbb{q}}_0[\mathbb{G}], \underset{\sim}{\mathbb{q}}_1[\mathbb{G}]$. By Forcing Theorem 1.12, (2), there is $\mathbb{p}_2 \in \mathbb{G}$ such that

(3.22) $\mathbb{p}_2 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}_2 \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0, \underset{\sim}{\mathbb{q}}_1$". two-11

Without loss of generality, we may assume that

(3.23) $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{r}$. two-12

By (3.22) we have $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \in \mathbb{G} * \mathbb{H}$ by (3.21) and $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle, \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$ by (3.19), (3.23) and (3.22).

To prove that $\mathbb{G} * \mathbb{H}$ is a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter, suppose that D is a dense subset of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ in V and we show that $\mathbb{G} * \mathbb{H} \cap D \neq \emptyset$.

Let

(3.24) $D_1 = \{\underset{\sim}{\mathbb{q}}[\mathbb{G}] : \text{there is } \mathbb{p} \in \mathbb{G} \text{ such that } \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in D\}$. two-13

Claim 3.9.1 *$D_1 \in \mathsf{V}[\mathbb{G}]$ and D_1 a dense subset of $\underset{\sim}{\mathbb{Q}}[\mathbb{G}]$.* *Cl-P-two-4-0*

$\vdash$ $D_1 \in \mathsf{V}[\mathbb{G}]$ is clear from the definition (3.24) of D_1.

To prove that it is dense in $\underset{\sim}{\mathbb{Q}}[\mathbb{G}]$, let $\mathbb{q} \in \underset{\sim}{\mathbb{Q}}[\mathbb{G}]$. Then there is $\langle \underset{\sim}{\mathbb{r}}, \mathbb{p}_0 \rangle \in \underset{\sim}{\mathbb{Q}}$ such that $\mathbb{q} = \underset{\sim}{\mathbb{r}}[\mathbb{G}]$ and $\mathbb{p}_0 \in \mathbb{G}$. Note that $\mathbb{p}_0 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \ \varepsilon \ \underset{\sim}{\mathbb{Q}}$" and hence $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$.

Let $D_0 = \{\mathbb{p} \in \mathbb{P} : \text{there is } \underset{\sim}{\mathbb{s}} \text{ such that } \langle \mathbb{p}, \underset{\sim}{\mathbb{s}} \rangle \in D \text{ and } \langle \mathbb{p}, \underset{\sim}{\mathbb{s}} \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{r}} \rangle\}$. Then D_0 is dense below $\mathbb{p}_0$ in $\mathbb{P}$. Since $\mathbb{p}_0 \in \mathbb{G}$. It follows that there is $\mathbb{p}_1 \in \mathbb{G} \cap D_0$. Let $\underset{\sim}{\mathbb{s}}$ be such that

(3.25) $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{s}} \rangle \in D$ and two-14

(3.26) $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{s}} \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{r}} \rangle$. two-15

Then $\underset{\sim}{\mathbb{s}}[\mathbb{G}] \in D_1$ by (3.25) and by the definition (3.24) of D_1. $\underset{\sim}{\mathbb{s}}[\mathbb{G}] \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \underset{\sim}{\mathbb{r}}[\mathbb{G}] = \mathbb{q}$ by $\mathbb{p}_1 \in \mathbb{G}$ and by (3.26). $\dashv$ (Claim 3.9.1)

By Claim 3.9.1, there is $\mathbb{q} \in \mathbb{H} \cap D_1$. By the definition (3.24) of D_1, there is $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in D$ such that $\mathbb{p} \in \mathbb{G}$ and $\underset{\sim}{\mathbb{q}}[\mathbb{G}] = \mathbb{q} \in \mathbb{H}$. Thus $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{G} * \mathbb{H} \cap D$.

Since $\mathbb{G}, \mathbb{H} \in \mathsf{V}[\mathbb{G}][\mathbb{H}]$, we have $\mathbb{G} * \mathbb{H} \in \mathsf{V}[\mathbb{G}][\mathbb{H}]$. It follows that $\mathsf{V}[\mathbb{G} * \mathbb{H}] \subseteq \mathsf{V}[\mathbb{G}][\mathbb{H}]$ by Lemma 1.9.

On the other hand, Since $\mathbb{G} = i^{-1}{}''\mathbb{G} * \mathbb{H}$ and $\mathbb{H} = \{\underset{\sim}{\mathbb{q}}[\mathbb{G}] : \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{G} * \mathbb{H}\}$, We have $\mathbb{G}, \mathbb{H} \in \mathsf{V}[\mathbb{G} * \mathbb{H}]$. Similarly as above, it follows that $\mathsf{V}[\mathbb{G}][\mathbb{H}] \subseteq \mathsf{V}[\mathbb{G} * \mathbb{H}]$.

(2): $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter by Lemma 2.22, (1) and since $i : \mathbb{P} \to \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is a complete embedding by Lemma 3.8.

$\mathbb{H}$ is upward closed: Suppose that $\mathbb{q} \in \mathbb{H}$ and $\mathbb{q} \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \mathbb{q}_0$. We want to show that $\mathbb{q}_0 \in \mathbb{H}$. Let $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{K}$ be such that $\mathbb{q} = \underset{\sim}{\mathbb{q}}[\mathbb{G}]$. Since $\mathbb{q}_0 \in \underset{\sim}{\mathbb{Q}}[\mathbb{G}]$, there is $\langle \underset{\sim}{\mathbb{q}}_0, \mathbb{p}_1 \rangle \in \underset{\sim}{\mathbb{Q}}$ such that $\mathbb{p}_1 \in \mathbb{G}$ and $\underset{\sim}{\mathbb{q}}_0[\mathbb{G}] = \mathbb{q}_0$. Since we have $\underset{\sim}{\mathbb{q}}[\mathbb{G}] = \mathbb{q} \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \mathbb{q}_0 = \underset{\sim}{\mathbb{q}}_0[\mathbb{G}]$, there is $\mathbb{p}_1 \in \mathbb{G}$ such that $\mathbb{p}_1 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}_0$". Without loss of generality, we may assume $\mathbb{p}_1 \leq_{\mathbb{P}} \mathbb{p}$. By the definition of $\mathbb{G}$ there is $\underset{\sim}{\mathbb{q}}_1$ such that $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{K}$. Since $\mathbb{K}$ is a filter, there is $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \in \mathbb{K}$ such that $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$. We have $\mathbb{p}_2 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}_2 \leq_{\underset{\sim}{\mathbb{Q}}}$ ". Thus $\mathbb{K} \ni \langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_0 \rangle$. It follows that $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{K}$ and $\mathbb{q}_0 = \underset{\sim}{\mathbb{q}}_0[\mathbb{G}] \in \mathbb{H}$.

Suppose now that $\mathbb{q}_0, \mathbb{q}_1 \in \mathbb{H}$. We want to show that there is $\mathbb{q}_2 \in \mathbb{H}$ with $\mathbb{q}_2 \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \mathbb{q}_0$, $\mathbb{q}_1$. Let $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_\rangle \in \mathbb{K}$ be such that $\mathbb{q}_0 = \underset{\sim}{\mathbb{q}}_0[\mathbb{G}]$ and $\mathbb{q}_1 = \underset{\sim}{\mathbb{q}}_1[\mathbb{G}]$. Let $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \in \mathbb{K}$ be such that $\langle \mathbb{p}_2, \underset{\sim}{\mathbb{q}}_2 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle$. Then we have

$$\mathbb{H} \ni \underset{\sim}{\mathbb{q}}_2[\mathbb{G}] \leq_{\underset{\sim}{\mathbb{Q}}[\mathbb{G}]} \underbrace{\underset{\sim}{\mathbb{q}}_0[\mathbb{G}]}_{= \mathbb{q}_0}, \underbrace{\underset{\sim}{\mathbb{q}}_1[\mathbb{G}]}_{= \mathbb{q}_1}. \tag{3.27}$$

To prove that $\mathbb{H}$ is a $(\mathsf{V}[\mathbb{G}], \underset{\sim}{\mathbb{Q}}[\mathbb{G}])$-generic filter, let D be a dense subset of $\underset{\sim}{\mathbb{Q}}[\mathbb{G}]$ in $\mathsf{V}[\mathbb{G}]$. Let $\underset{\sim}{D}$ be a $\mathbb{P}$-name of D. We can choose $\underset{\sim}{D}$ such that

$$\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{D} \text{ is a dense subset of } \underset{\sim}{\mathbb{Q}} \text{”}. \tag{3.28}$$ two-15-0

Let

$$D^* = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}} \ : \ \mathbb{p} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}} \ \varepsilon \ \underset{\sim}{D} \text{”}\}. \tag{3.29}$$ two-16

Claim 3.9.2 *$D^* \in \mathsf{V}$ and D^* is dense in $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$.*

$\vdash$ $D^* \in \mathsf{V}$ is seen in the definition (3.29) of D^*.

Suppose $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. By (3.28), we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is a dense subset of $\underset{\sim}{\mathbb{Q}}$". Hence there are $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}' \in \mathcal{H}(\mu(\underset{\sim}{\mathbb{Q}}))$ such that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}' \leq_{\underset{\sim}{\mathbb{Q}}} \underset{\sim}{\mathbb{q}}$ and $\underset{\sim}{\mathbb{q}}' \in \underset{\sim}{D}$". Then $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}' \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$ and $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}' \rangle \in D^*$. $\dashv$ (Claim 3.9.2)

Let $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{K} \cap D^*$. Then $\mathbb{p} \in \mathbb{G}$ by the definition (3.13) of $\mathbb{G}$ and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \ \varepsilon \ \underset{\sim}{D}$" by the definition (3.29) of D^*. Thus $\underset{\sim}{\mathbb{q}}[\mathbb{G}] \in \underset{\sim}{D}[\mathbb{G}] = D$ and $\underset{\sim}{\mathbb{q}}[\mathbb{G}] \in \mathbb{H}$ by the definition (3.14) of $\mathbb{H}$.

By the definition (3.12) of $\mathbb{G} * \mathbb{H}$, it is clear that $\mathbb{K} \subseteq \mathbb{G} * \mathbb{H}$. Since $\mathbb{G} * \mathbb{H}$ is a filter by (2) and since $\mathbb{K}$ is a maximal filter (see Lemma 1.1). It follows that $\mathbb{K} = \mathbb{G} * \mathbb{H}$.

❑ (Lemma 3.9)

In the following, we shall call a poset $\mathbb{P}$ *conjunctive* if for any two compatible $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$ there is $\mathbb{p}_2 \leq_{\mathbb{P}} \mathbb{p}_0$, $\mathbb{p}_1$ which is maximal (with respect to $\leq_{\mathbb{P}}$) with this property. We call such a $\mathbb{p}_2$ a conjunction of $\mathbb{p}_0$ and $\mathbb{p}_1$ and denote it by $\mathbb{p}_0 \wedge \mathbb{p}_1$. Though $\mathbb{p}_2$ might

not be unique in general, it is enough to take one of such $\mathbb{p}_2$ as $\mathbb{p}_0 \wedge \mathbb{p}_1$ in the following except that we always assume that $\mathbb{p} \wedge \mathbb{1}_{\mathbb{P}} = \mathbb{1}_{\mathbb{P}} \wedge \mathbb{p} = \mathbb{p}$ holds for all $\mathbb{p} \in \mathbb{P}$.

$\mathbb{P}$ is conjunctive if $\mathbb{P}$ is the partial ordering consisting of all the positive elements of a Boolean algebra $\mathbb{B}$.

Exercise 3.10 (1) $\mathrm{Fn}(X, Y, <\mu)$ *is conjunctive for any sets X, Y and infinite cardinal μ.* *Ex-two-0*

(2) *If $\mathbb{P}$ is a conjunctive poset and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is a conjunctive poset", then $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is also conjunctive.* ❑

Suppose that $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ be the standard $\mathbb{P}$-name of the generic filter and let $\underset{\sim}{\mathbb{R}}$ be the $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathbb{R}} = \{q \in \check{\mathbb{Q}}_{\mathbb{P}} : q \text{ is compatible with all } r \,\varepsilon\, \check{i}_{\mathbb{P}}{}''\underset{\sim}{\mathbb{G}}_{\mathbb{P}} \text{ in } \check{\mathbb{Q}}_{\mathbb{P}}\}\text{”}. \tag{3.30}$$ two-17

Let

$$\mathbb{1}_{\underset{\sim}{\mathbb{R}}} = \surd_{\mathbb{P}}(\mathbb{1}_{\mathbb{Q}}). \tag{3.31}$$ two-18

Let $\leq_{\underset{\sim}{\mathbb{R}}}$ be the $\mathbb{P}$-name of the relation $\leq_{\mathbb{Q}}$ restricted to $\underset{\sim}{\mathbb{R}}$. Then we have $\Vdash_{\mathbb{P}}$ "$\mathbb{1}_{\underset{\sim}{\mathbb{R}}}$ is a maximal element of $\underset{\sim}{\mathbb{R}}$ with respect to $\leq_{\underset{\sim}{\mathbb{R}}}$" and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{R}} = \langle \underset{\sim}{\mathbb{R}}, \leq_{\underset{\sim}{\mathbb{R}}}, \mathbb{1}_{\underset{\sim}{\mathbb{R}}} \rangle$ is a poset".

Lemma 3.11 *Suppose that $\mathbb{P}$ is a conjunctive sub-Boolean poset, $\mathbb{Q}$ a conjunctive poset and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Then, for $\underset{\sim}{\mathbb{R}}$ as defined above, $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ is forcing equivalent to $\mathbb{Q}$. Furthermore, $RO(\mathbb{Q})$ is isomorphic to $RO(\mathbb{P} * \underset{\sim}{\mathbb{R}})$ over i and the complete embedding of $\mathbb{P}$ in $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ (as defined in Lemma 3.8, (1)).* *P-two-5*

Proof. Note that i is 1-1 and order reversing by Lemma 2.19, (1). Also note that, by footnote (31) , i preserves $\wedge$.

Let

$$\mathbb{D} = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}} : \text{there is } \mathbb{q} \in \mathbb{Q} \text{ such that } \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}\text{”}\}. \tag{3.32}$$ two-19

By (i) of the following Claim 3.11.2, $\mathbb{D}$ with the preordering $\leq_{\mathbb{P} * \underset{\sim}{\mathbb{R}}}$ restricted to $\mathbb{D}$ is a poset with the maximal element $\mathbb{1}_{\mathbb{D}} = \langle \mathbb{1}_{\mathbb{P}}, \surd_{\mathbb{P}}(\mathbb{1}_{\mathbb{Q}}) \rangle$.

Claim 3.11.1 (i) *If $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{D}$ and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$" then $\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$.* *Cl-P-two-5-a*

(ii) *If $\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$ then $i(\mathbb{p}) \top_{\mathbb{P}} \mathbb{q}$.*

(iii) *$\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$ and $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ imply $\langle \mathbb{p}', \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$.*

(iv) *$\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$ if and only if $\mathbb{p}$ is a projection of $\mathbb{q}$ for i.*

$\vdash$ (i): Clear by the definitions of $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ and $\mathbb{D}$.

(ii): Suppose that $\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$. Then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\forall x\, (x \varepsilon \underset{\sim}{\mathbb{G}} \to \check{i}_{\mathbb{P}}(x) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}})$". Since $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{\mathbb{p}}_{\mathbb{P}} \varepsilon \underset{\sim}{\mathbb{G}}$", it follows that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{i}_{\mathbb{P}}(\check{\mathbb{p}}_{\mathbb{P}}) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}}$" and hence $i(\mathbb{p}) \top_{\mathbb{Q}} \mathbb{q}$.

(iii): By Forcing Lemma 1.11, (1).

(iv): Suppose that $\langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \in \mathbb{D}$. Then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\forall x\, (x \varepsilon \underset{\sim}{\mathbb{G}} \to i(x) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}})$". Thus, for all $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$, we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\forall x\, (x \varepsilon \underset{\sim}{\mathbb{G}} \to i(x) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}})$". Since $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\mathbb{r} \varepsilon \underset{\sim}{\mathbb{G}}$", we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\check{i}_{\mathbb{P}}(\check{\mathbb{r}}_{\mathbb{P}}) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}}$" and hence $i(\mathbb{r}) \top_{\mathbb{Q}} \mathbb{q}$.

Suppose now that $\mathbb{p}$ is a projection of $\mathbb{q}$ for i and suppose for contradiction that $\mathbb{p} \nVdash_{\mathbb{P}}$ "$\forall x\, (x \varepsilon \underset{\sim}{\mathbb{G}} \to i(x) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}})$". Then there are $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p}'' \in \mathbb{P}$ such that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\check{\mathbb{p}}''_{\mathbb{P}} \varepsilon \underset{\sim}{\mathbb{G}} \wedge i(\check{\mathbb{p}}''_{\mathbb{P}}) \perp_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}}$". $\mathbb{p}'$ and $\mathbb{p}''$ are compatible in $\mathbb{P}$ (this is proved just as (ii)). Let $\mathbb{p}^* \leq_{\mathbb{P}} \mathbb{p}', \mathbb{p}''$. Then we have $\mathbb{p}^* \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p}^* \Vdash_{\mathbb{P}}$ "$i(\check{\mathbb{p}}^*_{\mathbb{P}}) \perp_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}}$". It follows $i(\mathbb{p}^*) \perp_{\mathbb{Q}} \mathbb{q}$. This is a contradiction to the choice of $\mathbb{p}$ as a projection of $\mathbb{q}$ for i.

$\dashv$ (Claim 3.11.1)

The next claim finishes the proof by Lemma 2.19, (3):

Claim 3.11.2 (i) $\langle \mathbb{p}, \mathbb{1}_{\underset{\sim}{\mathbb{R}}} \rangle \in \mathbb{D}$ *for all* $\mathbb{p} \in \mathbb{P}$. *In particular,* $\mathbb{1}_{\mathbb{P} * \underset{\sim}{\mathbb{R}}} \in \mathbb{D}$. *Cl-P-two-5-0*

(ii) $\mathbb{D}$ *(as a subset of* $\mathbb{P} * \underset{\sim}{\mathbb{R}}$*) is dense in* $\mathbb{P} * \underset{\sim}{\mathbb{R}}$.

(iii) *Let*

(3.33) $\quad i^\dagger : \mathbb{P} \to \mathbb{D};\ \mathbb{p} \mapsto \langle \mathbb{p}, \mathbb{1}_{\underset{\sim}{\mathbb{R}}} \rangle$. *two-19-0*

Then $i^\dagger$ *is a complete embedding of* $\mathbb{P}$ *into* $\mathbb{D}$.

(iv) $id : \mathbb{D} \to \mathbb{P} * \underset{\sim}{\mathbb{R}}$ *is a dense embedding.*

(v) *Let*

(3.34) $\quad i^* : \mathbb{D} \to \mathbb{Q};\ \langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle \mapsto i(\mathbb{p}) \wedge \mathbb{q}$ *where* $\mathbb{q} \in \mathbb{Q}$ *is such that* $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$". *two-20*

Then i *is a dense embedding from* $\mathbb{D}$ *into* $\mathbb{Q}$.

$\vdash$ (i): Clear by the definition (3.32) of $\mathbb{D}$.

(ii): Suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$. Then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{R}}$". In particular, $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \check{\mathbb{Q}}_{\mathbb{P}}$". By Lemma 1.25, there are $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{q} \in \mathbb{Q}$ such that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$". We have bbd Lemma3.5
$\langle \mathbb{p}', \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{D}$ and $\langle \mathbb{p}', \underset{\sim}{\mathbb{r}} \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle$.

(iii): By Lemma 3.8, $i^\dagger$ as a mapping from $\mathbb{P}$ to $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ is a complete embedding. By (3.11.2), (2), it follows that $i^\dagger$ as a mapping from $\mathbb{P}$ to $\mathbb{D}$ is also a complete embedding.

(iv) follows from (i), (ii) and Exercise 2.1.

(v): We first show that i^* is well-defined. Suppose $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}} \rangle \in \mathbb{D}$. We have to show that $i(\mathbb{p}) \wedge \mathbb{q} \in \mathbb{P}$ for $\mathbb{q} \in \mathbb{Q}$ as in (3.34). Since $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{R}}$" and $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{G}}$", we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$i(\mathbb{p}) \top_{\check{\mathbb{Q}}_{\mathbb{P}}} \underset{\sim}{\mathbb{r}}$". Thus $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$" implies $i(\mathbb{p}) \top_{\mathbb{P}} \mathbb{q}$.

$i^* \models$ (2.3): $i^*(\mathbb{1}_{\mathbb{D}}) = i^*(\langle \mathbb{1}_{\mathbb{P}}, \surd_{\mathbb{P}}(\mathbb{1}_{\mathbb{Q}})\rangle) = i(\mathbb{1}_{\mathbb{P}}) \wedge \mathbb{1}_{\mathbb{Q}} = \mathbb{1}_{\mathbb{Q}} \wedge \mathbb{1}_{\mathbb{Q}} = \mathbb{1}_{\mathbb{Q}}$.

i^* is order preserving: Suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle \in \mathbb{D}$ are such that

$$\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle \tag{3.35}$$ two-21

and let $\mathbb{q}'$, $\mathbb{q}' \in \mathbb{Q}$ be such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$" and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}}' \equiv \check{\mathbb{q}}'_{\mathbb{P}}$". Since (3.35) means $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}}' \leq_{\check{\mathbb{Q}}_{\mathbb{P}}} \underset{\sim}{\mathbb{r}}$", it follows that $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$. Thus $i(\mathbb{p}') \wedge \mathbb{q}' \leq_{\mathbb{Q}} i(\mathbb{p})$, $\mathbb{q}$ and hence $i^*(\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle) = i(\mathbb{p}') \wedge \mathbb{q}' \leq_{\mathbb{Q}} i(\mathbb{p}) \wedge \mathbb{q} = i^*(\langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle)$.

i^* is incompatibility preserving: Suppose $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle \in \mathbb{D}$. Let $\mathbb{q}$, $\mathbb{q}' \in \mathbb{Q}$ be such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$" and $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}}' \equiv \check{\mathbb{q}}'_{\mathbb{P}}$". Suppose further that $i^*(\langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle) = i(\mathbb{p}) \wedge \mathbb{q}$ and $i^*(\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle) = i(\mathbb{p}') \wedge \mathbb{q}'$ are compatible. Then there is $\mathbb{q}^* \in \mathbb{Q}$ such that $\mathbb{q}^* = i(\mathbb{p}) \wedge \mathbb{q} \wedge i(\mathbb{p}') \wedge \mathbb{q}'$.

We want to show that $\langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle$ and $\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle$ are compatible in $\mathbb{D}$.

Since i is incompatibility preserving and $i(\mathbb{p})$ and $i(\mathbb{p}')$ are compatible in $\mathbb{Q}$, $\mathbb{p}$ and $\mathbb{p}'$ are compatible. Let $\mathbb{p}'' = \mathbb{p} \wedge \mathbb{p}'$. Then, by footnote (31), $i(\mathbb{p}'') = \mathbb{p} \wedge \mathbb{p}'$. Let $\mathbb{p}^*$ be a projection of $\mathbb{q}^*$. Since $i(\mathbb{p}'') \geq_{\mathbb{Q}} q^*$, $i(\mathbb{p}^*) \top_{\mathbb{Q}} i(\mathbb{p}'')$. Since i is incompatibility preserving we have $\mathbb{p}^* \top_{\mathbb{P}} \mathbb{p}''$. Let $\mathbb{p}^{**} = \mathbb{p}^* \wedge \mathbb{p}''$. Then $\mathbb{p}^{**} \leq_{\mathbb{P}} \mathbb{p}^*$, $\mathbb{p}''$ and $\mathbb{p}^{**}$ is a projection of $\mathbb{q} \wedge \mathbb{q}'$ ($\geq_{\mathbb{Q}} q^*$). Thus $\langle \mathbb{p}, \surd_{\mathbb{P}}(\mathbb{q} \wedge \mathbb{q}')\rangle \in \mathbb{D}$ by Claim 3.11.1, (iv) and $\langle \mathbb{p}, \surd_{\mathbb{P}}(\mathbb{q} \wedge \mathbb{q}')\rangle \leq_{\mathbb{D}} \langle \mathbb{p}, \underset{\sim}{\mathbb{r}}\rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{r}}'\rangle$. ⊣ (Claim 3.11.2)

Using the mappings introduced above and Lemma 2.19, (3), we obtain the following commutative diagram:

$$\begin{tikzcd}
RO(\mathbb{P} * \underset{\sim}{\mathbb{R}}) \arrow[r, "\cong"] & RO(\mathbb{Q}) \\
\mathbb{P} * \underset{\sim}{\mathbb{R}} \arrow[u] \arrow[r, phantom, "\circlearrowright"] & \mathbb{Q} \arrow[u] \\
\mathbb{D} \arrow[u, "id"] \arrow[r, "id"] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & \mathbb{D} \arrow[u, "i^*"'] \\
\mathbb{P} \arrow[u, "i^\dagger"] \arrow[r, "id"'] & \mathbb{P} \arrow[u, "i^\dagger"']
\end{tikzcd} \tag{3.36}$$ two-22

Again by Lemma 2.19, (3), the diagram above can be further lifted to

$$\begin{tikzcd}
RO(\mathbb{P} * \underset{\sim}{\mathbb{R}}) \arrow[r, "\cong"] & RO(\mathbb{Q}) \\
RO(\mathbb{P}) \arrow[u, "\tilde{i}^\dagger"] \arrow[r, "id"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & RO(\mathbb{P}) \arrow[u, "\tilde{i}"']
\end{tikzcd} \tag{3.37}$$ two-23

where $\tilde{i}^\dagger$ and $\tilde{i}$ are the unique complete embeddings extending $i^\dagger$ and i respectively.

❑ (Lemma 3.11)

Now, we can give the promised proof of Lemma 2.22, (2).[37] We restate the assertion in the following as Corollary 3.12.

Corollary 3.12 *Suppose that $i : \mathbb{P} \overset{\lessdot}{\to} \mathbb{Q}$ for posets $\mathbb{P}$ and $\mathbb{Q}$. Then, for any $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there is a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $i''\mathbb{G} \subseteq \mathbb{H}$.* P-two-5-a-0

Proof. Suppose that i, $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{G}$ are as above. Let $\mathbb{Q}^*$ be a cBa poset with a dense embedding $d : \mathbb{Q} \to \mathbb{Q}^*$ (such a pair $(\mathbb{Q}^*, d)$ exists by Lemma 2.9). Let $\mathbb{B}$ be the complete Boolean algebra with $\mathbb{Q}^* = \mathbb{B}^+$.

Let $\mathbb{P}' = d \circ i''\mathbb{P}$. Since $d_0 = d \circ i : \mathbb{P} \to \mathbb{Q}^*$ is a complete embedding, we have

$$\mathbb{P}' \lessdot \mathbb{Q}^*. \tag{3.38}$$ two-23-a-a-0

Let $\mathbb{P}^*$ be the positive elements of the complete subalgebra of the complete Boolean algebra $\mathbb{B}$. By (3.38), $\mathbb{P}'$ is a dense poset of $\mathbb{P}^*$ and $d_0 : \mathbb{P} \to \mathbb{P}^*$ is a dense embedding.

By Lemma 3.11, $\mathbb{Q}^*$ is forcing equivalent to $\mathbb{P}^* * \underset{\sim}{\mathbb{R}}$ over $\mathbb{P}^*$ for some $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{R}}$ of a poset. Since $\mathbb{Q}^*$ is a cBa poset, there is a dense embedding e of $\mathbb{P}^* * \underset{\sim}{\mathbb{R}}$ to $\mathbb{Q}^*$ over $\mathbb{P}^*$.

By Lemma 2.3, (1), $d_0''\mathbb{G}$ generates a $(\mathsf{V}, \mathbb{P}^*)$-generic filter $\mathbb{G}^*$. Let $\mathbb{K}^*$ be a $(\mathsf{V}[\mathbb{G}^*], \underset{\sim}{\mathbb{R}}[\mathbb{G}^*])$-generic filter. By Lemma 3.9, (1), $\mathbb{H}^* = \mathbb{G}^* * \mathbb{K}^*$ is a $(\mathsf{V}, \mathbb{P}^* * \underset{\sim}{\mathbb{R}})$-generic filter.

Let $\mathbb{H}'$ be $(\mathsf{V}, \mathbb{Q}^*)$-generic filter corresponding to $\mathbb{H}^*$ via e and $\mathbb{H} = d^{-1}{}''\mathbb{H}'$. Then $\mathbb{H}$ is a $(\mathsf{V}, \mathbb{Q})$-generic filter by Lemma 2.3, (2). and $i''\mathbb{G} \subseteq \mathbb{H}$ by the construction. Thus, this $\mathbb{H}$ is as desired. ❑ (Corollary 3.12)

Remember that, for a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$, the equivalence relation $\sim_{\mathbb{P}}$ on $\mathbb{P}$ is defined by:

$$\mathbb{p} \sim_{\mathbb{P}} \mathbb{q} \;\Leftrightarrow\; \mathbb{p} \leq_{\mathbb{P}} \mathbb{q} \text{ and } \mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \tag{2.14}$$ forcing-eq-5-a

for $\mathbb{p}$, $\mathbb{q} \in \mathbb{P}$.

For a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and $\mathbb{p}, \mathbb{q}, \mathbb{r} \in \mathbb{P}$, we write "$\mathbb{p} = \mathbb{q} \wedge_{\mathbb{P}} \mathbb{r}$", or simply "$\mathbb{p} = \mathbb{q} \wedge \mathbb{r}$", to denote the assertion that

$$\text{“}\mathbb{p} \leq_{\mathbb{P}} \mathbb{q}, \mathbb{r} \text{ and, for any } \mathbb{p}' \leq_{\mathbb{P}} \mathbb{q}, \mathbb{r}, \text{ we have } \mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}\text{”}. \tag{3.39}$$ land-notation

Since $\leq_{\mathbb{P}}$ is merely a pre-ordering in general, such $\mathbb{p}$ need not to be unique. "$\mathbb{q} \wedge \mathbb{r}$ exists" is then the statement that "there is a $\mathbb{p} \in \mathbb{P}$ with $\mathbb{p} = \mathbb{q} \wedge \mathbb{r}$".

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[37] Lemma 2.22, (2) has been used already several times in earlier places. However no vicious circle occurs in its usage: The earlier places where this Lemma is applied, it was in connection with some argument with Forcing Theorem. In contrast, the proofs of Lemma 2.22 and the following Corollary 3.12 as well as in all other earlier Lemmas used in them, every thing is done without any semantic arguments deploying generic extensions.

Lemma 3.13 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets.* (1) *If $f : \mathbb{P} \to \mathbb{Q}$ is order-preserving then for any $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}$ with $\mathbb{p} \sim_{\mathbb{P}} \mathbb{p}'$ we have $f(\mathbb{p}) \sim_{\mathbb{Q}} f(\mathbb{p}')$.* P-two-5-a-1

(2) *If $f : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and $\mathbb{Q}$ is separative, then, for $\mathbb{p}$, $\mathbb{p}'$, $\mathbb{p}'' \in \mathbb{P}$, $\mathbb{p} = \mathbb{p}' \wedge_{\mathbb{P}} \mathbb{p}''$ implies $\mathbb{p} = \mathbb{p}' \wedge_{\mathbb{Q}} \mathbb{p}''$.*

(3) *For $\mathbb{p}$, $\mathbb{p}'$, $\mathbb{p}''$, $\mathbb{p}_0$, $\mathbb{p}_0'$, $\mathbb{p}_0'' \in \mathbb{P}$, If $\mathbb{p}' \sim_{\mathbb{P}} \mathbb{p}_0'$, $\mathbb{p}'' \sim_{\mathbb{P}} \mathbb{p}_0''$, $\mathbb{p} = \mathbb{p}' \wedge \mathbb{p}''$ and $\mathbb{p}_0 = \mathbb{p}_0' \wedge \mathbb{p}_0''$, then we have $\mathbb{p} \sim_{\mathbb{P}} \mathbb{p}_0$.*

Proof. (1): Suppose that $\mathbb{p} \sim_{\mathbb{P}} \mathbb{p}'$. This means that $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}'$ and $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$. Since f is order-preserving, if follows that $f(\mathbb{p}) \leq_{\mathbb{Q}} f(\mathbb{p}')$ and $f(\mathbb{p}') \leq_{\mathbb{Q}} f(\mathbb{p})$. That is, $f(\mathbb{p}) \sim_{\mathbb{Q}} f(\mathbb{p}')$.

(2): If $\mathbb{p} = \mathbb{p}' \wedge_{\mathbb{P}} \mathbb{p}''$, then $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}'$ and $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}''$. Since f is order-preserving, it follows that $f(\mathbb{p}) \leq_{\mathbb{Q}} f(\mathbb{p}')$, $f(\mathbb{p}'')$. Suppose that $f(\mathbb{p}) = f(\mathbb{p}') \wedge_{\mathbb{Q}} f(\mathbb{p}'')$ does not hold. This means that there is a $\mathbb{q} \leq_{\mathbb{Q}} f(\mathbb{p}')$, $f(\mathbb{p}'')$ such that $\mathbb{q} \not\leq_{\mathbb{Q}} f(\mathbb{p})$. Since $\mathbb{Q}$ is separative, there is $\mathbb{r} \leq_{\mathbb{Q}} \mathbb{q}$ such that

$$\mathbb{r} \perp_{\mathbb{Q}} f(\mathbb{p}). \tag{3.40}$$ two-23-a-a-0-0

Let

$$D \subseteq \{\mathbb{r} \in \mathbb{P} : \mathbb{r} \perp_{\mathbb{P}} \mathbb{p}' \text{ or } \mathbb{r} \perp_{\mathbb{P}} \mathbb{p}' \text{ or } \mathbb{r} \leq \mathbb{p}\} \tag{3.41}$$ two-23-a-a-1

be an maximal anti-chain. Since the right side of $\subseteq$ in (3.41) is dense in $\mathbb{P}$, D is a maximal anti-chain. Since f is a complete embedding, it follows that $f''D$ is a maximal anti-chain in $\mathbb{Q}$. Thus there is $\mathbb{d} \in D$ such that

$$f(\mathbb{d}) \top_{\mathbb{Q}} \mathbb{r}. \tag{3.42}$$ two-23-a-a-2

Since $\mathbb{d} \in D$, we have either $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}'$ or $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}''$ or $\mathbb{d} \leq_{\mathbb{P}} \mathbb{p}$. $\mathbb{d} \leq_{\mathbb{P}} \mathbb{p}$ is impossible: Since, if so, we would have $f(\mathbb{d}) \leq_{\mathbb{Q}} f(\mathbb{p})$. By (3.40), $f(\mathbb{d}) \perp_{\mathbb{Q}} \mathbb{r}$. This is a contradiction to the choice (3.42) of $\mathbb{d}$.

Thus, we have either $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}'$ or $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}''$. Suppose $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}'$. Then, since f is incompatibility preserving, we would have $f(\mathbb{d}) \perp_{\mathbb{Q}} f(\mathbb{p}')$. Since we have $\mathbb{r} \leq_{\mathbb{Q}} \mathbb{q} \leq_{\mathbb{Q}} f(\mathbb{p}')$, it follows that $f(\mathbb{d}) \perp_{\mathbb{Q}} \mathbb{r}$. This is a contradiction to (3.42). Similarly, $\mathbb{d} \perp_{\mathbb{P}} \mathbb{p}''$ leads also to a contradiction.

(3): Since $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}_0'$ and $\mathbb{p}'' \leq_{\mathbb{P}} \mathbb{p}_0''$, we have $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_0'$ and $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_0''$. It follows that $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}_0$. Similarly we also have $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}$. ❑ (Lemma 3.13)

Note that the assumption on $\mathbb{P}$ and $\mathbb{Q}$ in the following Lemma corresponds to the situation with a poset constructed by iteration discussed in Section 4 over an intermediate stage of the iteration.

Lemma 3.14 (Factor Lemma) *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets with mappings $i : \mathbb{P} \to \mathbb{Q}$ and $p : \mathbb{Q} \to \mathbb{P}$ such that* Factor Lemma

(3.43) $i : \mathbb{P} \to \mathbb{Q}$ *is a complete embedding;* two-23-a

(3.44) $p : \mathbb{Q} \to \mathbb{P}$ *is a projection;* two-23-a-0

(3.45) $p \circ i \sim_{\mathbb{P}} id_{\mathbb{P}}$;[38] *and* two-23-a-1

(3.46) *For any $\mathbb{p} \in \mathbb{P}$ and $\mathbb{q} \in \mathbb{Q}$, if $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$ then $i(\mathbb{p}) \wedge \mathbb{q}$ exists.* two-23-a-2

Let $\underset{\sim}{\mathbb{R}}$ be a $\mathbb{P}$-name of poset such that

(3.47) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{R}} \equiv$ *the sub-poset of $\check{\mathbb{Q}}$ with the underlying set:* two-23-a-3
$\{\mathbb{q} \in \check{\mathbb{Q}} : \mathbb{q} \top_{\check{\mathbb{Q}}} r$ *for all* $r \in i''\underset{\sim}{\mathbb{G}}_{\mathbb{P}}\}$".

Then $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ is forcing equivalent to $\mathbb{Q}$.

Proof. Let Scan Dec 21, 2019 18.59.pdf

(3.48) $D = \{\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$: two-23-a-4

$\mathbb{p} \in \mathbb{P}$, $\underset{\sim}{\mathbb{q}}$ is a canonical $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \varepsilon \underset{\sim}{\mathbb{R}}$", and
there is $\mathbb{q}_0 \in \mathbb{Q}$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \equiv \check{\mathbb{q}}_0$", and $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q}_0)\}$.

Claim 3.14.1 $\mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{R}}} \in D$ *and D is a dense subset of $\mathbb{P} * \underset{\sim}{\mathbb{R}}$.* Cl-two-0

$\vdash$ $\mathbb{1}_{\mathbb{P}*\underset{\sim}{\mathbb{R}}} \in D$ immediately follows from the definition of D.

To prove the denseness of D, suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$. Then $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \varepsilon \underset{\sim}{\mathbb{R}}$". Let $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ be such that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} = \check{\mathbb{q}}$" for some $\mathbb{q} \in \mathbb{Q}$.

Since $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\mathbb{p}' \varepsilon \underset{\sim}{\mathbb{G}}_{\mathbb{P}}$", we have $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \top_{\check{\mathbb{Q}}} \check{i}(\check{\mathbb{p}}')$". It follows that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\check{\mathbb{q}} \top_{\check{\mathbb{Q}}} \check{i}(\check{\mathbb{p}}')$". Thus $\mathbb{q} \top_{\mathbb{Q}} i(\mathbb{p}')$ and $p(\mathbb{q}) \top_{\mathbb{P}} p(i(\mathbb{p}'))$. Since $p(i(\mathbb{p}')) \sim_{\mathbb{P}} \mathbb{p}'$ by (3.45), it follows that $p(\mathbb{q}) \top_{\mathbb{P}} \mathbb{p}'$.

Let $\mathbb{p}'' \in \mathbb{P}$ be such that $\mathbb{p}'' \leq_{\mathbb{P}} p(\mathbb{q}), \mathbb{p}'$. Then $\langle \mathbb{p}'', \underset{\sim}{\mathbb{q}} \rangle \in \mathcal{D}$ and $\langle \mathbb{p}'', \underset{\sim}{\mathbb{q}} \rangle \leq_{\mathbb{P}*\underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$.

$\dashv$ (Claim 3.14.1)

Claim 3.14.2 *For $\mathbb{p} \in \mathbb{P}$ and $\mathbb{q} \in \mathbb{Q}$, if $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$, then $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{\mathbb{q}} \varepsilon \underset{\sim}{\mathbb{R}}$".* Cl-two-1

Proof. Suppose that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{G}}_{\mathbb{P}}$". We have to show that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$i(\underset{\sim}{\mathbb{r}}) \top_{\mathbb{Q}} \check{\mathbb{q}}$". Suppose that $\mathbb{p}_0 \leq_{\mathbb{P}} \mathbb{p}$. Since $\mathbb{p}_0 \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{r}} \varepsilon \underset{\sim}{\mathbb{G}}_{\mathbb{P}}$", there is $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}_0$ such that $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\check{\mathbb{p}}' \leq_{\mathbb{P}} \underset{\sim}{\mathbb{r}}$". Since $\mathbb{p}' \leq \mathbb{p} \leq p(\mathbb{q})$, $i(\mathbb{p}') \wedge \mathbb{q}$ exists by (3.46). Thus $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$i(\underset{\sim}{\mathbb{r}}) \top_{\check{\mathbb{Q}}} \check{\mathbb{q}}$". Since $\mathbb{p}_0$ was arbitrary, it follows that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$i(\underset{\sim}{\mathbb{r}}) \top_{\mathbb{Q}} \check{\mathbb{q}}$". Thus $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{\mathbb{q}} \top_{\check{\mathbb{Q}}} r$ for all $r \varepsilon \check{i}''\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$"or $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{\mathbb{q}} \varepsilon \underset{\sim}{\mathbb{R}}$".

$\dashv$ (Claim 3.14.2)

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[38] I.e., $p(i(\mathbb{p})) \sim_{\mathbb{P}} \mathbb{p}$ for all $\mathbb{p} \in \mathbb{P}$.

Let

$$(3.49)\quad f : D \to \mathbb{Q};\ \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \ \mapsto\ \mathbb{r} \ \text{ such that } \mathbb{r} = i(\mathbb{p}) \wedge \mathbb{q} \quad \text{where } \mathbb{q}_0 \in \mathbb{Q} \text{ is such that } \mathbb{p} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}} \equiv \check{\mathbb{q}}_0 \text{”}.$$ two-23-a-5

Note that, since $\mathbb{Q}$ is not necessarily a partial ordering but merely a pre-ordering in general, the condition $\mathbb{r}$ is the definition of f above may not be unique. Thus we have to choose one of such $\mathbb{r}$ for each $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in D$. In case of $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle = \mathbb{1}_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$, we add the condition in the definition of f that we choose $\mathbb{r}$ to be $\mathbb{1}_{\mathbb{Q}}$.

By Claim 3.14.1, the following Claim finishes the proof of the Lemma.

Claim 3.14.3 *$f : D \to \mathbb{Q}$ is a dense embedding.* *Cl-two-2*

$\vdash$ $f(\mathbb{1}_{\mathbb{P} * \underset{\sim}{\mathbb{R}}}) = \mathbb{1}_{\mathbb{Q}}$ follows immediately from the definition of f.

f is order preserving: Suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle \in D$ with $\mathbb{q}$, $\mathbb{q}' \in \mathbb{Q}$ are such that $\mathbb{p} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}_0 \equiv \mathbb{q} \text{”}$, $\mathbb{p}' \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}'_0 \equiv \mathbb{q}' \text{”}$, and $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle$. Then $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$. Hence, $f(\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle) = \underbrace{i(\mathbb{p}') \wedge \mathbb{q}' \leq_{\mathbb{Q}} i(\mathbb{p}) \wedge \mathbb{q}}_{(39)} = f(\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle)$.

f is incompatibility preserving: Suppose again that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle \in D$ with $\mathbb{q}$, $\mathbb{q}' \in \mathbb{Q}$ satisfying $\mathbb{p} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}_0 \equiv \mathbb{q} \text{”}$ and $\mathbb{p}' \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}'_0 \equiv \mathbb{q}' \text{”}$. Assume $f(\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle) \top_{\mathbb{Q}} f(\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle)$. This is equivalent to $i(\mathbb{p}') \wedge \mathbb{q}' \top_{\mathbb{Q}} i(\mathbb{p}) \wedge \mathbb{q}$.

Let $\mathbb{q}^* \leq_{\mathbb{Q}} i(\mathbb{p}') \wedge \mathbb{q}'$, $i(\mathbb{p}) \wedge \mathbb{q}$ and $\mathbb{p}^* = p(\mathbb{q}^*)$. Then $\mathbb{p}^* \leq_{\mathbb{P}} p \circ i(\mathbb{p}') \sim_{\mathbb{P}} \mathbb{p}'$, $p \circ i(\mathbb{p}) \sim_{\mathbb{P}} \mathbb{p}$. By Claim 3.14.2, $\mathbb{p}^* \Vdash_{\mathbb{P}} \text{“} \check{\mathbb{q}}^* \ \varepsilon\ \underset{\sim}{\mathbb{R}} \text{”}$. Let $\underset{\sim}{\mathbb{q}}^{**}$ be a *canonical* $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}^{**} \text{”} \ \varepsilon\ \underset{\sim}{\mathbb{R}}$ and $\mathbb{p}^* \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}}^{**} \equiv \check{\mathbb{q}} \text{”}$ (see Lemma 3.2). Then $\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^{**} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$ and $\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^{**} \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle$, $\langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle$. Since D is dense in $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ by Claim 3.14.1, if follows that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle \top_D \langle \mathbb{p}', \underset{\sim}{\mathbb{q}}'_0 \rangle$.

$f''D$ is dense in $\mathbb{Q}$: For $\mathbb{q} \in \mathbb{Q}$, let $\mathbb{p} = p(\mathbb{q})$. By Claim 3.14.2, there is a canonical $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}$ such that $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}} \ \varepsilon\ \underset{\sim}{\mathbb{R}} \text{”}$ and $\mathbb{p} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{q}} \equiv \check{\mathbb{q}} \text{”}$. We have $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{R}}$. Let $\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^*_0 \rangle \in D$ be such that $\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^*_0 \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{R}}} \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle$. Let $\mathbb{q}^* \in \mathbb{Q}$ be such that $\mathbb{p}^* \Vdash_{\mathbb{P}} \text{“} \check{\mathbb{q}}^* \equiv \underset{\sim}{\mathbb{q}}^*_0 \text{”}$. Then we have $f(\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^*_0 \rangle) = i(\mathbb{p}^*) \wedge \mathbb{q}^* \leq_{\mathbb{Q}} \mathbb{q}^* \leq_{\mathbb{Q}} \mathbb{q}$.

$\dashv$ (Claim 3.14.3)

❑ (Lemma 3.14)

If $\mathbb{P}$ and $\mathbb{Q}$ are Boolean posets, $\underset{\sim}{\mathbb{R}}$ in (3.30) is the set of representatives of positive elements of the quotient algebra $\mathbb{B}_{\mathbb{Q}}/\mathbb{H}$ where $\mathbb{H}$ is the filter on $\mathbb{B}_{\mathbb{Q}}$ generated from $i''\mathbb{G}$ for a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$. Hence $\underset{\sim}{\mathbb{R}}$ is forced to be forcing equivalent to $(\mathbb{B}_{\mathbb{Q}}/\mathbb{H})^+$.

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(39) This means, for arbitrary $\mathbb{r}$ and $\mathbb{r}'$ with $\mathbb{r} = i(\mathbb{p}) \wedge \mathbb{q}$ and $\mathbb{r}' = i(\mathbb{p}') \wedge \mathbb{q}'$, we have $\mathbb{r}' \leq_{\mathbb{Q}} \mathbb{r}$. This can be proved as follows: If $\mathbb{r}' = i(\mathbb{p}') \wedge \mathbb{q}'$ then $\mathbb{r}' \leq_{\mathbb{Q}} i(\mathbb{p}') \leq i(\mathbb{p})$. and $\mathbb{r}' \leq_{\mathbb{Q}} \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$. Thus, if $\mathbb{r}' = i(\mathbb{p}') \wedge \mathbb{q}'$ then $\mathbb{r}' \leq_{\mathbb{Q}} \mathbb{r}$.

More in details, we have the following: Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are Boolean posets and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let $\underset{\sim}{\mathbb{R}}$ be a $\mathbb{P}$-name as in (3.30).

Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter and $\mathbb{H}$ be the filter on $\mathbb{B}_{\mathbb{Q}}$ generated from $i''\mathbb{G}$ (in $\mathsf{V}[\mathbb{G}]$). Let $\sim_{\mathbb{H}}$ be the equivalence relation on $\mathbb{B}_{\mathbb{Q}}$ defined by

$$\mathbb{c} \sim_{\mathbb{H}} \mathbb{c}' \;\Leftrightarrow\; \neg(\mathbb{c} \triangle \mathbb{c}') \in \mathbb{H} \tag{3.50}$$ two-23-0

for $\mathbb{c}$, $\mathbb{c}' \in \mathbb{B}_{\mathbb{Q}}$.[40] Since $\mathbb{H}$ is a filter, $\sim_{\mathbb{H}}$ is an equivalence relation congruent with Boolean operations on $\mathbb{B}_{\mathbb{Q}}$ and hence we can build the quotient Boolean algebra $\mathbb{B}_{\mathbb{Q}}/\sim_{\mathbb{H}}$. For $\mathbb{c} \in \mathbb{B}_{\mathbb{Q}}$, let $[\mathbb{c}]_{\sim_{\mathbb{H}}}$ denote the equivalence class of $\mathbb{c}$ with respect to the equivalence relation $\sim_{\mathbb{H}}$.

Lemma 3.15 (1) *For* $\mathbb{q} \in \mathbb{Q}$, $\mathbb{q} \in \underset{\sim}{\mathbb{R}}[\mathbb{G}] \;\Leftrightarrow\; [\mathbb{q}]_{\sim_{\mathbb{H}}} \neq [\mathbb{0}_{\mathbb{B}_{\mathbb{Q}}}]_{\sim_{\mathbb{H}}}$. *P-two-5-0*

(2) *The mapping* $i : \underset{\sim}{\mathbb{R}}[\mathbb{G}] \to (\mathbb{B}_{\mathbb{Q}}/\sim_{\mathbb{H}})^+$; $\mathbb{p} \mapsto [\mathbb{p}]_{\sim_{\mathbb{H}}}$ *is a complete embedding and hence* $\underset{\sim}{\mathbb{R}}[\mathbb{G}] \approx (\mathbb{B}_{\mathbb{Q}}/\sim_{\mathbb{H}})^+$.

Proof. (1): For $\mathbb{q} \in \underset{\sim}{\mathbb{R}}[\mathbb{G}]$

$$\begin{aligned}
[\mathbb{q}]_{\sim_{\mathbb{H}}} = [\mathbb{0}_{\mathbb{B}_{\mathbb{Q}}}]_{\sim_{\mathbb{H}}} \;&\Leftrightarrow\; \neg(\mathbb{c} \triangle \mathbb{1}_{\mathbb{B}_{\mathbb{Q}}}) \in \mathbb{H} \\
&\Leftrightarrow\; \neg\mathbb{c} \in \mathbb{H} \\
&\Leftrightarrow\; i(\mathbb{p}) \leq_{\mathbb{Q}} \neg\mathbb{c} \text{ for some } \mathbb{p} \in \mathbb{G} \\
&\Leftrightarrow\; \mathbb{c} \notin \underset{\sim}{\mathbb{R}}[\mathbb{G}].
\end{aligned}$$

(2): i is well-defined by (1). The rest is clear. ❑ (Lemma 3.15)

Lemma 3.16 *Suppose that* κ *is a regular uncountable cardinal. If a poset* $\mathbb{P}$ *is* κ*-cc*[41] *and* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ *is* $\check{\kappa}$*-cc*" *then* $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ *is* κ*-cc.* *P-two-6* *bbd Lemma 11.7*

Proof. Otherwise there would be a pairwise incompatible sequence $\langle \mathbb{p}_\alpha, \underset{\sim}{\mathbb{q}}_\alpha \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}$, for $\alpha < \kappa$. Without loss of generality, we may assume $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}_\alpha \in \underset{\sim}{\mathbb{Q}}$"[42].

Let $\underset{\sim}{Z}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{Z} = \{\alpha \in \kappa : \check{\mathbb{p}}_\alpha \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}}\}$" where $\underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ is the standard $\mathbb{P}$-name of the generic filter.

We have $\Vdash_{\mathbb{P}}$ "$\{\check{\mathbb{p}}_\alpha : \alpha \in \underset{\sim}{Z}\}$ is pairwise compatible in $\mathbb{P}$". It follows that $\Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{\mathbb{q}}_\alpha : \alpha \in \underset{\sim}{Z}\}$ is pairwise incompatible".[43] By the assumption on $\underset{\sim}{\mathbb{Q}}$, it follows that $\Vdash_{\mathbb{P}}$ "$|\{\underset{\sim}{\mathbb{q}}_\alpha : \alpha \in \underset{\sim}{Z}\}| < \check{\kappa}$".

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[40] $\mathbb{c} \triangle \mathbb{c}' = ((\mathbb{c} \wedge \neg\mathbb{c}') \vee (\neg\mathbb{c} \wedge \mathbb{c}'))$.

[41] Note that, by Lemma 1.28, we know that $\Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is a regular cardinal" follows from this assumption.

[42] Otherwise we may replace $\underset{\sim}{\mathbb{q}}_\alpha$ with a $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}'_\alpha \in \mathcal{H}(\mu(\underset{\sim}{\mathbb{Q}}))$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}'_\alpha \in \underset{\sim}{\mathbb{Q}}$" and $\mathbb{p}_\alpha \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}_\alpha \equiv \underset{\sim}{\mathbb{q}}'_\alpha$".

[43] This statement can be formulated e.g. by using the $\mathbb{P}$-name $\{\langle odp(\check{\alpha}, \underset{\sim}{\mathbb{q}}_\alpha), \mathbb{1}_{\mathbb{P}}\rangle : \alpha < \kappa\}$.

Let $A \subseteq \mathbb{P}$ be a maximal antichain such that, for each $\mathbb{p} \in A$, there is $\lambda_{\mathbb{p}} < \kappa$ such that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$|\{\underset{\sim}{\mathbb{q}}_\alpha : \alpha \in \underset{\sim}{Z}\}| \leq \check{\lambda}_{\mathbb{p}}$".

Since $\mathbb{P}$ has the κ-cc, $|A| < \kappa$. Since κ is regular it follows that $\lambda = \max\{\lambda_{\mathbb{p}} : \mathbb{p} \in A\} < \kappa$. Since A is a maximal antichain we obtain

$$\Vdash_{\mathbb{P}} \text{"}|\{\underset{\sim}{\mathbb{q}}_\alpha : \alpha \in \underset{\sim}{Z}\}| \leq \check{\lambda} < \check{\kappa}\text{"}. \tag{3.51}$$ two-24

By Corollary 1.27 there is $S \in [\kappa]^{<\kappa}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{Z} \subseteq \check{S}$". Let $\alpha^* \in \kappa \setminus S$. Since $\mathbb{p}_{\alpha^*} \Vdash_{\mathbb{P}}$ "$\mathbb{p}_{\check{\alpha}^*} \in \underset{\sim}{\mathbb{G}}$", we have $\mathbb{p}_{\alpha^*} \Vdash_{\mathbb{P}}$ "$\check{\alpha}^* \in \underset{\sim}{Z}$". This is a contradiction. ❑ (Lemma 3.16)

Lemma 3.17 *Suppose that $\mathbb{P}$ is a poset and $\underset{\sim}{\mathbb{Q}}_0$, $\underset{\sim}{\mathbb{Q}}_1$ are $\mathbb{P}$-names of posets.* P-two-7

(1) *If $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i} : \underset{\sim}{\mathbb{Q}}_0 \to \underset{\sim}{\mathbb{Q}}_1$ is a complete embedding", then there is a complete embedding $i : \mathbb{P} * \underset{\sim}{\mathbb{Q}}_0 \to \mathbb{P} * \underset{\sim}{\mathbb{Q}}_1$ such that, for any $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{\mathbb{q}}_0$, $\underset{\sim}{\mathbb{q}}_1$ with $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P}_0$ and $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{P}_1$,*

$$\text{if } \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle = i(\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle), \text{ then } \mathbb{p}_1 = \mathbb{p}_0 \text{ and } \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{i}(\underset{\sim}{\mathbb{q}}_0) \equiv \underset{\sim}{\mathbb{q}}_1\text{"}. \tag{3.52}$$ two-25

(2) *If $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i} : \underset{\sim}{\mathbb{Q}}_0 \to \underset{\sim}{\mathbb{Q}}_1$ is a dense embedding", then there is a dense embedding $i : \mathbb{P} * \underset{\sim}{\mathbb{Q}}_0 \to \mathbb{P} * \underset{\sim}{\mathbb{Q}}_1$ such that, for any $\mathbb{p}_0$, $\mathbb{p}_1 \in \mathbb{P}$ and $\mathbb{P}$-names $\underset{\sim}{\mathbb{q}}_0$, $\underset{\sim}{\mathbb{q}}_1$ with $\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P}_0$ and $\langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{P}_1$,*

$$\text{if } \langle \mathbb{p}_1, \underset{\sim}{\mathbb{q}}_1 \rangle = i(\langle \mathbb{p}_0, \underset{\sim}{\mathbb{q}}_0 \rangle), \text{ then } \mathbb{p}_1 = \mathbb{p}_0 \text{ and } \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{i}(\underset{\sim}{\mathbb{q}}_0) \equiv \underset{\sim}{\mathbb{q}}_1\text{"}. \tag{3.53}$$ two-26

Proof. (1): For each $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}_0$, we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i}(\underset{\sim}{\mathbb{q}}_0) \in \underset{\sim}{\mathbb{Q}}_1$". By Maximal Principle (Lemma 1.23), there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i}(\underset{\sim}{\mathbb{q}}_0) \equiv \underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0}$". We may assume that $\underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0}$ is canonical. Also, since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i}(\mathbb{1}_{\underset{\sim}{\mathbb{Q}}_0}) \equiv \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_1}$", we may choose $\underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0}$ in case of $\underset{\sim}{\mathbb{q}}_0 = \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_0}$ to be $\mathbb{1}_{\underset{\sim}{\mathbb{Q}}_1}$.

Let $i : \mathbb{P} * \underset{\sim}{\mathbb{Q}}_0 \to \mathbb{P} * \underset{\sim}{\mathbb{Q}}_1$; $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle \mapsto \langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0} \rangle$. i is well-defined by the choice of $\underset{\sim}{\mathbb{q}}_{\underset{\sim}{\mathbb{q}}_0}$. Clearly this i satisfies (3.52). It is also easy to check that i is a complete embedding. We show here that i satisfies (2.34) and leave the rest as an exercise of the reader.

Suppose that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_1 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}_1$. Since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{i} : \underset{\sim}{\mathbb{Q}}_0 \to \underset{\sim}{\mathbb{Q}}_1$ is a complete embedding", there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}_0$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}_0 \in \underset{\sim}{\mathbb{Q}}_0 \wedge \forall r \in \underset{\sim}{\mathbb{Q}}_0 \, (r \leq_{\underset{\sim}{\mathbb{Q}}_0} \underset{\sim}{\mathbb{q}}_0 \to \underset{\sim}{i}(r) \top_{\underset{\sim}{\mathbb{Q}}_1} \underset{\sim}{\mathbb{q}}_1)$" by Maximal Principle. We may assume that $\underset{\sim}{\mathbb{q}}_0 \in \mathcal{H}(\mu(\underset{\sim}{\mathbb{Q}}_0))$. Thus $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle \in \mathbb{P} * \underset{\sim}{\mathbb{Q}}_0$ and it can be checked similarly to the argument above (applying again the Maximal Principle) that $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_0 \rangle$ is a projection of $\langle \mathbb{p}, \underset{\sim}{\mathbb{q}}_1 \rangle$ for i.

(2): Let i be defined just as in (1). It is then easy to show that i is a dense embedding. ❑ (Lemma 3.17)

Lemma 3.18 *Suppose that $i : \mathbb{P}_0 \to \mathbb{P}_1$ is a complete embedding, $\tilde{i} : \mathsf{V}^{\mathbb{P}_0} \to \mathsf{V}^{\mathbb{P}_1}$ is defined as in (2.51), and $\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}_0$-name of a poset.* (1) *The mapping* P-two-8

$$i^* : \mathbb{P}_0 * \underset{\sim}{\mathbb{Q}} \to \mathbb{P}_1 * \tilde{i}(\underset{\sim}{\mathbb{Q}}) : \langle \mathbb{p}, \underset{\sim}{\mathbb{q}} \rangle \mapsto \langle i(\mathbb{p}), \tilde{i}(\underset{\sim}{\mathbb{q}}) \rangle \tag{3.54}$$ two-27

is well-defined and it is a complete embedding.

(2) *If i in* (1) *is a dense embedding, then i^* is also a dense embedding.*

Proof. By Lemma 2.24. ❑ (Lemma 3.18)

3.2 Product of posets

Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets. We define the product $\mathbb{P} \times \mathbb{Q} = (\mathbb{P} \times \mathbb{Q}, \leq_{\mathbb{P}\times\mathbb{Q}}, \mathbb{1}_{\mathbb{P}\times\mathbb{Q}})$ of $\mathbb{P}$ and $\mathbb{Q}$ as follows: prod

$$\mathbb{P} \times \mathbb{Q} = \{\langle \mathbb{p}, \mathbb{q} \rangle \,:\, \mathbb{p} \in \mathbb{P},\, \mathbb{q} \in \mathbb{Q}\}; \tag{3.55}$$ prod-0

For $\langle \mathbb{p}, \mathbb{q} \rangle$, $\langle \mathbb{p}', \mathbb{q}' \rangle \in \mathbb{P} \times \mathbb{Q}$.

$$\langle \mathbb{p}', \mathbb{q}' \rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle \mathbb{p}, \mathbb{q} \rangle \;\Leftrightarrow\; \mathbb{p}' \leq_{\mathbb{P}} \mathbb{p} \text{ and } \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}. \tag{3.56}$$ prod-1

$$\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} = \langle \mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\mathbb{Q}} \rangle. \tag{3.57}$$ prod-2

Lemma 3.19 *For posets $\mathbb{P}$, $\mathbb{Q}$, the canonical embedding* P-prod-0

$$i : \mathbb{P} \times \mathbb{Q} \to \mathbb{P} * \check{\mathbb{Q}}_{\mathbb{P}};\ \langle \mathbb{p}, \mathbb{q} \rangle \mapsto \langle \mathbb{p}, \check{\mathbb{q}}_{\mathbb{P}} \rangle \tag{3.58}$$ prod-3

is a dense embedding.
In particular we have $\mathbb{P} \times \mathbb{Q} \approx \mathbb{P} * \check{\mathbb{Q}}_{\mathbb{P}}$.

Proof. $i(\mathbb{1}_{\mathbb{P}\times\mathbb{Q}}) = \langle \mathbb{1}_{\mathbb{P}}, \check{\mathbb{1}}_{\mathbb{Q}} \rangle = \mathbb{1}_{\mathbb{P}*\check{\mathbb{Q}}_{\mathbb{P}}}$.

i is order preserving: Suppose $\langle \mathbb{p}', \mathbb{q}' \rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle \mathbb{p}, \mathbb{q} \rangle$. This means $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ $\Leftrightarrow$ $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{p}' \Vdash_{\mathbb{P}}$ "$\check{\mathbb{q}}'_{\mathbb{P}} \leq_{\surd_{\mathbb{P}}(\mathbb{Q})} \check{\mathbb{q}} - \mathbb{P}$" $\Leftrightarrow$ $i(\langle \mathbb{p}', \mathbb{q}' \rangle) \leq_{\mathbb{P}*\check{\mathbb{Q}}_{\mathbb{P}}} i(\langle \mathbb{p}, \mathbb{q} \rangle)$.

i is incompatibility preserving: Suppose $\langle \mathbb{p}, \mathbb{q} \rangle$, $\langle \mathbb{p}', \mathbb{q}' \rangle \in \mathbb{P}\times\mathbb{Q}$ and $i(\langle \mathbb{p}, \mathbb{q} \rangle)$, $i(\langle \mathbb{p}', \mathbb{q}' \rangle)$ are compatible in $\mathbb{P} * \check{\mathbb{Q}}_{\mathbb{P}}$. Say, $\langle \mathbb{p}'', \underset{\sim}{\mathbb{q}} \rangle \leq_{\mathbb{P}*\check{\mathbb{Q}}_{\mathbb{P}}} i(\langle \mathbb{p}, \mathbb{q} \rangle)$, $i(\langle \mathbb{p}', \mathbb{q}' \rangle)$. Since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \in \check{\mathbb{Q}}_{\mathbb{P}}$" by Lemma 1.25, there are $\mathbb{p}^* \in \mathbb{P}$ and $\mathbb{q}^* \in \mathbb{Q}$ such that $\mathbb{p}^* \leq_{\mathbb{P}} \mathbb{p}''$ and $\mathbb{p}^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \equiv \check{\mathbb{q}}_{\mathbb{P}}$". Since $\mathbb{p}'' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}} \leq_{\check{\mathbb{Q}}_{\mathbb{P}}} \check{\mathbb{q}}_{\mathbb{P}}, \check{\mathbb{q}}'_{\mathbb{P}}$", it follows that $\mathbb{q}^* \leq_{\mathbb{Q}} \mathbb{p}, \mathbb{p}'$. Thus $\langle \mathbb{p}^*, \mathbb{q}^* \rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle \mathbb{q}, \mathbb{q} \rangle$, $\langle \mathbb{p}', \mathbb{q}' \rangle$ and $\langle \mathbb{q}, \mathbb{q} \rangle$ and $\langle \mathbb{p}', \mathbb{q}' \rangle$ are compatible in $\mathbb{P} \times \mathbb{Q}$.

$i''\mathbb{P} \times \mathbb{Q}$ is dense in $\mathbb{P} * \check{\mathbb{Q}}_{\mathbb{P}}$: The construction of $\langle \mathbb{p}^*, \mathbb{q}^* \rangle$ in the proof of (3) shows that $i''\mathbb{P} \times \mathbb{Q}$ is dense in $\mathbb{P} * \check{\mathbb{Q}}_{\mathbb{P}}$. ❑ (Lemma 3.19)

Cohen reals adds a Suslin tree
(Jech Millennium book p.563)

Lemma 3.20 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are posets.* Hydrolemma 11.6

(0) $\mathbb{P} \times \mathbb{Q}$ *is isomorphic to* $\mathbb{Q} \times \mathbb{P}$.

(1) *If* $\mathbb{K}$ *is a* $(\mathsf{V}, \mathbb{P} \times \mathbb{Q})$*-generic filter, then* $\mathbb{G} = \{\mathbb{p} \in \mathbb{P} : \langle \mathbb{p}, \mathbb{q} \rangle \in \mathbb{K}$ for some $\mathbb{q} \in \mathbb{Q}\}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter and* $\mathbb{H} = \{\mathbb{p} \in \mathbb{P} : \langle \mathbb{p}, \mathbb{q} \rangle \in \mathbb{K}$ for some $\mathbb{p} \in \mathbb{P}\}$ *is a* $(\mathsf{V}[\mathbb{G}], \mathbb{Q})$*-generic filter. By* (0)*, the assertion is also true if we exchange* $(\mathbb{P}, \mathbb{G})$ *with* $(\mathbb{Q}, \mathbb{H})$ *while we keep* $\mathbb{P} \times \mathbb{Q}$.

(2) *If* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter and* $\mathbb{H}$ *a* $(\mathsf{V}[\mathbb{G}], \mathbb{Q})$*-generic filter, then* $\mathbb{G} \times \mathbb{H} = \{\langle \mathbb{p}, \mathbb{q} \rangle : \mathbb{p} \in \mathbb{G}, \mathbb{q} \in \mathbb{H}\}$ *is a* $(\mathsf{V}, \mathbb{P} \times \mathbb{Q})$*-generic filter. By* (0)*, the assertion is also true if we exchange* $(\mathbb{P}, \mathbb{G})$ *with* $(\mathbb{Q}, \mathbb{H})$ *while we keep* $\mathbb{P} \times \mathbb{Q}$.

Proof. By Lemma 3.19 and Lemma 3.9. (Exercise: Give a direct proof of Lemma 3.20.)

❑ (Lemma 3.20)

3.3 Solovay's theorems on characterizations of forcing equivalence and forcing subordinance

Lemma 3.21 (1) *Suppose that $\mathbb{P}$ is a poset and $\mathbb{G}$ is a maximal filter on $\mathbb{P}$. For any* solovay P-forcing-eq-sub-0 $\mathbb{p}^* \in \mathbb{G}$, $\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$ *is a maximal filter on* $\mathbb{P} \downarrow \mathbb{p}^*$ *and* $\mathbb{G}$ *coincides with the filter on* $\mathbb{P}$ *generated from* $\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$.

(2) *For* $\mathbb{p}^* \in \mathbb{P}$*, if* $\mathbb{G}^*$ *is a maximal filter in* $\mathbb{P} \downarrow \mathbb{p}^*$*, then the filter* $\mathbb{G}$ *on* $\mathbb{P}$ *generated from* $\mathbb{G}^*$ *is a maximal filter.*

(3) *Suppose that $\mathbb{P}$ is a poset and $\mathbb{G}$ a* $(\mathsf{V}, \mathbb{P})$*-generic filter. Then, for all* $\mathbb{p}^* \in \mathbb{G}$, $\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$ *is a* $(\mathsf{V}, \mathbb{P} \downarrow \mathbb{p}^*)$*-generic filter and* $V[\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)] = V[\mathbb{G}]$.

(4) *For* $\mathbb{p}^* \in \mathbb{P}$*, if* $\mathbb{G}^*$ *is a* $(\mathsf{V}, \mathbb{P} \downarrow \mathbb{p}^*)$*-generic filter then the filter* $\mathbb{G}$ *on* $\mathbb{P}$ *generated from* $\mathbb{G}^*$ *is a* $(\mathsf{V}, \mathbb{P})$*-generic filter and* $\mathsf{V}[\mathbb{G}^*] = \mathsf{V}[\mathbb{G}]$.

Proof. (1): Let $\mathbb{G}^* = \mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$. Then the filter $\mathbb{G}_0$ on $\mathbb{P}$ generated from $\mathbb{G}^*$ can be represented as

$$\mathbb{G}_0 = \{\mathbb{p} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \text{ for some } \mathbb{q} \in \mathbb{G}^*\}. \tag{3.59}$$ forcing-eq-sub-a-a-0

Clearly $\mathbb{G}_0 \subseteq \mathbb{G}$.

$\mathbb{G}_0 = \mathbb{G}$: Suppose $\mathbb{r} \in \mathbb{G}$. Then there is $\mathbb{s} \in \mathbb{G}$ such that $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}, \mathbb{p}^*$. It follows that $\mathbb{s} \in \mathbb{G}^*$ and $\mathbb{s}$ witnesses that $\mathbb{r} \in \mathbb{G}_0$.

If $\mathbb{G}^*$ were not a maximal filter on $\mathbb{P} \downarrow \mathbb{p}^*$, then there would be a filter $\mathbb{H}^*$ on $\mathbb{P} \downarrow \mathbb{p}^*$ such that $\mathbb{G}^* \subsetneqq \mathbb{H}^*$. It follows that $\mathbb{G} = \mathbb{G}_0 \subsetneqq \{\mathbb{p} \in \mathbb{P} : \mathbb{r} \leq_{\mathbb{P}} \mathbb{p}$ for some $\mathbb{r} \in \mathbb{H}^*\}$. This is a contradiction to the maximality of $\mathbb{G}$.

(2): Let $\mathbb{G}_0$ be defined by (3.59). If $\mathbb{G}_0$ is not a maximal filter, there is a filter $\mathbb{H}$ on $\mathbb{P}$ with $\mathbb{G}_0 \subsetneqq \mathbb{H}$. Then $\mathbb{H}^* = \mathbb{H} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$ is a filter on $\mathbb{P} \downarrow \mathbb{p}^*$ and $\mathbb{G}^* \subsetneqq \mathbb{H}^*$. This is a contradiction to the maximality of $\mathbb{G}^*$.

(3): $\mathbb{G}^* = \mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)$ is $(\mathsf{V}, \mathbb{P})$-generic by Exercise 1.10, (2). The equality $V[\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*)] = V[\mathbb{G}]$ follows from the mutual definability of $\mathbb{G}$ and $\mathbb{G}^*$ ($\mathbb{G}$ is definable from $\mathbb{G}^*$ by (2) and Lemma 1.1).

(4): By Exercise 1.10, (1). ❑ (Lemma 3.21)

Theorem 3.22 (Solovay, unpublished[44]) *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are cBa posets. If there are $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ and $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ then there are $\mathbb{p}^* \in \mathbb{G}$, $\mathbb{q}^* \in \mathbb{H}$ and a complete embedding $i : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}^*$ in V such that $i''\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*) \subseteq \mathbb{H}$.* *P-forcing-eq-sub-1*

Proof. We may assume that $\mathbb{P}$ and $\mathbb{Q}$ are atomless. Suppose that $\underset{\sim}{\mathbb{G}}$ is a $\mathbb{Q}$-name such that $\underset{\sim}{\mathbb{G}}[\mathbb{H}] = \mathbb{G}$.

Let

$$\mathbb{q}_0 = [\![\text{“}\underset{\sim}{\mathbb{G}} \text{ is a } (\mathsf{V}, \mathbb{P})\text{-generic filter”}]\!]^{\mathbb{B}_\mathbb{Q}}. \tag{3.60}$$ forcing-eq-sub-a-0

We have $\mathbb{q}_0 \in \mathbb{H}$ (see Lemma 2.13, (2)).

Claim 3.22.1 *There is $\mathbb{p}^* \in \mathbb{P}$ such that, for all $\mathbb{r} \leq_\mathbb{P} \mathbb{p}^*$, $\mathbb{q}_0 \wedge^{\mathbb{B}_\mathbb{Q}} [\![\check{\mathbb{r}}_\mathbb{Q} \,\varepsilon\, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_\mathbb{Q}} \neq \mathbb{0}_{\mathbb{B}_\mathbb{Q}}$.* *Cl-forcing-eq-sub-0*

$\vdash$ For each $\mathbb{r} \in \mathbb{G}$, we have $\overbrace{\mathbb{q}_0}^{\mathbb{H} \ni} \wedge^{\mathbb{B}_\mathbb{Q}} \overbrace{[\![\check{\mathbb{r}}_\mathbb{Q} \,\varepsilon\, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_\mathbb{Q}}}^{\mathbb{H} \ni} \in \mathbb{H}$.

Thus, by Forcing Theorem 1.12, (2), there is $\mathbb{p}^* \in \mathbb{G}$ such that

$$\mathbb{p}^* \Vdash_\mathbb{P} \text{“}\forall x \,\varepsilon\, \underset{\sim}{\mathbb{G}}_\mathbb{P} \Big((\surd_\mathbb{P}(\mathbb{q}_0) \surd_\mathbb{P}(\wedge^{\mathbb{B}_\mathbb{Q}}) [\![\check{}(x) \,\varepsilon\, \surd_\mathbb{P}(\underset{\sim}{\mathbb{G}})]\!]^{\surd_\mathbb{P}(\mathbb{B}_\mathbb{Q})}) \not\equiv \surd_\mathbb{P}(\mathbb{0}_{\mathbb{B}_\mathbb{Q}}) \Big) \text{”} \tag{3.61}$$ forcing-eq-sub-0

where $\check{(\cdot)}$ is the standard $\mathbb{P}$-name for the function (in V) $\check{}\, : \mathbb{P} \to \mathsf{V}^\mathbb{Q}$; $\mathbb{s} \mapsto \check{\mathbb{s}}_\mathbb{Q}$.[45] For $\mathbb{r} \leq_\mathbb{P} \mathbb{p}^*$, we have $\mathbb{r} \Vdash_\mathbb{P} \text{“}\check{\mathbb{r}}_\mathbb{P} \,\varepsilon\, \underset{\sim}{\mathbb{G}}_\mathbb{P}\text{”}$ and $\Vdash_\mathbb{P} \text{“}\check{}(\check{\mathbb{r}}_\mathbb{P}) \equiv \surd_\mathbb{P}(\check{\mathbb{r}}_\mathbb{Q})\text{”}$. By (3.61), it follows that

$$\mathbb{r} \Vdash_\mathbb{P} \text{“}\surd_\mathbb{P}(\mathbb{q}_0) \surd_{\mathbb{B}_\mathbb{Q}}(\wedge^{\mathbb{B}_\mathbb{Q}}) [\![\surd_\mathbb{P}(\check{\mathbb{r}}_\mathbb{Q}) \,\varepsilon\, \surd_\mathbb{P}(\underset{\sim}{\mathbb{G}})]\!]^{\surd_\mathbb{P}(\mathbb{B}_\mathbb{Q})} \not\equiv \surd_\mathbb{P}(\mathbb{0}_{\mathbb{B}_\mathbb{Q}})\text{”}. \tag{3.62}$$ forcing-eq-sub-1

Thus $\mathbb{q}_0 \wedge^{\mathbb{B}_\mathbb{Q}} [\![\check{\mathbb{r}}_\mathbb{Q} \,\varepsilon\, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_\mathbb{Q}} \neq \mathbb{0}_{\mathbb{B}_\mathbb{Q}}$. ⊣ (Claim 3.22.1)

Let $i : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}_0$ be defined by

$$i(\mathbb{r}) = \mathbb{q}_0 \wedge^{\mathbb{B}_\mathbb{Q}} [\![\check{\mathbb{r}}_\mathbb{Q} \,\varepsilon\, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_\mathbb{Q}} \tag{3.63}$$ forcing-eq-sub-2

for all $\mathbb{r} \in \mathbb{P} \downarrow \mathbb{p}^*$. i is well-defined by Claim 3.22.1. Let $\mathbb{q}^* = i(\mathbb{p}^*)$.

The next Claim shows that these $\mathbb{p}^*$ and $\mathbb{q}^*$ together with i are as desired.

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[44] see [Hamkins 2014].

[45] “$[\![\cdot \,\varepsilon\, \cdot]\!]^{\cdot}$” in (3.61) is treated as subformula of the whole formula forced corresponding to the formula defining the class $\{\langle x, y, z\rangle : [\![x \,\varepsilon\, y]\!]^z\}$.

Claim 3.22.2 (i) $i(\mathbb{1}_{\mathbb{P}\downarrow\mathbb{p}^*}) = \mathbb{1}_{\mathbb{Q}\downarrow\mathbb{q}^*}$. *Cl-forcing-eq-sub-0-0*

(ii) *i is order preserving.*

(iii) $i : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}^*$.

(iv) *i is incompatibility preserving.*

(v) *i preserves $\neg$.*

(vi) $i : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}^*$ *satisfies* (2.34).

(vii) $i''\mathbb{G} \subseteq \mathbb{H}$.

$\vdash$ (i): This is clear since $\mathbb{1}_{\mathbb{P}\downarrow\mathbb{p}^*} = \mathbb{p}^*$ and $\mathbb{1}_{\mathbb{Q}\downarrow\mathbb{q}^*} = \mathbb{q}^*$.

(ii): Suppose $\mathbb{r}' \leq_{\mathbb{P}} \mathbb{r}$. Then for any $\mathbb{q} \leq_{\mathbb{Q}} \mathbb{q}_0$, $\mathbb{q} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{r}}'_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$" implies $\mathbb{q} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{r}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$" by (3.60). By the definition (3.63) of i, it follows that $i(\mathbb{r}') \leq_{\mathbb{Q}} i(\mathbb{r})$.

(iii) follows from (i) and (ii).

(iv): Suppose that $\mathbb{r}, \mathbb{r}' \leq_{\mathbb{P}} \mathbb{p}_0$ are such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{r}'$. Then, by the definition (3.60) of $\mathbb{q}_0$, no two $\mathbb{q}, \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}_0$ with $\mathbb{q} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{r}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$" and $\mathbb{q}' \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{r}}'_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$" can be By the definition (3.63) of i, it follows that $i(\mathbb{r}) \perp_{\mathbb{Q}} i(\mathbb{r}')$.

(v): Note that, for $\mathbb{r} \in \mathbb{P} \downarrow \mathbb{p}^*$, we have

$$i(\mathbb{r}) = i(\mathbb{p}^*) \wedge i(\mathbb{r}) = \mathbb{q}^* \wedge [\![\check{\mathbb{r}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \tag{3.64}$$

by (ii). Since $\mathbb{q}^* = \mathbb{q}_0 \wedge [\![\check{\mathbb{p}^*}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}}$, it follows that

$$\begin{aligned} i(\neg^{\mathbb{B}_{\mathbb{P}}\downarrow\mathbb{p}^*}\mathbb{r}) &= \mathbb{q}^* \wedge^{\mathbb{B}_{\mathbb{Q}}} [\![\surd_{\mathbb{Q}}(\mathbb{p}^* \wedge \neg\mathbb{r}) \in \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \\ &= \mathbb{q}^* \wedge^{\mathbb{B}_{\mathbb{Q}}} [\![\surd_{\mathbb{Q}}(\neg\mathbb{r}) \in \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \\ &= \mathbb{q}^* \wedge^{\mathbb{B}_{\mathbb{Q}}} [\![\check{\mathbb{r}}_{\mathbb{Q}} \notin \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \\ &= \mathbb{q}^* \wedge^{\mathbb{B}_{\mathbb{Q}}} \neg^{\mathbb{B}_{\mathbb{Q}}}[\![\check{\mathbb{r}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \\ &= \neg^{\mathbb{B}_{\mathbb{Q}}\downarrow\mathbb{q}^*} i(\mathbb{r}). \end{aligned} \tag{3.65}$$

(vi): Suppose $\mathbb{q} \in \mathbb{P} \downarrow \mathbb{q}^*$. Let

$$\mathbb{p} = \prod^{\mathbb{B}_{\mathbb{P}}} \underbrace{\{\mathbb{s} \in \mathbb{P} \,:\, i(\mathbb{s}) \geq_{\mathbb{Q}} \mathbb{q}\}}_{\substack{\| \\ S}}. \tag{3.66}$$ forcing-eq-sub-2-0

Since $i(\mathbb{p}^*) = \mathbb{q}^* \geq_{\mathbb{Q}} \mathbb{q}$, $\mathbb{p} \leq_{\mathbb{P}} \mathbb{p}^*$ by (ii). For each $\mathbb{s} \in S$ we have $\mathbb{q} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{s}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$". Hence, by Lemma 1.2, $\mathbb{q} \Vdash_{\mathbb{Q}}$ "$\check{\mathbb{p}}_{\mathbb{Q}} \in \underset{\sim}{\mathbb{G}}$". In particular, $\mathbb{p} \neq \mathbb{0}_{\mathbb{B}_{\mathbb{P}}}$.

We show that $\mathbb{p}$ satisfies (2.34).

Suppose that

$$\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}. \tag{3.67}$$ forcing-eq-sub-3

We have to show that $i(\mathbb{r}) \top_{\mathbb{Q}} \mathbb{q}$. Otherwise $\mathbb{q} \leq_{\mathbb{Q}} \neg^{\mathbb{B}_{\mathbb{Q}} \downarrow \mathbb{q}^*} i(\mathbb{r}) = i(\neg^{\mathbb{B}_{\mathbb{P}} \downarrow \mathbb{p}^*} \mathbb{r})$ by (v). Thus $\mathbb{p} \leq_{\mathbb{P}} \neg^{\mathbb{B}_{\mathbb{P}} \downarrow \mathbb{p}^*} \mathbb{r}$ by (3.66). But this is a contradiction to (3.67).

(vii): Suppose that $\mathbb{p} \in \mathbb{G} \downarrow \mathbb{p}^*$. Then

$$(3.68) \quad i(p) = \underbrace{\mathbb{q}^*}_{\substack{\cap \\ \mathbb{H}}} \wedge^{\mathbb{B}_{\mathbb{Q}}} \underbrace{[\![\check{\mathbb{p}}_{\mathbb{Q}} \, \varepsilon \, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}}}_{\substack{\cap \\ \mathbb{H}}} \in \mathbb{H} \downarrow \mathbb{q}^*.$$

$\dashv$ (Claim 3.22.2)

❑ (Theorem 3.22)

Theorem 3.23 (Solovay, unpublished[46]) *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are cBa posets. If there are $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ and $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$ then there are $\mathbb{p}^* \in \mathbb{G}$, $\mathbb{q}^* \in \mathbb{H}$ and an isomorphism $i^* : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}^*$ in V such that $i^{*}{}''\mathbb{G} \cap (\mathbb{P} \downarrow \mathbb{p}^*) = \mathbb{H} \cap (\mathbb{Q} \downarrow \mathbb{q}^*)$.* *P-forcing-eq-sub-2*

Proof. The proof is almost identical with that of Theorem 3.22.

We may assume that $\mathbb{P}$ and $\mathbb{Q}$ are atomless. Suppose that $\underset{\sim}{\mathbb{G}}$ is a $\mathbb{Q}$-name and $\underset{\sim}{\mathbb{H}}$ be a $\mathbb{P}$-name such that $\underset{\sim}{\mathbb{G}}[\mathbb{H}] = \mathbb{G}$ and $\underset{\sim}{\mathbb{H}}[\mathbb{G}] = \mathbb{H}$.

Let

$$(3.69) \quad \mathbb{q}_0 = [\![\text{“}\underset{\sim}{\mathbb{G}} \text{ is a } (\mathsf{V}, \mathbb{P})\text{-generic filter and } \underset{\sim}{\mathbb{H}}_{\mathbb{Q}} \equiv \surd_{\mathbb{Q}}(\underset{\sim}{\mathbb{H}})[\underset{\sim}{\mathbb{G}}]]\!]^{\mathbb{B}_{\mathbb{Q}}}$$ *forcing-eq-sub-4*

where $\underset{\sim}{\mathbb{H}}_{\mathbb{Q}}$ is the standard $\mathbb{P}$-name for $(\mathsf{V}, \mathbb{Q})$-generic filter.

Claim 3.23.1 *There is $\mathbb{p}^* \in \mathbb{P}$ such that, for all $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}^*$,* *Cl-forcing-eq-sub-1*

$$(3.70) \quad \mathbb{q}_0 \wedge^{\mathbb{B}_{\mathbb{Q}}} [\![\check{\mathbb{r}}_{\mathbb{Q}} \, \varepsilon \, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \wedge^{\mathbb{B}_{\mathbb{Q}}} \prod^{\mathbb{B}_{\mathbb{Q}}} \{\mathbb{q} \in \mathbb{Q} : \mathbb{r} \Vdash_{\mathbb{P}} \text{“}\mathbb{q} \in \underset{\sim}{\mathbb{H}}\text{”}\} \neq \mathbb{0}_{\mathbb{B}_{\mathbb{Q}}}.$$ *forcing-eq-sub-4-0*

$\vdash$ For each $\mathbb{r} \in \mathbb{G}$, we have

$$(3.71) \quad \overbrace{\mathbb{q}_0}^{\substack{\mathbb{H} \\ \Psi}} \wedge^{\mathbb{B}_{\mathbb{Q}}} \overbrace{[\![\check{\mathbb{r}}_{\mathbb{Q}} \, \varepsilon \, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}}}^{\substack{\mathbb{H} \\ \Psi}} \wedge^{\mathbb{B}_{\mathbb{Q}}} \overbrace{\prod^{\mathbb{B}_{\mathbb{Q}}} \{\mathbb{q} \in \mathbb{Q} : \mathbb{r} \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{q}} \, \varepsilon \, \underset{\sim}{\mathbb{H}}\text{”}\}}^{\substack{\mathbb{H} \\ \Psi}} \text{ (by Lemma 1.2)} \in \mathbb{H}.$$ *forcing-eq-sub-4-1*

Thus there is a condition $\mathbb{p}^* \in \mathbb{P}$ which forces the universal statement corresponding to the one in (3.62). The rest of the proof is similar to that of Claim 3.22.1. $\dashv$ (Claim 3.23.1)

Let $i : \mathbb{P} \downarrow \mathbb{p}^* \to \mathbb{Q} \downarrow \mathbb{q}_0$ be defined by

$$(3.72) \quad i^*(\mathbb{r}) = \mathbb{q}_0 \wedge^{\mathbb{B}_{\mathbb{Q}}} [\![\check{\mathbb{r}}_{\mathbb{Q}} \, \varepsilon \, \underset{\sim}{\mathbb{G}}]\!]^{\mathbb{B}_{\mathbb{Q}}} \wedge^{\mathbb{B}_{\mathbb{Q}}} \sum^{\mathbb{B}_{\mathbb{Q}}} \{\mathbb{q} \in \mathbb{Q} : \mathbb{r} \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{q}} \, \varepsilon \, \underset{\sim}{\mathbb{H}}\text{”}\}.$$ *forcing-eq-sub-5*

for all $\mathbb{r} \in \mathbb{P} \downarrow \mathbb{p}$. i^* is well-defined by Claim 3.23.1. Let $\mathbb{q}^* = i^*(\mathbb{p}^*)$. Note that $\mathbb{q}^* \leq_{\mathbb{Q}} \mathbb{q}_0$.

By Lemma 2.12, the next Claim implies that these $\mathbb{p}^*$ and $\mathbb{q}^*$ together with i as above is as desired:

[46] see [Hamkins 2014].

Claim 3.23.2 (i) $i^*(\mathbb{1}_{\mathbb{P}\downarrow\mathbb{p}^*}) = \mathbb{1}_{\mathbb{Q}\downarrow\mathbb{q}^*}$.

(ii) *i^* is order preserving.*

(iii) $i^* : \mathbb{P}\downarrow\mathbb{p}^* \to \mathbb{Q}\downarrow\mathbb{q}^*$.

(iv) *i^* is incompatibility preserving.*

(v) *$i^*{}''(\mathbb{P}\downarrow\mathbb{p}^*)$ is dense in $\mathbb{Q}\downarrow\mathbb{q}^*$.*

(vi) *i^* is an isomorphism form $\mathbb{P}\downarrow\mathbb{p}^*$ to $\mathbb{Q}\downarrow\mathbb{q}^*$.*

(vii) $i^a * {}''\mathbb{G} = \mathbb{H}$.

$\vdash$ (i) $\sim$ (iv) can be proved just as in Claim 3.22.2.

(v): Suppose $\mathbb{q} \in \mathbb{Q}\downarrow\mathbb{q}^*$. Then we have

$$\mathbb{q} \Vdash_{\mathbb{Q}} \text{“}\exists p\ \varepsilon\ \underset{\sim}{\mathbb{G}}\ (p \leq_{\surd_{\mathbb{Q}}(\mathbb{P})} \surd_{\mathbb{Q}}(\mathbb{p}^*)\ \wedge\ p \Vdash_{\surd_{\mathbb{Q}}(\mathbb{P})} \text{“}\surd_{\mathbb{Q}}(\check{\mathbb{q}}_{\mathbb{P}})\ \varepsilon\ \surd_{\mathbb{Q}}(\underset{\sim}{\mathbb{H}})\text{”})\text{”}. \tag{3.73}$$ forcing-eq-sub-6

Let $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ and $\mathbb{p} \in \mathbb{P}\restriction\mathbb{p}^*$ be such that

$$\mathbb{q}' \Vdash_{\mathbb{Q}} \text{“}\surd_{\mathbb{Q}}(\mathbb{p})\ \varepsilon\ \underset{\sim}{\mathbb{G}} \wedge \surd_{\mathbb{Q}}(\mathbb{p}) \leq_{\surd_{\mathbb{Q}}(\mathbb{p})} \surd_{\mathbb{Q}}(\mathbb{p}^*) \wedge \surd_{\mathbb{Q}}(\mathbb{p}) \Vdash_{\surd_{\mathbb{Q}}(\mathbb{P})} \text{“}\surd_{\mathbb{Q}}(\check{\mathbb{q}}_{\mathbb{P}})\ \varepsilon\ \surd_{\mathbb{Q}}(\underset{\sim}{\mathbb{H}})\text{”}\text{”}. \tag{3.74}$$ forcing-eq-sub-7

It follows that $i^*(\mathbb{p}) \leq_{\mathbb{Q}} \mathbb{q}$.

(vi): Since $\mathbb{P}\downarrow\mathbb{p}^*$ and $\mathbb{Q}\downarrow\mathbb{q}^*$ are cBa posets and $i^* : \mathbb{P}\downarrow\mathbb{p}^* \to \mathbb{Q}\downarrow\mathbb{q}^*$ is a dense embedding by (i) $\sim$ (iv), it follows that i^* is an isomorphism form $\mathbb{P}\downarrow\mathbb{p}^*$ to $\mathbb{Q}\downarrow\mathbb{q}^*$ by Lemma 2.12, (3).

(vii): Similarly to Claim 3.22.2, we have $i^*{}''\mathbb{G} \subseteq \mathbb{H}$. Since $i^* : \mathbb{P}\downarrow\mathbb{p}^* \to \mathbb{Q}\downarrow\mathbb{q}^*$ is an isomorphism, it follows that $i^*{}''\mathbb{G} = \mathbb{H}$ by maximality of generic filters (Lemma 1.1).

❑ (Theorem 3.23)

A poset $\mathbb{P}$ is said to be *locally homogeneous* if $\mathbb{P}\downarrow\mathbb{p}$ is isomorphic to $\mathbb{P}$ for all $\mathbb{p} \in \mathbb{P}$. If $\mathbb{P}$ is locally homogeneous then it is easy to see that its sub-Booleanization is also locally homogeneous. Moreover, if $\mathbb{P}$ is locally homogeneous then $RO(\mathbb{P})$ and $RO(\mathbb{P})^+$ are also locally homogeneous (see [Koppelberg 1989], Corollary 12.5).

Corollary 3.24 *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are locally homogeneous posets.* *P-forcing-eq-sub-3*

(1) *If there are $(\mathsf{V},\mathbb{P})$-generic filter $\mathbb{G}$ and $(\mathsf{V},\mathbb{Q})$-generic filter $\mathbb{H}$ such that $\mathsf{V}[\mathbb{G}] \subseteq \mathsf{V}[\mathbb{H}]$ Then there is a complete embedding of $RO(\mathbb{P})$ into $RO(\mathbb{Q})$.*

(2) *If there are $(\mathsf{V},\mathbb{P})$-generic filter $\mathbb{G}$ and $(\mathsf{V},\mathbb{Q})$-generic filter $\mathbb{H}$ such that $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{H}]$ Then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent ($\mathbb{P} \approx \mathbb{Q}$), that is, $RO(\mathbb{P})$ and $RO(\mathbb{Q})$ are isomprohic.*

Proof. Let $\mathbb{P}^* = RO(\mathbb{P})^+$ and $\mathbb{Q}^* = RO(\mathbb{Q})^+$. Then $\mathbb{P}^*$ and $\mathbb{Q}^*$ cBa posets and they are also locally homogeneous. Let $\mathbb{G}^*$ be the filter on $\mathbb{P}^*$ generated from $\mathbb{G}$ and $\mathbb{H}^*$ the filter on $\mathbb{Q}^*$ generated from $\mathbb{H}$. By Lemma 2.3, (1), $\mathbb{G}^*$ is a $(\mathsf{V},\mathbb{P}^*)$-generic filter and $\mathbb{H}^*$ a $(\mathsf{V},\mathbb{Q}^*)$-generic filter. Thus, in case of (1), there are $\mathbb{p}^* \in \mathbb{P}^*$, $\mathbb{q}^* \in \mathbb{Q}^*$ and a complete embedding $i : \mathbb{P}^*\downarrow\mathbb{p}^* \to \mathbb{Q}^*\downarrow\mathbb{q}^*$ by Theorem 3.22. Since $\mathbb{P}^* \cong \mathbb{P}^*\downarrow\mathbb{p}^*$ and $\mathbb{Q}^* \cong \mathbb{Q}^*\downarrow\mathbb{q}^*$ by local homogeneity, we find a complete embedding of $\mathbb{P}^*$ into $\mathbb{Q}^*$. For (2), we can argue similarly using Theorem 3.23. ❑ (Corollary 3.24)

3.4 Lévy collapse and set-generic multiverse

For a set $S \subseteq \mathsf{On}$ and infinite regular cardinal λ, let multi

(3.75) $\mathrm{Col}(\lambda, S) = \{f \,:\, f$ is a mapping with $\mathrm{dom}(f) \subseteq (S \setminus 2) \times \lambda$, $\mathrm{range}(f) \subseteq \sup S$, $|\, f \,| < \lambda$, for all $\langle \alpha, \xi \rangle \in \mathrm{dom}(f)\, (f(\langle \alpha, \xi \rangle < \alpha))\}$, collapse-0

(3.76) $\mathbb{1}_{\mathrm{Col}(\lambda,S)} = \emptyset$ collapse-1

and

(3.77) $f \leq_{\mathrm{Col}(\lambda,S)} g \;\Leftrightarrow\; g \subseteq f$ collapse-2

for f, $g \in \mathrm{Col}(\lambda, S)$.

The poset $\mathrm{Col}(\lambda, S) = (\mathrm{Col}(\lambda, S), \leq_{\mathrm{Col}(\lambda,S)}, \mathbb{1}_{\mathrm{Col}(\lambda,S)})$ is called a *Lévy Collapse*. Lévy's original poset was $\mathrm{Col}(\omega, \kappa)$ for an inaccessible κ (note that κ here is considered as a set of ordinals).

Lemma 3.25 (1) $\mathrm{Col}(\lambda, S)$ *is separative.* *P-collapse-0*

(2) *For* $\alpha \geq 2$, $\mathrm{Col}(\lambda, \{\alpha\}) \cong \mathrm{Fn}(\lambda, \alpha, <\lambda)$.

(3) $\mathrm{Col}(\lambda, S)$ *is* $<\lambda$*-closed.*

(4) *If* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathrm{Col}(\lambda, S))$*-generic filter and* $g = \bigcup \mathbb{G}$*, then we have* $g : (S \setminus 2) \times \lambda \to \sup(S)$ *and* $g(\alpha, \cdot)$ *is a surjection from* λ *onto* α *for all* $\alpha \in S \setminus 2$.

(5) *In* $\mathsf{V}[\mathbb{G}]$ *for* $\mathbb{G}$ *as above,* $\mathsf{V}[\mathbb{G}] \models$ "$|\, \alpha \,| \leq \lambda$ *for all* $\alpha \in S$".

(6) *Suppose* κ, λ *are infinite regular cardinal with* $\lambda < \kappa$*. If* κ *is an inaccessible cardinal or* $\lambda = \omega$*, then* $\mathrm{Col}(\lambda, \kappa)$ *has the* κ*-cc.*

(7) *Suppose* κ, λ *are infinite regular cardinal with* $\lambda < \kappa$*. If* κ *is an inaccessible cardinal or* $\lambda = \omega$*, then* $\mathrm{Col}(\lambda, \kappa)$ *forces that all ordinals* α *with* $\lambda \leq \alpha < \kappa$ *to be of cardinality* λ *and preserves all cardinals and cofinality* $\geq \kappa$.

(8) *If* $S = X \mathbin{\dot\cup} Y$ *then* $\mathrm{Col}(\lambda, S) \cong \mathrm{Col}(\lambda, X) \times \mathrm{Col}(\lambda, Y)$.

Proof. (1): We show that $\mathrm{Col}(\lambda, S)$ satisfies (2.8).

Suppose that f, $g \in \mathrm{Col}(\lambda, S)$ and $f \not\leq_{\mathrm{Col}(\lambda,S)} g$. If $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ then this means that f and g are incompatible to each other as functions. Thus f and g are incompatible in $\mathrm{Col}(\lambda, S)$. Otherwise there is $\langle \alpha, \xi \rangle \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$. Let $\eta < \alpha$ be such that $\eta \neq g(\langle \alpha, \xi \rangle)$ and[(47)] let $r = f \cup \{\langle \langle \alpha, \xi \rangle, \eta \rangle\}$. Then $r \leq_{\mathrm{Col}(\lambda,S)} f$ and r is incompatible with g in $\mathrm{Col}(\lambda, S)$.

(2): The mapping

(3.78) $h : \mathrm{Col}(\lambda, \{\alpha\}) \to \mathrm{Fn}(\lambda, \alpha, <\lambda);\ f \mapsto f(\langle \alpha, \cdot \rangle)$ collapse-3

[(47)] By the definition of $\mathrm{Col}(\lambda, S)$, $\alpha \geq 2$ and hence we can choose always such η.

is an isomorphism from $Col(\lambda, \{\alpha\})$ to $\mathrm{Fn}(\lambda, \alpha, <\lambda)$.

(3): Clear by the definition (3.75) $\sim$ (3.77) of $\mathrm{Col}(\lambda, S)$.

(4): Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathrm{Col}(\lambda, S))$-generic filter. In $\mathsf{V}[\mathbb{G}]$, since $\mathbb{G}$ is a filter, $\mathbb{g} = \bigcup \mathbb{G}$ is a (possibly partial) function from $(S \setminus 2) \times \lambda$ to $\sup(S)$.

$\mathbb{g}$ is a total function: For each $\langle \alpha, \xi \rangle \in (S \setminus 2) \times \lambda$,

$$D_{\langle \alpha, \xi \rangle} = \{f \in \mathrm{Col}(\lambda, S) \,:\, \langle \alpha, \xi \rangle \in \mathrm{dom}(f)\} \tag{3.79}$$ collapse-4

is dense in $\mathrm{Col}(\lambda, S)$. By genericity of $\mathbb{G}$, there is an $f \in \mathbb{G} \cap D_{\alpha, \xi}$. Thus $\langle \alpha, \xi \rangle \in \mathrm{dom}(f) \subseteq \mathrm{dom}(\mathbb{g})$.

By the definition of $\mathrm{Col}(\lambda, S)$, we have $\mathbb{g}(\langle \alpha, \cdot \rangle) : \lambda \to \alpha$ for $\alpha \in S \setminus 2$.

$\mathbb{g}$ is a surjection: For each $\eta < \alpha$,

$$E_{\langle \alpha, \eta \rangle} = \{f \in \mathrm{Col}(\lambda, S) \,:\, f(\langle \alpha, \xi \rangle) = \eta \text{ for some } \xi < \lambda \text{ with } \langle \alpha, \xi \rangle \in \mathrm{dom}(f)\} \tag{3.80}$$ collapse-5

is dense in $\mathrm{Col}(\lambda, S)$. By genericity of $\mathbb{G}$, there is $f \in \mathbb{G} \cap E_{\langle \alpha, \eta \rangle}$. Thus we have $\eta \in \mathrm{dom}(f(\langle \alpha, \cdot \rangle)) \subseteq \mathrm{dom}(\mathbb{g}(\langle \alpha, \cdot \rangle))$.

(5): follows from (4).

(6): Suppose that $f_\alpha \in \mathrm{Col}(\lambda, \kappa)$ for $\alpha < \kappa$. We have to show that there are distinct α, $\alpha' < \kappa$ such that f_α and $f_{\alpha'}$ are compatible.

Let $a_\alpha = \mathrm{dom}(f_\alpha)$ for $\alpha < \kappa$. By the Generalized Δ-System Lemma (Lemma 1.32), there is $I \in [\kappa]^\kappa$ such that a_α, $\alpha \in I$ form a Δ-system with the root d. Since $|\, d \,| < \kappa$ there are strictly less than κ possibilities of $f_\alpha \restriction d$ for $d \in I$. Thus there are distinct α, $\alpha' \in I$ such that $f_\alpha \restriction d = f_{\alpha'} \restriction d$. We have $f_\alpha \cup f_{\alpha'} \in \mathrm{Col}(\lambda, \kappa)$ and $f_\alpha \cup f_{\alpha'} \leq_{\mathrm{Col}(\lambda, \kappa)} f_\alpha$, $f_{\alpha'}$.

(7): follows from (5) and (6) by Lemma 1.28.

(8): The mapping defined by

$$i : \mathrm{Col}(\lambda, S) \to \mathrm{Col}(\lambda, X) \times \mathrm{Col}(\lambda, Y);\ f \mapsto \langle f \restriction (X \setminus 2) \times \lambda,\ f \restriction (Y \setminus 2) \times \lambda \rangle \tag{3.81}$$ collapse-6

is an an isomorphism. ❑ (Lemma 3.25) from the beginning of the section to here $\kappa \leftrightarrow \lambda$

Lemma 3.26 *For a regular infinite κ, $\mathrm{Col}(\kappa, S)$ is locally homogeneous.* *L-collapse-0*

Proof. If $S \setminus 2 =$ then $\mathrm{Col}(\kappa, S) = \{\emptyset\}$ which is trivially locally homogeneous.

Suppose $S \setminus 2 \neq \emptyset$. For an arbitrary $f \in \mathrm{Col}(\kappa, S)$ let

$$D = \mathrm{dom}(f) \subseteq (S \setminus 2) \times \kappa. \tag{3.82}$$

Since $|\, D \,| < \kappa$, there is a bijection

$$i : ((S \setminus 2) \times \kappa) \setminus D \to (S \setminus 2) \times \kappa \tag{3.83}$$ collapse-6-a

such that, for $\langle \alpha, \xi \rangle \in ((S \setminus 2) \times \kappa) \setminus D$, $i(\langle \alpha, \xi \rangle) = \langle \alpha, i_\alpha(\xi) \rangle$ for some $i_\alpha : \kappa \to \kappa$.

Let

(3.84) $\quad i^* : \mathrm{Col}(\kappa, S) \downarrow f \to \mathrm{Col}(\kappa, S);\ g \mapsto i^*(g)$

be defined by $\mathrm{dom}(i^*(g)) = i''\,\mathrm{dom}(g) \setminus \mathrm{dom}(f)$ and

(3.85) $\quad (i^*(g)(\langle \alpha, \xi \rangle)) = g(\langle \alpha, i_\alpha^{-1}(\xi)) \rangle)$

for $\langle \alpha, \xi \rangle \in \mathrm{dom}(i^*(g))$.

i^* is then an isomorphism from $\mathrm{Col}(\kappa, S) \downarrow f$ to $\mathrm{Col}(\kappa, S)$. ❑ (Lemma 3.26)

Lemma 3.27 *Let α be an ordinal and suppose that $\mathbb{P}$ is a separable poset such that* P-collapse-1

(3.86) $\quad |\,\mathbb{P}\,| = |\,\alpha\,|$ *and* collapse-6-0

(3.87) $\quad \Vdash_{\mathbb{P}}$ "$\exists f\,(f : \omega \to \check{\alpha}_{\mathbb{P}} \wedge f$ *is surjective* $\wedge\ f \notin \surd_{\mathbb{P}}({}^{\omega}\alpha))$".[48] collapse-7

Then, there is a dense subposet of $\mathrm{Col}(\omega, \{\alpha\})$ *(*$\cong \mathrm{Fn}(\omega, \alpha)$*) which can be densely embedded into* $\mathbb{P}$.
In particular, we have $\mathbb{P} \approx \mathrm{Col}(\omega, \{\alpha\})$.

Proof. Let $\nu = |\,\alpha\,|$. By Maximal Principle (Lemma 1.23, (2)), there is a $\mathbb{P}$-name $\underset{\sim}{f}$ such that

(3.88) $\quad \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f} : \omega \to \check{\alpha}_{\mathbb{P}} \wedge \underset{\sim}{f}$ is surjective $\wedge\ \underset{\sim}{f} \notin \surd_{\mathbb{P}}({}^{\omega}\alpha)$". collapse-8

Claim 3.27.1 *For each $\mathbb{p} \in \mathbb{P}$, there are ν-many pairwise incompatible conditions below $\mathbb{p}$.* Cl-collapse-0

$\vdash$ **Case I.** $\nu = \omega$: The Claim follows from the atomlessness of $\mathbb{P}$.

Case II. ν is uncountable and of cofinality $> \omega$: For each $n \in \omega$ let

(3.89) $\quad E_n = \{\eta < \alpha\ :\ \text{there is a } \mathbb{q} \leq_{\mathbb{P}} \mathbb{p} \text{ such that } \mathbb{q} \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{f}(\check{n}) \equiv \check{\eta}\text{".}\}$ collapse-9

Since $\bigcup_{n\in\omega} E_n = \alpha$, there is $n^* < \omega$ such that $|\,E_{n^*}\,| = \nu$. For each $\eta \in E_{n^*}$ let $\mathbb{q}_\eta$ be such that $\mathbb{q}_\eta \leq_{\mathbb{P}} \mathbb{p}$ and $\mathbb{q}_\eta \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(n^*) \equiv \check{\eta}$". $\mathbb{q}_\eta$, $\eta \in E_{n^*}$ are pairwise incompatible since they force pairwise contradictory statements.

Case III. ν is uncountable and of cofinality ω. Let $\nu = \sup\{\nu_n\ :\ n \in \omega\}$ where $\langle \nu_n\ :\ n \in \omega \rangle$ is a strictly increasing sequence of cardinals $< \nu$. For $i \in \omega$, let $\mathbb{p}_i$, $\mathbb{q}_i \in \mathbb{P}$ be such that

(3.90) $\quad \mathbb{p}_i$ and $\mathbb{q}_i$ are incompatible in $\mathbb{P}$ for all $i \in \omega$; collapse-10

(3.91) $\quad \mathbb{p}_0$, $\mathbb{q}_0 \leq \mathbb{p}$; and collapse-11

(3.92) $\quad \mathbb{p}_{i+1}$, $\mathbb{q}_{i+1} \leq_{\mathbb{P}} \mathbb{q}_i$ for all $i \in \omega$. collapse-12

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[48] Note that, if $|\,\alpha\,| > \aleph_0$ (in V), the assertion "$f \notin \surd_{\mathbb{P}}({}^{\omega}\alpha)$" is forced in any case. Note also that it follows from (3.87) that $\mathbb{P}$ is atomless.

Since $\mathbb{P}$ is atomless, it easy to see that there are such $\mathbb{p}_i$, $\mathbb{q}_i$ for $i \in \omega$.

For each $i \in \omega$ and $n \in \omega$ let

(3.93) $\quad E_{i,n} = \{\eta < \alpha \,:\, \text{there is a } \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_i \text{ such that } \mathbb{q} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}(\check{n}) \equiv \check{\eta} \text{”}.\}$ collapse-13

Since $\bigcup_{n\in\omega} E_{i,n} = \alpha$ for all $i \in \omega$, there is $n_i \in \omega$ for each $i \in \omega$ such that $|\, E_{i,n_i} \,| \geq \nu_i$.

Now for each $i \in \omega$ and $\eta \in E_{i,n_i}$, let $\mathbb{p}_{i,\eta} \leq_{\mathbb{P}} \mathbb{p}_i$ be such that $\mathbb{p}_{i,\eta} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}(n_i) \equiv \eta \text{”}$. Then, $\{\mathbb{p}_{i,\eta} \,:\, i \in \omega, \eta \in E_{i,n_i}\}$ is a pairwise incompatible set of conditions below $\mathbb{p}$ of cardinality ν. $\dashv$ (Claim 3.27.1)

Let

(3.94) $\quad \mathbb{D} = \{f \in \mathrm{Col}(\omega, \{\alpha\}) \,:\, \text{there is an } n \in \omega \text{ such that } \mathrm{dom}(f) = \{\alpha\} \times n\}.$

Clearly $\mathbb{D}$ is a dense subposet of $\mathrm{Col}(\omega, \{\alpha\})$.

Let $\underset{\sim}{g}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{g} : \omega \to \underset{\sim}{\mathbb{G}}_{\mathbb{P}}$ is surjective”. We define $\Phi : \mathbb{D} \to \mathbb{P}$ by specifying $\Phi(f)$ by induction on $|\, f \,|$ for $f \in \mathbb{D}$ and show that this Φ is a dense embedding. Let

(3.95) $\quad \Phi(\emptyset) = \mathbb{1}_{\mathbb{P}}.$ collapse-14

Suppose that $\Phi(f)$ has been defined for $f \in \mathbb{D}$ with $\mathrm{dom}(f) = \{\alpha\} \times n$. Let A_f be a maximal antichain below $\Phi(f)$ such that $|\, A_f \,| = \nu$ and each $\mathbb{p} \in A_f$ decides the value of $\underset{\sim}{g}(n)$. It is easy to make $|\, A_f \,| \geq \nu$ by Claim 3.27.1. By (3.86), we also have $|\, A_f \,| \leq \nu$. Let $\langle \mathbb{a}^f_\xi \,:\, \xi < \alpha \rangle$ be a 1-1 enumeration of A_f. Then define

(3.96) $\quad \Phi(f \cup \{\langle\langle \alpha, n\rangle, \xi\rangle\}) = \mathbb{a}^f_\xi$ collapse-15

for each $\xi < \alpha$.

Then $\Phi(\mathbb{1}_{\mathrm{Col}(\omega,\{\alpha\})}) = \Phi(\emptyset) = \mathbb{1}_{\mathbb{P}}$ and Φ is order and incompatibility preserving. Thus, we are done by showing that $\Phi''\mathbb{D}$ is dense in $\mathbb{P}$.

Suppose $\mathbb{p} \in \mathbb{P}$. Since $\mathbb{p} \Vdash_{\mathbb{P}} \text{“} \check{\mathbb{p}}_{\mathbb{P}} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}} \text{”}$, there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $n \in \omega$ such that $\mathbb{q} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{g}(\check{n}_{\mathbb{P}}) \equiv \check{\mathbb{p}}_{\mathbb{P}} \text{”}$.

Since $\{\mathbb{a}^f_\xi \,:\, \xi < \alpha,\, f \in \mathbb{D},\, |\, f \,| = n\}$ is a maximal antichain of $\mathbb{1}_{\mathbb{P}}$ by construction, we have $\mathbb{q} \top_{\mathbb{P}} \mathbb{a}^f_\xi$ for some $\xi < \alpha$ and $f \in \mathbb{D}$. Since $\mathbb{a}^f_\xi$ decides the value of $\underset{\sim}{g}(\check{n}_{\mathbb{P}})$, it should decide the value to be $\check{\mathbb{p}}_{\mathbb{P}}$. In particular, $\mathbb{a}^f_\xi \Vdash_{\mathbb{P}} \text{“} \check{\mathbb{p}}_{\mathbb{P}} \,\varepsilon\, \underset{\sim}{\mathbb{G}}_{\mathbb{P}} \text{”}$. Since $\mathbb{P}$ is separative, this is equivalent to $\Phi(f \cup \{\langle\langle \alpha, n\rangle, \xi\rangle\}) = \mathbb{a}^f_\xi \leq_{\mathbb{P}} \mathbb{p}$ (see Lemma 2.5, (2)). $\square$ (Lemma 3.27)

Recall that, for posets $\mathbb{P}$ and $\mathbb{Q}$, $\mathbb{P} \approx \mathbb{Q}$ denotes $RO(\mathbb{P}) \cong RO(\mathbb{P})$.

Corollary 3.28 (1) *For any $\alpha > \omega$, $\mathrm{Col}(\omega, \{\alpha\}) \approx \mathrm{Col}(\omega, \alpha + 1)$.* *P-collapse-2*

(2) *For a cardinal κ and $\alpha < \kappa$, $\mathrm{Col}(\omega, \kappa) \approx \mathrm{Col}(\omega, \kappa \setminus \alpha)$.*

Proof. (1): $\mathrm{Col}(\omega, \alpha + 1)$ satisfies the properties (3.86) and (3.87).

(2): By Lemma 3.25, (8) and (1) above,

$$\begin{aligned}\mathrm{Col}(\omega,\kappa) &\cong \mathrm{Col}(\omega,\alpha+1)\times\mathrm{Col}(\omega,\kappa\setminus(\alpha+1))\\ &\approx \mathrm{Col}(\omega,\{\alpha\})\times\mathrm{Col}(\omega,\kappa\setminus(\alpha+1))\\ &\cong \mathrm{Col}(\omega,\kappa\setminus\alpha).\end{aligned}$$

❑ (Corollary 3.28)

A lemma quite similar to Lemma 3.27 also holds with ω there replaced by a regular uncountable cardinal κ:

Lemma 3.29 *Let α be an ordinal with $\nu = |\alpha| > \kappa$ for a regular uncountable cardinal κ and suppose that $\mathbb{P}$ is a $<\kappa$-closed separable poset such that* P-collapse-1-x

(3.97) $|\mathbb{P}| = \nu$ *and* collapse-6-0-x

(3.98) $\Vdash_{\mathbb{P}}$ "$\exists f\,(f:\kappa\to\check{\alpha}_{\mathbb{P}} \wedge f$ *is surjective*)".[49] collapse-7-x

Then, there is a dense subposet of $\mathrm{Col}(\kappa,\{\alpha\})$ *(*$\cong \mathrm{Fn}(\kappa,\alpha,<\kappa)$*) which can be densely embedded into* $\mathbb{P}$*.*
In particular, we have $\mathbb{P}\approx\mathrm{Col}(\kappa,\{\alpha\})$*.*

Proof. Suppose that $\mathbb{P}$ is a $<\kappa$-closed separable poset such that $\mathbb{P}\models$ (3.97), (3.98). By Maximal Principle (Lemma 1.23, (2)), there is a $\mathbb{P}$-name $\underset{\sim}{f}$ such that

(3.99) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}:\kappa\to\check{\alpha}_{\mathbb{P}} \wedge \underset{\sim}{f}$ is surjective". collapse-8-x

Claim 3.29.1 *For each $\mathbb{p}\in\mathbb{P}$, there are ν-many pairwise incompatible conditions below $\mathbb{p}$.* Cl-collapse-0-x

$\vdash$ **Case I.** ν is of cofinality $>\kappa$: For each $\xi\in\kappa$ let

(3.100) $E_\xi = \{\beta<\alpha : \text{there is a } \mathbb{q}\le_{\mathbb{P}}\mathbb{p} \text{ such that } \mathbb{q}\Vdash_{\mathbb{P}}\text{"}\underset{\sim}{f}(\xi)\equiv\check{\beta}\text{".}\}$ collapse-9-x

Since $\bigcup_{\xi<\kappa}E_\xi=\alpha$, there is $\xi^*<\kappa$ such that $|E_{\xi^*}|=\nu$. For each $\beta\in E_{\xi^*}$ let $\mathbb{q}_\beta$ be such that $\mathbb{q}_\beta\le_{\mathbb{P}}\mathbb{p}$ and $\mathbb{q}_\beta\Vdash_{\mathbb{P}}$"$\underset{\sim}{f}(\xi^*)\equiv\check{\beta}$". $\mathbb{q}_\beta$, $\beta\in E_{n^*}$ are pairwise incompatible since they force pairwise contradictory statements.

Case II. $\mu = cf(\nu)\le\kappa$: Let $\nu=\sup\{\nu_\eta : \eta<\mu\}$ where $\langle\nu_\eta : \eta<\mu\rangle$ is a strictly increasing sequence of cardinals $<\nu$. For $\eta<\mu$, let $\mathbb{p}_{\eta+1}$, $\mathbb{q}_\eta\in\mathbb{P}$ be such that

(3.101) $\mathbb{q}_0\le\mathbb{p}$; collapse-11-x

(3.102) $\langle\mathbb{q}_\eta : \eta<\mu\rangle$ is a decreasing sequence in $\mathbb{P}$; collapse-10-x-0

(3.103) $\mathbb{p}_{\eta+1}\le_{\mathbb{P}}\mathbb{q}_\eta$ for all $\eta\in\mu$; and collapse-12-x

(3.104) $\mathbb{p}_{\eta+1}$ and $\mathbb{q}_{\eta+1}$ are incompatible in $\mathbb{P}$ for all $\eta<\mu$. collapse-10-x

[49] Note that it follows from (3.98) that $\mathbb{P}$ is atomless.

Since $\mu \leq \kappa$, $\mathbb{P}$ is κ-closed and atomless, it easy to see that there are such $\mathbb{p}_{\eta+1}$, $\mathbb{q}_\eta$ for $\eta < \mu$.

For each $\eta \in \mu$ and $\xi \in \mu$ let

(3.105) $E_{\eta,\xi} = \{\beta < \alpha \,:\, \text{there is a } \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_{\eta+1} \text{ such that } \mathbb{q} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{f}(\check{\xi}) \equiv \check{\eta}\text{”}.\}$ collapse-13-x

Since $\bigcup_{\xi<\mu} E_{\eta,\xi} = \alpha$ for all $\eta < \mu$, there is $\xi_\eta \in \mu$ for each $\eta < \mu$ such that $|\, E_{\eta,\xi_\eta} \,| \geq \nu_\eta$.

Now, for each $\eta < \mu$ and $\beta \in E_{\eta,\xi_\eta}$, let $\mathbb{p}_{\eta,\beta} \leq_{\mathbb{P}} \mathbb{p}_\eta$ be such that $\mathbb{p}_{\eta,\beta} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{f}(\check{\xi}_\eta) \equiv \beta\text{”}$. Then, $\{\mathbb{p}_{\eta,\beta} \,:\, \eta < \mu, \beta \in E_{\eta,\xi_\eta}\}$ is a pairwise incompatible set of conditions below $\mathbb{p}$ of cardinality ν. $\dashv$ (Claim 3.29.1)

Let

(3.106) $\mathbb{D}^* = \{f \in \mathrm{Col}(\kappa, \{\alpha\}) \,:\, \text{there is a } \xi < \kappa \text{ such that } \mathrm{dom}(f) = \{\alpha\} \times \xi\}$ and

(3.107) $\mathbb{D} = \{f \in \mathrm{Col}(\kappa, \{\alpha\}) \,:\, \text{there is a } \xi < \kappa \text{ such that } \mathrm{dom}(f) = \{\alpha\} \times (\xi+1)\} \cup \{\emptyset\}$.

Clearly $D \subseteq D^* \subseteq \mathrm{Col}(\kappa, \{\alpha\})$ and $\mathbb{D}$ is a dense subposet of $\mathrm{Col}(\kappa, \{\alpha\})$.

In the following, we abuse the standard terminology and say $otp(f) = \xi$ for $f \in \mathbb{D}^*$ such that $\mathrm{dom}(f) = \{\alpha\} \times \xi$.

Let $\underset{\sim}{g}$ be a $\mathbb{P}$-name such that

(3.108) $\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{g} : \check{\kappa} \to \underset{\sim}{\mathbb{G}}_{\mathbb{P}} \text{ is surjective”}$. collapse-10-x-0-0

We define $\Phi : \mathbb{D} \to \mathbb{P}$ by specifying $\Phi(f)$ by induction on $otp(\mathrm{dom}(f))$ for $f \in \mathbb{D}$ such that

(3.109) $\Phi(\mathbb{1}_{\mathrm{Col}(\omega,\{\alpha\})}) = \Phi(\emptyset) = \mathbb{1}_{\mathbb{P}}$; collapse-10-x-1

(3.110) Φ is order and incompatibility preserving; collapse-10-x-2

(3.111) For each $\xi < \kappa$, $\{\Phi(f) \,:\, f \in \mathbb{D},\, otp(f) = \xi + 1\}$ is a maximal antichain in $\mathbb{P}$; collapse-10-x-3

(3.112) For each $\xi < \kappa$ and for each $f \in \mathbb{D}$ with $otp(f) = \xi + 1$, $\Phi(f)$ decides the value of $\underset{\sim}{g}(\check{\xi}_{\mathbb{P}})$. collapse-10-x-4

The construction of Φ is possible by Claim 3.104, (3.97) and by the κ-closedness of $\mathbb{P}$. We are done by showing that $\Phi''\mathbb{D}$ is dense in $\mathbb{P}$.

Suppose $\mathbb{p} \in \mathbb{P}$. Since $\mathbb{p} \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{p}}_{\mathbb{P}} \in \underset{\sim}{\mathbb{G}}_{\mathbb{P}}\text{”}$, there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ and $\xi < \kappa$ such that $\mathbb{q} \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{g}(\check{\xi}_{\mathbb{P}}) \equiv \check{\mathbb{p}}_{\mathbb{P}}\text{”}$.

Since $\{\Phi(f) \,:\, f \in \mathbb{D},\, otp(f) = \xi + 1\}$ is a maximal antichain in $\mathbb{P}$ by (3.108), we have $\mathbb{q} \top_{\mathbb{P}} \Phi(f^*)$ for some $f^* \in \mathbb{D}$ with $otp(f) = \xi + 1$. Since $\Phi(f^*)$ decides the value of $\underset{\sim}{g}(\check{\xi}_{\mathbb{P}})$ by (3.112), it should decide the value to be $\check{\mathbb{p}}_{\mathbb{P}}$. In particular, $\Phi(f^*) \Vdash_{\mathbb{P}} \text{“}\check{\mathbb{p}}_{\mathbb{P}} \,\varepsilon\, \underset{\sim}{\mathbb{G}}_{\mathbb{P}}\text{”}$. Since $\mathbb{P}$ is separative, this is equivalent to $\Phi(f^*) \leq_{\mathbb{P}} \mathbb{p}$ (see Lemma 2.5, (2)). ❑ (Lemma 3.29)

Corollary 3.30 (1) *Suppose that κ is a regular uncountable cardinal. If $\alpha > \kappa$ is such that $|\alpha|^{<\kappa} = |\alpha|$. Then, for any $\beta < \alpha$, $\mathrm{Col}(\kappa, \{\alpha\}) \approx \mathrm{Col}(\kappa, (\alpha+1) \setminus \beta)$.* Cor-Col-0

(2) *Suppose that κ is a regular uncountable cardinal and $\lambda > \kappa$ is $<\kappa$-inaccessible (i.e. $\mu^{<\kappa} < \lambda$ holds for all $\mu < \lambda$). Then, for any $\alpha < \lambda$, $\mathrm{Col}(\kappa, \lambda) \approx \mathrm{Col}(\kappa, \lambda \setminus \alpha)$.*

Proof. (1): Let $\mathbb{P} = \mathrm{Col}(\kappa, (\alpha+1) \setminus \beta)$. Then $|\alpha| \leq |\mathbb{P}| \leq |\alpha|^{<\kappa}$, $\mathbb{P}$ is $<\kappa$-closed and adds a surjection from κ to $|\alpha|$. Thus, we have $\mathrm{Col}(\kappa, \{\alpha\}) \approx \mathrm{Col}(\kappa, (\alpha+1) \setminus \beta)$ by Theorem 3.29.

(2): For $\alpha < \lambda$, let $\alpha' = |\alpha|^{<\kappa} < \lambda$. Then we have $|\alpha'|^{<\kappa} = |\alpha'|$. Thus, by (1) above and Lemma 3.25, (8), $\mathrm{Col}(\kappa, \lambda) \approx \mathrm{Col}(\kappa, \lambda \setminus (\alpha'+1)) \times \mathrm{Col}(\kappa, \alpha'+1) \approx \mathrm{Col}(\kappa, \lambda \setminus (\alpha'+1)) \times \mathrm{Col}(\kappa, (\alpha'+1) \setminus \alpha) \approx \mathrm{Col}(\kappa, \lambda \setminus \alpha)$. ❑ (Corollary 3.30)

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4 Iterated forcing

4.1 Finite support iteration

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For an ordinal δ, a finite support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ is a sequence satisfying the following conditions: fin-sup

(4.1) Each $\mathbb{P}_\alpha$ for $\alpha \leq \delta$ is a poset and the underlying set of $\mathbb{P}_\alpha$, which is also denoted by $\mathbb{P}_\alpha$ as before, consists of sequences of length α. In particular $\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{1}_{\mathbb{P}_0} = \emptyset$; fs-0

(4.2) $\underset{\sim}{\mathbb{Q}}_\alpha$ for each $\alpha < \delta$ is a $\mathbb{P}_\alpha$-name and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \langle \underset{\sim}{\mathbb{Q}}_\alpha, \leq_{\underset{\sim}{\mathbb{Q}}_\alpha}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\alpha} \rangle$ is a poset"; fs-1

(4.3) For $\beta < \alpha \leq \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$, $\mathbb{p} \restriction \beta \in \mathbb{P}_\beta$; fs-2

(4.4) For $\alpha < \delta$, fs-2-0

$\mathbb{P}_{\alpha+1} = \{\mathbb{p}^\frown\langle \underset{\sim}{\mathbb{q}} \rangle : \mathbb{p} \in \mathbb{P}_\alpha,\ \underset{\sim}{\mathbb{q}}$ is a canonical $\mathbb{P}_\alpha$-name, $\underset{\sim}{\mathbb{q}} \in \mathcal{H}(\mu(\underset{\sim}{\mathbb{Q}}_\alpha)),\ \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{q}}\ \varepsilon\ \underset{\sim}{\mathbb{Q}}_\alpha$"$\}$; 定義に変更あり．要チェック．

(4.5) For all limit $\gamma \leq \delta$, fs-3

$\mathbb{P}_\gamma = \{\mathbb{p} : \mathbb{p}$ is a sequence of length γ, $\mathbb{p} \restriction \alpha \in \mathbb{P}_\alpha$ for all $\alpha < \gamma$,
$\{\beta < \gamma : \mathbb{p}(\beta) \neq \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}\}$ is finite[(50)] $\}$;

(4.6) $\mathbb{1}_{\mathbb{P}_\alpha}(\beta) = \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}$ for all $\beta < \alpha \leq \delta$; fs-4

(4.7) For $\alpha \leq \delta$ and $\mathbb{p}, \mathbb{p}' \in \mathbb{P}_\alpha$, $\mathbb{p}' \leq_{\mathbb{P}_\alpha} \mathbb{p}$ if and only if $\mathbb{p}' \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\mathbb{p}'(\beta) \leq_{\underset{\sim}{\mathbb{Q}}_\beta} \mathbb{p}(\beta)$" for all $\beta < \alpha$. fs-5

[(50)] The set $\{\beta < \gamma : \mathbb{p}(\beta) \neq \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}\}$ is called the support of $\mathbb{p} \in \mathbb{P}_\gamma$ and denoted by $\mathrm{supp}(\mathbb{p})$.

Remark 4.1 (1) $\underset{\sim}{\mathbb{Q}}_\delta$ *plays no role at all in this definition. Because of this, an iteration is also often introduced as* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ *in the literature. We sometimes also adopt this notation.* R-fs-0

(2) (4.3) *actually follows from* (4.1) $\sim$ (4.5) *without* (4.3) *(see below).*

Lemma 4.2 *Suppose that* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ *is a finite support iteration.* L-fs-0

(1) *If* $\delta_0 \leq \delta$*, then* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta_0 \rangle$ *is also a finite support iteration.*

(2) *For all* $\alpha \leq \delta$ *and* $\mathbb{p} \in \mathbb{P}_\alpha$*,* $\mathrm{supp}(\mathbb{p})$ *is finite.*

(3) $\mathbb{P}_0 = \{\emptyset\}$ *and* $\mathbb{1}_{\mathbb{P}_0} = \emptyset$*.*

(4) *For* $\beta \leq \alpha \leq \delta$ *and* $\mathbb{p}$*,* $\mathbb{p}' \in \mathbb{P}_\alpha$*, if* $\mathbb{p}' \leq_{\mathbb{P}_\alpha} \mathbb{p}$ *then* $\mathbb{p}' \restriction \beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$*.*

(5) *For any* $\beta < \delta$*,* $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$*.*

(6) *For any limit* $\eta < \delta$ *and* $\mathbb{p}$*,* $\mathbb{p}' \in \mathbb{P}_\eta$*,* $\mathbb{p}' \leq_{\mathbb{P}_\eta} \mathbb{p}$ *if and only if* $\mathbb{p}' \restriction \beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$ *for all* $\beta < \eta$*.*

Proof. (1): If (4.1) $\sim$ (4.7) hold for $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ then they also hold for $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta_0 \rangle$.

(2): If α is a limit ordinal then this is simply (4.5). If $\alpha = \gamma + n$ for a limit ordinal γ then, for $\mathbb{p} \in \mathbb{P}_\alpha$, $\mathbb{p} \restriction \gamma \in \mathbb{P}_\gamma$ by (4.3) and $\mathrm{supp}(\mathbb{p}) \subseteq \mathrm{supp}(\mathbb{p} \restriction \gamma) \cup \{\gamma, \gamma+1, ..., \gamma+(n-1)\}$.

(3): This follows from (4.1).

(4): This is obvious by (4.7).

(5): The mapping

$$i : \mathbb{P}_{\beta+1} \to \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta;\ \mathbb{p} \mapsto \langle \mathbb{p} \restriction \beta, \mathbb{p}(\beta) \rangle \tag{4.8}$$

is an isomorphism: i is 1-1 by definition and i is onto by (4.4). i strictly preserves the ordering by (4.7) and (4).

(6): The direction "$\Rightarrow$" follows from (4).

For "$\Leftarrow$", suppose that $\mathbb{p}' \restriction \beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$ holds for all $\beta < \eta$. Since η is a limit ordinal, we have $\beta + 1 < \eta$ for $\beta < \eta$. By the assumption above, we have $\mathbb{p}' \restriction (\beta+1) \leq_{\mathbb{P}_{\beta+1}} \mathbb{p} \restriction (\beta+1)$. By (4.7), it follows that $\mathbb{p}' \restriction \beta \Vdash_\beta$ "$\mathbb{p}'(\beta) \leq_{\underset{\sim}{\mathbb{Q}}_\beta} \mathbb{p}(\beta)$". Since $\beta < \eta$ was arbitrary, it follows that $\mathbb{p}' \leq_{\mathbb{P}_\eta} \mathbb{p}$ by (4.7). ❑ (Lemma 4.2)

For a finite support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$, $\alpha \leq \beta \leq \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$, let $\mathbb{p}^\frown \vec{\mathbb{1}}_\beta$ be the sequence s of length β defined by

$$s(\xi) = \begin{cases} \mathbb{p}(\xi) & \text{if } \xi < \alpha; \\ \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\xi} & \text{if } \xi \in \beta \setminus \alpha. \end{cases} \tag{4.9}$$ L-fs

Lemma 4.3 *Suppose that* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \le \delta \rangle$ *is a finite support iteration,* $\alpha \le \beta \le \gamma \le \delta$ *and* $\mathbb{p} \in \mathbb{P}_\alpha$. L-fs-1

(1) $\mathbb{p}^\frown \vec{\mathbb{1}}_\beta \in \mathbb{P}_\beta$.

(2) *If* $\mathbb{q} \in \mathbb{P}_\beta$ *and* $\mathbb{p} \le_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$, *then* $\mathbb{r} = \mathbb{p} \cup (\mathbb{q} \restriction (\beta \setminus \alpha)) \in \mathbb{P}_\beta$ *and*

(4.10) $\mathbb{r} = \mathbb{p}^\frown \vec{\mathbb{1}}_\beta \wedge \mathbb{q}$ *holds in* $\mathbb{P}_\beta$. fs-6

(3) $i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$; $\mathbb{p} \mapsto \mathbb{p}^\frown \vec{\mathbb{1}}_\beta$ *is a complete embedding.*

(4) $i_{\alpha,\gamma} = i_{\beta,\gamma} \circ i_{\alpha,\beta}$.

Proof. (1): By induction on β (for each fixed α and $\alpha \le \beta \le \gamma$).

(2): By induction on β (for each fixed α and $\alpha \le \beta \le \gamma$). We prove first $\mathbb{r} \in \mathbb{P}_\beta$ and then (4.10).

Case 0. $\beta = \alpha$: $\mathbb{r} = \mathbb{p} \in \mathbb{P}_\alpha = \mathbb{P}_\beta$ and $\mathbb{p} \le_{\mathbb{P}_\beta} \mathbb{q}$.

Case 1. $\beta = \beta_0 + 1$: Suppose that $\mathbb{q} \in \mathbb{P}_\beta$ is such that $\mathbb{p} \le_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. By induction hypothesis, we have $\mathbb{r} \restriction \beta_0 = \mathbb{p} \cup ((\mathbb{q} \restriction \beta_0) \restriction (\beta_0 \setminus \alpha)) \in \mathbb{P}_{\beta_0}$ and

(4.11) $\mathbb{r} \restriction \beta_0 = \mathbb{p}^\frown \vec{\mathbb{1}}_{\beta_0} \wedge (\mathbb{q} \restriction \beta_0)$. fs-7

In particular $(\mathbb{r} \restriction \beta_0) \le_{\mathbb{P}_{\beta_0}} (\mathbb{q} \restriction \beta_0)$. Since $\mathbb{r}(\beta_0) = \mathbb{q}(\beta_0)$, we have $(\mathbb{r} \restriction \beta_0) \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \in \underset{\sim}{\mathbb{Q}}_{\beta_0}$" and $(\mathbb{r} \restriction \beta_0) \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{q}(\beta_0)$". It follows that $\mathbb{r} \in \mathbb{P}_\beta$ by (4.4) and $\mathbb{r} \le_{\mathbb{P}_\beta} \mathbb{q}$ by (4.7).

By (4.11) and, since $\mathbb{r} \restriction \beta_0 \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_{\beta_0}}$", we have $\mathbb{r} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_\beta$.

Suppose that $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_\beta, \mathbb{q}$. Then we have $\mathbb{s} \restriction \beta_0 \le_{\mathbb{P}_{\beta_0}} \mathbb{p}^\frown \vec{\mathbb{1}}_{\beta_0}, \mathbb{q} \restriction \beta_0$. Hence, by (4.11), $\mathbb{s} \restriction \beta_0 \le_{\mathbb{P}_{\beta_0}} \mathbb{r} \restriction \beta_0$. Since $\mathbb{r}(\beta_0) = \mathbb{q}(\beta_0)$, we have $\mathbb{s} \restriction \beta_0 \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{s}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{r}(\beta_0)$". Thus $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{r}$.

Case 2. β is a limit ordinal: In this case, we have $|\operatorname{supp}(\mathbb{r})| < \aleph_0$ since $\operatorname{supp}(\mathbb{r}) \subseteq \operatorname{supp}(\mathbb{p}) \cup \sup(\mathbb{q})$. Since $\mathbb{r} \restriction \xi = \mathbb{p} \cup ((\mathbb{q} \restriction \xi) \restriction (\xi \setminus \alpha)) \in \mathbb{P}_\xi$ for all $\alpha \le \xi < \beta$, it follows that $\mathbb{r} \in \mathbb{P}_\beta$. Now (4.10) follows from Lemma 4.2, (6) and induction hypothesis.

(3): $i_{\alpha,\beta}(\mathbb{1}_{\mathbb{P}_\alpha}) = \mathbb{1}_{\mathbb{P}_\alpha}{}^\frown \vec{\mathbb{1}}_\beta = \mathbb{1}_{\mathbb{P}_\beta}$.

If $\mathbb{p}' \le_{\mathbb{P}_\alpha} \mathbb{p}$ then $i_{\alpha,\beta}(\mathbb{p}') = \mathbb{p}'^\frown \vec{\mathbb{1}}_\beta \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_\beta = i_{\alpha,\beta}(\mathbb{p})$ by (2).

If $i_{\alpha,\beta}(\mathbb{p}) = \mathbb{p}^\frown \vec{\mathbb{1}}_\beta$ and $i_{\alpha,\beta}(\mathbb{p}') = \mathbb{p}'^\frown \vec{\mathbb{1}}_\beta$ are compatible in $\mathbb{P}_\beta$, say $\mathbb{r} \le_{\mathbb{P}_\beta} i_{\alpha,\beta}(\mathbb{p})$, $i_{\alpha,\beta}(\mathbb{p}')$, then $\mathbb{r} \restriction \alpha \le_{\mathbb{P}_\alpha} i_{\alpha,\beta}(\mathbb{p}) \restriction \alpha = \mathbb{p}$, $i_{\alpha,\beta}(\mathbb{p}') \restriction \alpha = \mathbb{p}'$. and hence $\mathbb{p}$ and $\mathbb{p}'$ are compatible in $\mathbb{P}_\alpha$.

Now suppose that $\mathbb{q} \in \mathbb{P}_\beta$. We show that $\mathbb{q} \restriction \alpha$ is a reduction of q for $i_{\alpha,\beta}$ in the sense of (2.34). Let $\mathbb{r} \le_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. Then,

(4.12) $\mathbb{r} \cup \mathbb{q} \restriction (\beta \setminus \alpha) \le_{\mathbb{P}_\beta} \underbrace{\mathbb{r}^\frown \vec{\mathbb{1}}_\beta}_{= i_{\alpha,\beta}(\mathbb{r})}, \mathbb{q}$ fs-8

by (2). Thus $i_{\alpha,\beta}(\mathbb{r})$ and $\mathbb{q}$ are compatible in $\mathbb{P}_\beta$.

(4): Clear by the definition of the complete embeddings in (3). ❑ (Lemma 4.3)

One of the important properties of the finite support iteration is the preservation of chain conditions (see Proposition 4.5). The next lemma is the main building block for the proof of this preservation theorem. $\tilde{i}_{\alpha,\beta}$ の説明をここに加える.

Lemma 4.4 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ is a finite support iteration. If $\mathbb{p}$, $\mathbb{q} \in \mathbb{P}_\delta$ are such that, for an $\alpha < \delta$, $\mathbb{p} \restriction \alpha$ and $\mathbb{q} \restriction \alpha$ are compatible in $\mathbb{P}_\alpha$ and, $\mathrm{supp}(\mathbb{p}) \setminus \alpha$ and $\mathrm{supp}(\mathbb{q}) \setminus \alpha$ are disjoint. Then $\mathbb{p}$ and $\mathbb{q}$ are compatible in $\mathbb{P}_\delta$.* L-fs-2
Furthermore, if there is $\mathbb{r}_\alpha = \mathbb{p} \restriction \alpha \wedge^{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. Then we can find $\mathbb{r} = \mathbb{p} \wedge^{\mathbb{P}_\delta} \mathbb{q}$.

Proof. By induction on β with $\alpha \leq \beta \leq \delta$ we define a sequence $\langle \mathbb{r}_\beta : \alpha \leq \beta \leq \delta \rangle$ such that

(4.13) $\mathbb{r}_\beta \in \mathbb{P}_\beta$ for all $\alpha \leq \beta \leq \delta$; fs-9

(4.14) $\langle \mathbb{r}_\beta : \alpha \leq \beta \leq \delta \rangle$ is an increasing sequence of sequences fs-10

(4.15) $\mathbb{r}_\beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$, $\mathbb{q} \restriction \beta$; fs-11

(4.16) $\mathrm{supp}(\mathbb{r}_\beta) = \mathrm{supp}(\mathbb{r}_\alpha) \cup (\mathrm{supp}(\mathbb{p}) \cap (\beta \setminus \alpha)) \cup (\mathrm{supp}(\mathbb{q}) \cap (\beta \setminus \alpha))$. fs-11-0

Having constructed such a sequence, we are done since then we have

(4.17) $\mathbb{r}_\delta \leq_{\mathbb{P}_\delta} \mathbb{p}$, $\mathbb{q}$. fs-12

The following three cases show that the construction of $\mathbb{r}_\beta$'s as above is possible:

Case 0. $\beta = \alpha$: Let $\mathbb{r}_\alpha \in \mathbb{P}_\alpha$ be any common extension of $\mathbb{p} \restriction \alpha$ and $\mathbb{q} \restriction \alpha$ (which exists by the assumption).

Case I. $\beta = \beta_0 + 1$ and we have constructed $\langle \mathbb{r}_\beta : \alpha \leq \beta < \beta_0 \rangle$ in accordance with (4.13) $\sim$ (4.16). Then we have in particular $\mathbb{r}_{\beta_0} \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta_0$, $\mathbb{q} \restriction \beta_0$. If $\beta_0 \notin \mathrm{supp}(\mathbb{p})$ and $\beta_0 \notin \mathrm{supp}(\mathbb{q})$, then we may set $\mathbb{r}_\beta = \mathbb{r}_{\beta_0} \frown \vec{\mathbb{1}}_\beta$. Otherwise only one of $\beta_0 \in \mathrm{supp}(\mathbb{p})$ or $\beta_0 \in \mathrm{supp}(\mathbb{q})$ holds. Say $\beta_0 \in \mathrm{supp}(\mathbb{p})$. Then (by Lemma 4.3(2)) $\mathbb{r}_{\beta_0} \cup \{\langle \beta_0, \mathbb{p}(\beta_0) \rangle\}$ is as desired.

Case II. β is a limit ordinal and we have constructed $\langle \mathbb{r}_\beta : \alpha \leq \beta < \beta_0 \rangle$ in accordance with (4.13) $\sim$ (4.16). Then $\mathbb{r}_\beta = \bigcup_{\xi < \beta} \mathbb{r}_\xi \in \mathbb{P}_\xi$ by (4.16). By Lemma 4.2, (6), this $\mathbb{r}_\beta$ is as desired.

For the last paragraph of the statement of the Lemma, we obtain $\mathbb{r}_\delta = \mathbb{p} \wedge^{\mathbb{P}_\delta} \mathbb{q}$ by starting with $\mathbb{r}_\alpha = \mathbb{p} \restriction \alpha \wedge^{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. ❑ (Lemma 4.4)

Proposition 4.5 *Suppose that κ is a regular uncountable cardinal and $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ a finite support iteration such that $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is $\check{\kappa}$-cc" for all $\alpha < \delta$. Then $\mathbb{P}_\delta$ has the κ-cc. In particular, for an iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \leq \delta \rangle$ as above, $\mathbb{P}_\alpha$ is κ-cc for all $\alpha \leq \delta$.* L-fs-3

Proof. Note that the last sentence of the claim follows from the assertion above it by Lemma 4.2, (1).

By induction on $\xi \in \mathsf{On}$, we prove

$(*)_\xi$ For any finite support iteration of the length $\delta \le \xi$, the assertion of the proposition holds.

For $\xi = 0$, the assertion is trivial. Suppose that the assertion $(*)_{\xi'}$ holds for all $\xi' < \xi$. Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \le \xi \rangle$ be a finite support iteration such that $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is $\check{\kappa}$-cc" for all $\alpha < \xi$. We have to show that $\mathbb{P}_\xi$ has the κ-cc.

Note that, by induction hypothesis and Lemma 4.2, (1), each $\mathbb{P}_\alpha$, $\alpha < \xi$ has the κ-cc.

If $\xi = \xi_0 + 1$, then $\mathbb{P}_\xi \cong \mathbb{P}_{\xi_0} * \underset{\sim}{\mathbb{Q}}_{\xi_0}$ by Lemma 4.2, (5). $\mathbb{P}_{\xi_0}$ has the κ-cc by induction hypothesis. Thus, by Lemma 3.16, it follows that $\mathbb{P}_\xi$ is κ-cc.

Now, suppose that ξ is a limit ordinal and $\mathbb{p}_\eta \in \mathbb{P}_\xi$, $\eta < \kappa$. By Δ-System Lemma (Corollary 1.33) there is $I \in [\kappa]^\kappa$ such that $\{\mathrm{supp}(\mathbb{p}_\eta) : \eta \in I\}$ forms a Δ-system with the root r. Let $\xi_0 = \max(r) + 1$. Since $\mathbb{P}_{\xi_0}$ is κ-cc by induction hypothesis, there are distinct $\eta_0, \eta_1 \in I$ such that $\mathbb{p}_{\eta_0} \restriction \xi_0$ and $\mathbb{p}_{\eta_1} \restriction \xi_0$ are compatible in $\mathbb{P}_{\xi_0}$. By Lemma 4.4, $\mathbb{p}_{\eta_0}$ and $\mathbb{p}_{\eta_1}$ are compatible in $\mathbb{P}_\delta$. ❑ (Proposition 4.5)

Lemma 4.6 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \le \delta \rangle$ is a finite support iteration with the commentating system $\langle i_{\alpha,\beta} : \alpha < \beta \le \delta \rangle$ of complete embeddings as defined in Lemma 4.3, (3). Then there is a finite support iteration $\langle \mathbb{P}^*_\alpha, \underset{\sim}{\mathbb{Q}}^*_\alpha : \alpha \le \delta \rangle$ with the corresponding commentating system $\langle i^*_{\alpha,\beta} : \alpha < \beta \le \delta \rangle$ of complete embeddings and dense embeddings $k_\alpha : \mathbb{P}_\alpha \to \mathbb{P}^*_\alpha$ such that* L-fs-3-a

(4.18) $\Vdash_{\mathbb{P}^*_\alpha}$ "$\underset{\sim}{\mathbb{Q}}^*_\alpha$ *is a cBa poset" for all $\alpha \le \delta$ and* fs-12-0

(4.19)
$$\begin{tikzcd} \mathbb{P}^*_\alpha \arrow[r, "i^*_{\alpha,\beta}"] & \mathbb{P}^*_\beta \\ \mathbb{P}_\alpha \arrow[u, "k_\alpha"] \arrow[r, "i_{\alpha,\beta}"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{P}_\beta \arrow[u, "k_\beta"'] \end{tikzcd} \qquad \text{for all } \alpha \le \beta \le \delta.$$
fs-12-1

Proof. ❑ (Lemma 4.6)

The following Proposition shows that any non-trivial finite support iteration adds always many Cohen reals. This implies that we have to choose some other type of iteration different from the finite support iteration to obtain a generic extension in which no Cohen reals should be added.

Proposition 4.7 *Suppose that δ is a limit ordinal with $cf(\alpha) = \kappa$ and $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \le \delta \rangle$ is a finite support iteration. If $\{\alpha < \delta : \Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is non-trivial atomless poset"$\}$ is cofinal in δ, then, for any $\alpha^* < \delta$, $\mathbb{P}_{\alpha^*} \times \mathrm{Fn}(\kappa, 2)$ is completely embeddable in $RO(\mathbb{P}_\delta)$ over $\mathbb{P}_{\alpha^*}$.* L-fs-3-a-0

Proof. By Lemma 4.6, we may assume that $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is cBa poset" for all $\alpha < \delta$ holds.

Let $D \subseteq \delta \setminus \alpha^*$ be a set of order type κ cofinal in δ such that $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is a non-trivial poset" for all $\alpha \in D$. Let $\langle \alpha_\xi : \xi < \kappa \rangle$ be a strictly increasing enumeration of D.

For each $\xi < \kappa$ let $\underset{\sim}{\mathfrak{q}}_{\alpha_\xi}$ and $\underset{\sim}{\mathfrak{r}}_{\alpha_\xi}$ be a $\mathbb{P}_{\alpha_\xi}$-name such that

(4.20) $\Vdash_{\mathbb{P}_{\alpha_\xi}}$ "$\underset{\sim}{\mathfrak{q}}_{\alpha_\xi}, \underset{\sim}{\mathfrak{r}}_{\alpha_\xi} \in \underset{\sim}{\mathbb{Q}}_{\alpha_\xi}$" and fs-12-1-0

(4.21) $\Vdash_{\mathbb{P}_{\alpha_\xi}}$ "$\underset{\sim}{\mathfrak{q}}_{\alpha_\xi} \perp_{\underset{\sim}{\mathbb{Q}}_{\alpha_\xi}} \underset{\sim}{\mathfrak{r}}_{\alpha_\xi}$ and $\underset{\sim}{\mathfrak{q}}_{\alpha_\xi} \vee \underset{\sim}{\mathfrak{r}}_{\alpha_\xi} \equiv \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_{\alpha_\xi}}$". fs-12-1-1

For each $\langle \mathbb{s}, \mathbb{f} \rangle \in \mathbb{P}_{\alpha^*} \times \mathrm{Fn}(\kappa, 2)$, let $i(\langle \mathbb{s}, \mathbb{f} \rangle)$ be the condition $\mathbb{p} \in \mathbb{P}_\delta$ such that

(4.22) $\mathrm{supp}(\mathbb{p}) = \mathrm{supp}(\mathbb{s}) \cup \{\alpha_\xi : \xi \in \mathrm{dom}(\mathbb{f})\}$; fs-12-1-2

(4.23) $\mathbb{p} \restriction \alpha^* = \mathbb{s}$; fs-12-1-3

(4.24) $$\mathbb{p}(\alpha_\xi) = \begin{cases} \underset{\sim}{\mathfrak{q}}_{\alpha_\xi}, & \text{if } \mathbb{f}(\xi) = 1, \\ \underset{\sim}{\mathfrak{r}}_{\alpha_\xi}, & \text{otherwise.} \end{cases}$$ fs-12-1-4
for $\xi \in \mathrm{dom}(\mathbb{f})$,

We want to show that $i : \mathbb{P}_{\alpha^*} \times \mathrm{Fn}(\omega, 2) \to \mathbb{P}_\delta$ is a complete embedding.

Clearly we have $i(\langle \mathbb{1}_{\mathbb{P}_{\alpha^*}}, \emptyset \rangle) = \mathbb{1}_{\mathbb{P}_\delta}$ and i is order and incompatibility preserving.

For $\mathbb{p} \in \mathbb{P}_\delta$, let $\mathbb{p}' \leq_{\mathbb{P}_\delta} \mathbb{p}$ be such that, for all $\alpha \in \mathrm{supp}(\mathbb{p}') \cap D$ with $\alpha = \alpha_\xi$ for some $\xi < \kappa$, $\mathbb{p}' \restriction \alpha$ decides whether $\mathbb{p}'(\alpha) \top_{\mathbb{P}_\alpha} \underset{\sim}{\mathfrak{q}}_{\alpha_\xi}$. Let $D_0 = \{\xi < \kappa : \alpha_\xi \in \mathrm{supp}(\mathbb{p}') \cap D\}$ and let $\mathbb{f} : D_0 \to 2$ be such that

(4.25) $$\mathbb{f}(\xi) = \begin{cases} 1, & \text{if } \mathbb{p}' \restriction \alpha_\xi \Vdash_{\mathbb{P}_{\alpha_\xi}} \text{"}\mathbb{p}'(\alpha) \top_{\mathbb{P}_\alpha} \underset{\sim}{\mathfrak{q}}_{\alpha_\xi}\text{"}; \\ 0, & \text{otherwise.} \end{cases}$$ fs-12-1-5

Then $\langle \mathbb{p}' \restriction \alpha^*, \mathbb{f} \rangle$ is a projection of $\mathbb{p}'$ and hence also of $\mathbb{p}$ for i. ❑ (Proposition 4.7)

For a poset $\mathbb{P}$ and $\mathbb{P}$-name $\underset{\sim}{a}$, the hereditary range $hr_{\mathbb{P}}(\underset{\sim}{a})$ of $\underset{\sim}{a}$ is defined analogously to the hereditary domain of $\underset{\sim}{a}$ (see 33). Here we define both hereditary domain and hereditary range simultaneously by induction as:

(4.26) $hr^0_{\mathbb{P}}(\underset{\sim}{a}) = \mathrm{range}(\underset{\sim}{a})$; fs-12-2

(4.27) $hd^0_{\mathbb{P}}(\underset{\sim}{a}) = \mathrm{dom}(\underset{\sim}{a})$;[51] fs-12-3

(4.28) $hr^{n+1}_{\mathbb{P}}(\underset{\sim}{a}) = hr^n_{\mathbb{P}}(\underset{\sim}{a}) \cup \mathrm{range}\,''\, hd^n_{\mathbb{P}}(\underset{\sim}{a})$; fs-12-4

(4.29) $hd^{n+1}_{\mathbb{P}}(\underset{\sim}{a}) = hd^n_{\mathbb{P}}(\underset{\sim}{a}) \cup \mathrm{dom}\,''\, hd_n(\underset{\sim}{a})$. fs-12-5

Let

(4.30) $hr_{\mathbb{P}}(\underset{\sim}{a}) = \bigcup_{n \in \omega} hr^n_{\mathbb{P}}(\underset{\sim}{a})$; fs-12-6

(4.31) $hd_{\mathbb{P}}(\underset{\sim}{a}) = \bigcup_{n \in \omega} hd^n_{\mathbb{P}}(\underset{\sim}{a})$. fs-12-7

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[51] Note that $\mathrm{dom}(\underset{\sim}{a})$ is a set of $\mathbb{P}$-names.

Note that $hr_{\mathbb{P}}(\underset{\sim}{a})$ is a set of conditions in $\mathbb{P}$ while $hd_{\mathbb{P}}(\underset{\sim}{a})$ is a set of $\mathbb{P}$-names.

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \le \delta \rangle$ be a finite support iteration. For a $\mathbb{P}_\delta$-name $\underset{\sim}{a}$ the hereditary support $\mathrm{hsupp}(\underset{\sim}{a})$ of $\underset{\sim}{a}$ is defined by:

(4.32) $\quad \mathrm{hsupp}(\underset{\sim}{a}) = \bigcup(\mathrm{supp}\,''\, hr_{\mathbb{P}_\delta}(\underset{\sim}{a}))$. fs-12-8

For $\mathbb{P}_\delta$-name $\underset{\sim}{a}$ and $\alpha < \delta$, the restriction $\underset{\sim}{a} \restriction \mathbb{P}_\alpha$ of $\underset{\sim}{a}$ to $\mathbb{P}_\alpha$ is defined recursively by

(4.33) $\quad \underset{\sim}{a} \restriction \mathbb{P}_\alpha = \{\langle \underset{\sim}{b} \restriction \mathbb{P}_\alpha, \mathbb{p} \restriction \alpha \rangle : \langle \underset{\sim}{b}, \mathbb{p} \rangle \in \underset{\sim}{a}\}$. fs-12-9

For $\alpha \le \beta \le \delta$ let $i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$ be the complete embedding given in Lemma 4.3, (3) and $\tilde{i}_{\alpha,\beta} : \mathsf{V}^{\mathbb{P}_\alpha} \to \mathsf{V}^{\mathbb{P}_\beta}$ the class mapping associated with i (in the sense of (2.51)).

Lemma 4.8 (1) $\underset{\sim}{a} \restriction \mathbb{P}_\alpha$ *is a* $\mathbb{P}_\alpha$*-name.* *L-fs-3-a-1*

(2) *Suppose* $\mathrm{hsupp}(\underset{\sim}{a}) \subseteq \alpha$ *for some* $\alpha < \kappa$. *Then* $\tilde{i}_{\alpha,\alpha'}(\underset{\sim}{a} \restriction \mathbb{P}_\alpha) = \underset{\sim}{a} \restriction \mathbb{P}_{\alpha'}$ *for any* $\alpha < \alpha' \le \delta$. *In particular,* $\tilde{i}_{\alpha,\delta}(\underset{\sim}{a} \restriction \mathbb{P}_\alpha) = \underset{\sim}{a}$.

(3) *Suppose that* $\mathbb{G}$ *is a* $(\mathsf{V}, \mathbb{P}_\delta)$*-generic filter,* $\mathbb{G}_\alpha = (i_{\alpha,\delta})^{-1}\,''\mathbb{G}$ *and* $\mathbb{G}_{\alpha'} = (i_{\alpha',\delta})^{-1}\,''\mathbb{G}$ *for* α, α' *as above. Then we have* $\underset{\sim}{a} \restriction \mathbb{P}_\alpha[\mathbb{G}_\alpha] = \underset{\sim}{a} \restriction \mathbb{P}_{\alpha'}[\mathbb{G}_{\alpha'}] = \underset{\sim}{a}[\mathbb{G}]$.

Proof. By induction on $rank(\underset{\sim}{a})$. ❑ (Lemma 4.8)

4.2 Martin's Axiom

For a poset $\mathbb{P}$ and a set (family) $\mathcal{D}$ of dense subsets of $\mathbb{P}$, a filter $\mathbb{G} \subseteq \mathbb{P}$ is said to be a *$\mathcal{D}$-generic filter* (over $\mathbb{P}$) if $\mathbb{G} \cap D \ne \emptyset$ for all $D \in \mathcal{D}$. Using this terminology, $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter if and only if it is a $\mathcal{D}$-generic filter over $\mathbb{P}$ for $\mathcal{D} = \{D : D$ is a dense subset of $\mathbb{P}\}$. MA

Martin's Axiom (MA) is the following assertion:

(MA): For any ccc poset $\mathbb{P}$ if $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ of size $< 2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter (in V).

The condition "of size $< 2^{\aleph_0}$" cannot be weakened: Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$. $\mathbb{P}$ is ccc. For $f \in {}^\omega 2$, let $D_f = \{\mathbb{p} \in \mathbb{P} : \mathbb{p} \ne f \restriction \mathrm{dom}(\mathbb{p})\}$. For $n \in \omega$, let $E_n = \{\mathbb{p} \in \mathbb{P} : n \in \mathrm{dom}(\mathbb{p})\}$. Let $\mathcal{D} = \{D_f : f \in {}^\omega 2\} \cup \{E_n : n \in \omega\}$. Then $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ of cardinality $2^{\aleph_0}$. There is no $\mathcal{D}$-generic filter over $\mathbb{P}$ (in V): If $\mathbb{G}$ ($\in \mathsf{V}$) were a $\mathcal{D}$-generic filter then by the genericity for E_n,s, $g = \bigcup \mathbb{G}$ would a mapping from ω to 2. But by genericity for D_f's, $g \ne f$ for all $f \in {}^\omega 2$.

The same idea as the proof of Lemma 1.6 shows also the following:

Lemma 4.9 CH *implies* MA. *L-fs-3-0*

Proof. Under CH, $|\mathcal{D}| < 2^{\aleph_0}$ is equivalent to $|\mathcal{D}| \le \aleph_0$.

Suppose that $\mathbb{P}$ is a poset ($\mathbb{P}$ can be even an arbitrary poset not necessarily satisfying ccc) and $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < 2^{\aleph_0}$.

Let $\langle D_n : n \in \omega \rangle$ be an enumeration of $\mathcal{D}$. Then we can construct a decreasing sequence $\mathbb{p}_0 \geq_{\mathbb{P}} \mathbb{p}_1 \geq_{\mathbb{P}} \cdots$ of elements of $\mathbb{P}$ such that $\mathbb{p}_n \in D_n$ for all $n \in \omega$. This is possible since each D_n is dense in $\mathbb{P}$.

The filter $\mathbb{G}$ generated from $\{\mathbb{p}_n : n \in \omega\}$ is $\mathcal{D}$-generic. ❑ (Lemma 4.9)

For a cardinal κ, let $\mathsf{MA}(\kappa)$ be the following statement:

$\mathsf{MA}(\kappa)$: For any ccc poset $\mathbb{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$.

Thus, MA is equivalent to the assertion that $\mathsf{MA}(\kappa)$ holds for all $\kappa < 2^{\aleph_0}$.

This notation can be yet generalized to the following "Forcing Axiom" notation: For a class $\mathcal{C}$ of posets and a cardinal κ, let

$\mathsf{FA}(\mathcal{C}, \kappa)$: For any $\mathbb{P} \in \mathcal{C}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|D| \leq \kappa$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$.

If we denote the class of all ccc posets with ccc then $MA(\kappa)$ is equivalent to $\mathsf{FA}(\mathsf{ccc}, \kappa)$.

Let

$$\mathsf{ma} = \min\{\kappa \in \mathsf{Card} : \mathsf{MA}(\kappa) \text{ does not hold}\} \tag{4.34}$$ fs-13

or more generally, for a class $\mathcal{C}$ of posets , let

$$\mathsf{ma}(\mathcal{C}) = \min\{\kappa \in \mathsf{Card} : \mathsf{FA}(\mathcal{C}, \kappa) \text{ does not hold}\}. \tag{4.35}$$ fs-13-0

Lemma 4.10 (1) $\mathsf{FA}(\mathcal{C}, \aleph_0)$ *holds for any family of posets.* *L-fs-4*

(2) $\mathsf{FA}(\{\mathrm{Fn}(\omega, 2)\}, 2^{\aleph_0})$ *does not hold.*

(3) *If* $\mathrm{Fn}(\omega, 2) \in \mathcal{C}$ *then* $\aleph_0 < \mathsf{ma}(\mathcal{C}) \leq 2^{\aleph_0}$. *In particular,* $\aleph_0 < \mathsf{ma} \leq 2^{\aleph_0}$.

Proof. (1): By the proof of Lemma 4.9.

(2): By the argument above Lemma 4.9.

(3): By (1) and (2). ❑ (Lemma 4.10)

The cardinal invariant ma is known to be smaller than all the classical cardinal invariants of reals. In particular it is less than or equal to the towering number $\mathfrak{t}$ defined as follows:

For A, $B \in [\omega]^{\aleph_0}$, A is *almost included in* B (notation: $A \subseteq^* B$) if $A \setminus B$ is finite. $D \subseteq [\omega]^{\aleph_0}$ has the *strong finite intersection property* (abbreviation: s.f.i.p.) if $\bigcap D_0$ is infinite for any $D_0 \in [D]^{<\aleph_0}$. $A^* \in [\omega]^{\aleph_0}$ is a *pseudo intersection* of $D \subseteq [\omega]^{\aleph_0}$ if $A^* \subseteq^* B$ for all $B \in D$. $D \subseteq [\omega]^{\aleph_1}$ is a *tower* if D is linearly ordered with respect to $\subseteq^*$. Note that any tower has the s.f.i.p.

(4.36) $\mathfrak{p} = \min\{|\,D\,| : D$ has the s.f.i.p. but without any pseudo intersection$\}$. fs-14

(4.37) $\mathfrak{t} = \min\{|\,D\,| : D$ is a tower without any pseudo intersection$\}$. fs-15

Lemma 4.11 (1) $\aleph_0 < \mathfrak{p} \le \mathfrak{t} \le 2^{\aleph_0}$.[52] *L-fs-4-0*

(2) *If D is a tower without any pseudo intersection then there is a regular cardinal $\kappa \le |\,D\,|$ and a sequence $\vec{d} = \langle d_\alpha : \alpha < \kappa\rangle$ in D such that $\vec{d}$ is a decreasing sequence with respect to $\subseteq^*$ and there is no pseudo intersection of $D_0 = \{d_\alpha : \alpha < \kappa\}$.*

(3) *$\mathfrak{t}$ is a regular cardinal.*

(4) *$\aleph_0 \le \kappa < \mathfrak{t}$ implies that $2^\kappa \le 2^{\aleph_0}$. In particular, if $\mathfrak{t} = 2^{\aleph_0}$ then $2^\kappa = 2^{\aleph_0}$ for all infinite $\kappa < 2^{\aleph_0}$ and $2^{\aleph_0}$ is regular.*

Proof. (1): $\aleph_0 < \mathfrak{p}$ follows from Lemma 4.10, (3) and Lemma 4.12, (1) below. (Exercise: Find a direct proof).

$\mathfrak{p} \le \mathfrak{t}$ follows from the definitions (4.36), (4.37) and the fact that each tower has the s.f.i.p.

To show that $\mathfrak{t} \le 2^{\aleph_0}$, let $D \subseteq [\omega]^{\aleph_0}$ be a tower which is maximal with respect to $\subseteq$ among towers. Then D does not have any pseudo intersection. It follows that $\mathfrak{t} \le |\,D\,| \le 2^{\aleph_0}$.

(2): By induction, we construct first a $\subseteq^*$-descending sequence $\vec{c} = \langle c_\xi : \xi < \delta\rangle$ cofinal in D for some ordinal δ such that $\vec{c}$ cannot be further extended anymore. Then δ is a limit ordinal and $\{c_\xi : \xi < \delta\}$ does not have any pseudo intersection. Let $\kappa = cf(\delta)$ and let $\langle \xi_\alpha : \alpha < \kappa\rangle$ be strictly increasing sequence cofinal in δ. Letting $d_\alpha = c_{\xi_\alpha}$ for $\alpha < \kappa$, $\vec{d} = \langle d_\alpha : \alpha < \kappa\rangle$ is as desired.

(3) follows from (2).

(4): Suppose that $\aleph_0 \le \kappa \le \mathfrak{t}$. $2^\kappa \ge 2^{\aleph_0}$ is trivial and holds without the assumption $\kappa < \mathfrak{t}$. To prove $2^\kappa \le 2^{\aleph_0}$, we construct inductively a mapping $f : {}^{\kappa\ge}2 \to \mathcal{P}(\omega)$ and show that f is 1-1.

Let $f : {}^{\kappa\ge}2 \to \mathcal{P}(\omega)$ be a mapping such that

(4.38) $f(\emptyset) = \omega$; fs-20

(4.39) for $t, t' \in {}^{\kappa\ge}2$, if $t \subseteq t'$ then we have $f(t) \supseteq^* f(t')$; fs-21

(4.40) $f(t^\frown\langle 0\rangle)$ and $f(t^\frown\langle 1\rangle)$ are disjoint. fs-22

The $\subseteq$-increasing sequence $\langle f \restriction {}^\alpha 2 : \alpha \le \kappa\rangle$ is constructed by induction on $\alpha \le \kappa$: Suppose that $\langle f(t) : t \in {}^{\alpha>}2\rangle$ for some $\alpha \le \kappa$ has been constructed in accordance with (4.38) $\sim$ (4.40). If $\alpha = \alpha_0 + 1$, then for each $t \in {}^{\alpha_0}2$, let $a_t \subseteq f(t)$ be such that a_t and $f(t) \setminus a_t$ are both infinite. Let $f(t^\frown\langle 0\rangle) = a_t$ and $f(t^\frown\langle 1\rangle) = f(t) \setminus a_t$.

If α is a limit, then $\{f(t \restriction \beta) : \beta < \alpha\}$ for each $t \in {}^{\alpha>}2$ is a tower by (4.39). Since $\alpha \le \kappa < \mathfrak{t}$, there is a pseudo intersection b_t of $\{f(t \restriction \beta) : \beta < \alpha\}$. Let $f(t) = b_t$.

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[52] Actually $\mathfrak{t} = \mathfrak{p}$ is recently proved by Milliaris and Shelah (in ZFC).

In both cases, the resulting $f(t \restriction \beta)$, $\beta \leq \alpha$ are in accordance with (4.38) $\sim$ (4.40).

To prove that f is 1-1, let t, $t' \in {}^{\kappa\geq}2$ with $t \neq t'$. If $t \subsetneqq t'$, let $\alpha = \ln(t)$ and suppose that $t'(\alpha) = 0$. Then $f(t') \subseteq^* a_t \subsetneqq^* f(t)$ and thus $f(t) \neq f(t')$. The case with $t'(\alpha) = 1$ can be treated similarly.

If t and t' are incompatible, then, $t'' \frown \langle 0 \rangle \subseteq t$ and $t'' \frown \langle 1 \rangle \subseteq t'$, say, for some t''. $f(t) \subseteq^* a_{t''}$ and $f(t') \subseteq^* f(t'') \setminus a_{t''}$. Thus we again have $f(t) \neq f(t')$.

If $t = 2^{\aleph_0}$, then $2^{\aleph_0}$ is regular by (3). ❑ (Lemma 4.11)

Lemma 4.12 (1) $\mathsf{ma} \leq \mathfrak{p}$. *L-fs-5*

(2) MA *implies* $\mathfrak{t} = \mathfrak{p} = 2^{\aleph_0}$.

(3) MA *implies that the continuum is a regular cardinal and* $2^\kappa = 2^{\aleph_0}$ *holds for all* $\aleph_0 \leq \kappa < 2^{\aleph_0}$.

Proof.

(1): Suppose that $D \subseteq [\omega]^{\aleph_0}$ has the s.f.i.p. and $|\, D \,| < \mathsf{ma}$. We have to show that D has a pseudo intersection.

Let

$$\mathbb{P} = \{\langle s, t \rangle \,:\, s \in [\omega]^{<\aleph_0},\, t \in [D]^{<\aleph_0}\}. \tag{4.41}$$ fs-16

For $\langle s, t \rangle$ $\langle s', t' \rangle \in \mathbb{P}$

$$\begin{aligned} \langle s', t' \rangle \leq_{\mathbb{P}} \langle s, t \rangle \;\Leftrightarrow\; & (\mathrm{a})\;\; s \subseteq s', \\ & (\mathrm{b})\;\; s' \cap (max(s) + 1) = s, \\ & (\mathrm{c})\;\; t \subseteq t', \\ & (\mathrm{d})\;\; s' \setminus s \subseteq b \text{ for all } b \in t. \end{aligned} \tag{4.42}$$ fs-17

Claim 4.12.1 $\mathbb{P}$ *has the ccc.* *Cl-fs-0*

$\vdash$ Suppose that $A \subseteq \mathbb{P}$ is uncountable. Then there is $s_0 \in [\omega]^{<\aleph_0}$ such that $A_0 = \{\langle s, t \rangle \in A \,:\, s = s_0\}$ is uncountable. Let $\langle s_0, t \rangle$, $\langle s_0, t' \rangle \in A_0$. Then we have $\langle s_0, t \cup t' \rangle \leq_{\mathbb{P}} \langle s_0, t \rangle$, $\langle s_0, t' \rangle$. $\dashv$ (Claim 4.12.1)

For $n \in \omega$, let

$$D_n = \{\langle s, t \rangle \in \mathbb{P} \,:\, n \leq \max(s)\}. \tag{4.43}$$ fs-18

For $b \in D$, let

$$E_b = \{\langle s, t \rangle \in \mathbb{P} \,:\, b \in t\}. \tag{4.44}$$ fs-19

Claim 4.12.2 (i) D_n *is dense in* $\mathbb{P}$ *for all* $n \in \omega$. *Cl-fs-1*

(ii) E_b *is dense in* $\mathbb{P}$ *for all* $b \in D$.

$\vdash$ (i): Suppose $\langle s,t\rangle \in \mathbb{P}$. Let $a = \cap t$. Then $|\,a\,| = \aleph_0$ since $t \in [D]^{<\aleph_0}$. Let $m \in a \setminus (\max(\{n, \max(s)\}) + 1)$. Then $\langle s \cup \{m\}, t\rangle \leq_{\mathbb{P}} \langle s,t\rangle$ and $\langle s \cup \{m\}, t\rangle \in D_n$.

(ii): Suppose $\langle s,t\rangle \in \mathbb{P}$ and $b \in D$. Then $\langle s, t \cup \{b\}\rangle \leq_{\mathbb{P}} \langle s,t\rangle$ and $\langle s, t\cup\{b\}\rangle \in E_b$.

$\dashv$ (Claim 4.12.2)

Let $\mathcal{D} = \{D_n : n \in \omega\} \cup \{E_b : b \in D\}$. $|\,\mathcal{D}\,| < \mathsf{ma}$. Hence there is an $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$. Let $a^* = \bigcup\{s : \langle s,t\rangle \in \mathbb{G}$ for some $t \in [D]^{\aleph_0}\}$.

The next Claim finishes the proof:

Claim 4.12.3 *$a^* \in [\omega]^{\aleph_0}$ and a^* is a pseudo intersection of D.* *Cl-fs-2*

$\vdash$ $|\,a^*\,| = \aleph_0$ since, for each $n \in \omega$, there is $\langle s,t\rangle \in \mathbb{G} \cap D_n$. Thus there is $m \geq n$ such that $m \in s \subseteq a^*$. For each $b \in D$, there is $\langle s,t\rangle \in \mathbb{G} \cap E_b$. By (4.42) and since $\mathbb{G}$ is a filter, it follows that $a^* \setminus (\max(s) + 1) \subseteq b$. $\dashv$ (Claim 4.12.3)

(2): By (1) and Lemma 4.11, (1). (3): By (2) and Lemma 4.11, (4).

The rest of the subsection is devoted to the proof of the following theorem:

Theorem 4.13 *For any uncountable regular cardinal κ with $2^{<\kappa} = \kappa$,*[a] *there is a ccc poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "MA holds and $2^{\aleph_0} = \kappa$".* *L-fs-6*

[a] Note that, in particular, we have $\kappa \geq 2^{\aleph_0}$.

Note that the condition that κ is regular and $2^{<\kappa} = \kappa$ is necessary by Lemma 4.12, (4) if $\mathsf{MA} + 2^{\aleph_0} = \kappa$ is to be forced by a ccc poset.

Corollary 4.14 *If ZFC is consistent then $\mathsf{ZFC} + \neg\mathsf{CH} + \mathsf{MA}$ is also consistent. In particular, we can show for example:* *L-fs-7*

(1) *If ZFC is consistent then $\mathsf{ZFC} + 2^{\aleph_0} \equiv \aleph_2 + \mathsf{MA}$ is also consistent.*

(2) *If ZFC + "there is an inaccessible cardinal" is consistent, then ZFC "$+ 2^{\aleph_0}$ is a weakly inaccessible cardinal" + MA is consistent.*

Proof. (1): Let $\mathbb{P}_0 = \mathrm{Fn}(\aleph_1, 2^{\aleph_0}, < \aleph_1)$ and let $\mathbb{G}_0$ be a $(\mathsf{V}, \mathbb{P}_0)$-generic filter. By Proposition 1.47, we have $\mathsf{V}[\mathbb{G}_0] \models$ "CH". In $\mathsf{V}[\mathbb{G}_0]$ let $\mathbb{P}_1$ be the poset $\mathrm{Fn}(\aleph_2, 2^{\aleph_1}, < \aleph_2)$ and let $\mathbb{G}_1$ be a $(\mathsf{V}[\mathbb{G}_0], \mathbb{P}_1)$-generic filter. Then similar argument shows that $\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1] \models 2^{\aleph_1} \equiv \aleph_2$. Since $\mathsf{V}[\mathbb{G}_0] \models$ "$\mathbb{P}_1$ is $< \aleph_2$-closed", CH is preserved in the generic extension $\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1]$. Thus, in $\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1]$, $\aleph_2$ (which is not necessarily the $\aleph_2$ of V or $\mathsf{V}[\mathbb{G}_0]$) satisfies the condition for κ in Theorem 4.13. Thus, in $\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1]$, there is a ccc poset $\mathbb{P}_2$ such that, for a $(\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1], \mathbb{P}_2)$-generic filter $\mathbb{G}_2$, we have $\mathsf{V}[\mathbb{G}_0][\mathbb{G}_1][\mathbb{G}_2] \models$ "$\mathsf{MA} + 2^{\aleph_0} \equiv \aleph_2$".

The construction above can be recast into a finite step iteration argument and we obtain a single poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathsf{MA} + 2^{\aleph_0} \equiv \aleph_2$". This together with Theorem 1.38 implies the desired relative consistency statement.

(2): Let κ be an inaccessible cardinal. Then κ satisfies the condition for κ in Theorem 4.13. Let $\mathbb{P}$ be the ccc poset in Theorem 4.13 for this κ. Since ccc posets preserve

cardinals and cofinality, they preserve weakly inaccessibility of cardinals. Thus we have $\Vdash_{\mathbb{P}}$ "MA holds and $2^{\aleph_0} = \kappa$ is a weakly inaccessible". Now Theorem 1.38 applied to this implies the desired relative consistency statement. ❑ (Corollary 4.14)

The technique known as the "bookkeeping"[53] is one of the main ingredients of the proof of Theorem 4.13. For the corresponding proofs for "Proper Forcing Axiom" or "Martin's Maximum" introduced later, we need a supercompact cardinals to device a bookkeeping for a consistency proof by iterated forcing. In case of the proof of Theorem 4.13 the bookkeeping can be carried out in ZFC without any additional axioms thanks to the following Lemma:

Lemma 4.15 *For any infinite cardinal κ, $\mathsf{MA}(\kappa)$ is equivalent to* L-fs-8

$\mathsf{MA}'(\kappa)$: *For any ccc poset $\mathbb{P}$ with $|\mathbb{P}| \leq \kappa$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \kappa$, there is a $\mathcal{D}$-generic filter over $\mathbb{P}$.*

Proof. "$\mathsf{MA}(\kappa) \Rightarrow \mathsf{MA}'(\kappa)$" is clear by the definition of $\mathsf{MA}'(\kappa)$.

"$\mathsf{MA}'(\kappa) \Rightarrow \mathsf{MA}(\kappa)$": Assume $\mathsf{MA}'(\kappa)$. Suppose that $\mathbb{P}$ is an arbitrary ccc poset and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$.

Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be such that $\mathbb{P}, \mathcal{D} \in M$, $\mathcal{D} \subseteq M$ and $|M| \leq \kappa$. Let $\mathbb{P}' = \mathbb{P} \cap M$. By the elementarity of M, the embedding of $\mathbb{P}'$ into $\mathbb{P}$ is incompatibility preserving. Thus $\mathbb{P}'$ is also ccc. For each $D \in \mathcal{D}$, $D \cap M$ is a dense subset of $\mathbb{P}'$ (again by the elementarity of M). Since $|\mathbb{P}'| \leq \kappa$. There is a $\{D \cap M : D \in \mathcal{D}\}$-generic filter $\mathbb{G}'$ over $\mathbb{P}'$ by $MA'(\kappa)$. The filter $\mathbb{G}$ over $\mathbb{P}$ generated by $\mathbb{G}'$ is then a $\mathcal{D}$-generic filter over $\mathbb{P}$. ❑ (Lemma 4.15)

Proof of Theorem 4.13.

Let κ be as in the statement of the theorem. Note that we have $|\mathcal{H}(\kappa)| = \kappa$ by the assumption on κ (see Lemma 2.9, (3) in [Fuchino 2017]). Let $f : \kappa \to \mathcal{H}(\kappa)$ be a surjective mapping such that, for any $a \in \mathcal{H}(\kappa)$, we have $|f^{-1}{}''\{a\}| = \kappa$. The function f in connection with the following definition of the iteration of forcing is called the "*bookkeeping function*" for the construction of the iteration. The idea here is that we want to use f as an enumeration of all possible $\mathbb{P}_\kappa$-names of ccc posets(in V_κ in $V[\mathbb{G}_\kappa]$ — and f is fixed before $\mathbb{P}_\kappa$ has been defined!) and use it in the definition of the iteration to construct $\mathbb{P}_\kappa$.

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa \rangle$ be the finite support iteration such that

(4.45) fs-23

[53] Some non-native speakers of English may misunderstand the word "bookkeeping". If you are not native in English and are not very much convinced in the naming "bookkeeping" ("buchalteria" in Polish by the way) of the method used in the following, you should consult a dictionary to check the meaning of the word.

$$\underset{\sim}{\mathbb{Q}}_\alpha = \begin{cases} \tilde{i}_{\beta,\alpha}(\underset{\sim}{\mathbb{Q}}), & \text{if } f(\alpha) = \langle \beta, \underset{\sim}{\mathbb{Q}} \rangle, \text{ for some } \beta < \alpha, \\ & \underset{\sim}{\mathbb{Q}} \text{ is a } \mathbb{P}_\beta\text{-name and} \\ & \Vdash_{\mathbb{P}_\alpha} \text{“}\tilde{i}_{\beta,\alpha}(\underset{\sim}{\mathbb{Q}}) \text{ is a ccc poset”}; \\ otr_{\mathbb{P}_\alpha}(\surd_{\mathbb{P}_\alpha}(\{\emptyset\}), \emptyset, \emptyset), & \text{otherwise} \end{cases}$$

for all $\alpha < \kappa$. Here, $\tilde{i}_{\beta,\alpha} : \mathsf{V}^{\mathbb{P}_\beta} \to \mathsf{V}^{\mathbb{P}_\alpha}$ denotes the class function induced from the complete embedding $i_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ (defined in Lemma 4.3, (3)). See (2.51).

By Lemma 4.5, all $\mathbb{P}_\alpha$, $\alpha \leq \kappa$ are ccc posets. We are done by showing the following:

Claim 4.16 (1) $\mathbb{P}_\alpha \in \mathcal{H}(\kappa)$ *for all* $\alpha < \kappa$. *Cl-fs-3*

(2) $\Vdash_{\mathbb{P}_\kappa}$ “$2^{\aleph_0} = \kappa$”.

(3) $\Vdash_{\mathbb{P}_\kappa}$ “MA”.

$\vdash$ (1): By induction on $\alpha < \kappa$. Lemma 3.6 is used for the successor steps.

(2): $\Vdash_{\mathbb{P}_\kappa}$ “$2^{\aleph_0} \geq \kappa$” follows from Lemma 4.7.

We have $|\,\mathbb{P}_\kappa\,| \leq \kappa$ by (1). Hence, by Lemma 1.46 (and the ccc of $\mathbb{P}_\kappa$), it follows that $\Vdash_{\mathbb{P}_\kappa}$ “$2^{\aleph_0} \leq \kappa$”.

(3): Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. For each $\alpha < \kappa$, let $\mathbb{G}_\alpha = (i_{\alpha,\kappa})^{-1}{}''\mathbb{G}_\kappa$. $\mathbb{G}_\alpha$ is a $(\mathsf{V}, \mathbb{P}_\alpha)$-generic filter (see Lemma 2.22, (1)).

By Lemma 4.15, it is enough to show that $\mathsf{MA}'(\mu)$ holds in $\mathsf{V}[\mathbb{G}_\kappa]$ for all $\mu < \kappa$.

In $\mathsf{V}[\mathbb{G}_\kappa]$, let $\mathbb{P}$ be a ccc poset of cardinality $< \kappa$ and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\,\mathcal{D}\,| < \kappa$. We want to show that, in $\mathsf{V}[\mathbb{G}_\alpha]$ there is a $\mathcal{D}$-generic filter on $\mathbb{P}$. Without loss of generality, we may assume that $\mathbb{P} \in \mathcal{H}(\kappa)$ (and hence also $\mathcal{D} \in \mathcal{H}(\kappa)$).

By Lemma 1.24, there are $\mathbb{P}_\kappa$-names $\underset{\sim}{\mathbb{P}}$, $\underset{\sim}{\mathcal{D}}$, such that $\underset{\sim}{\mathbb{P}}[\mathbb{G}_\kappa] = \mathbb{P}$ and all the property of these objects in $\mathsf{V}[\mathbb{G}_\kappa]$ needed later are forced by $\mathbb{1}_{\mathbb{P}_\kappa}$. In particular, we assume

(4.46) $\Vdash_{\mathbb{P}_\kappa}$ “$\underset{\sim}{\mathbb{P}}$ is a ccc poset”. [54] fs-24

By Lemma 1.31, we may assume that $\underset{\sim}{\mathbb{P}}$, $\underset{\sim}{\mathcal{D}} \in \mathcal{H}(\kappa)$. In particular $\mathrm{hsupp}(\underset{\sim}{\mathbb{P}})$ and $\mathrm{hsupp}(\underset{\sim}{\mathcal{D}})$ have cardinality $< \kappa$. Since κ is regular, there is $\beta < \kappa$ such that $\mathrm{hsupp}(\underset{\sim}{\mathbb{P}})$, $\mathrm{hsupp}(\underset{\sim}{\mathcal{D}}) \subseteq \beta$. Let $\underset{\sim}{\mathbb{Q}} = \underset{\sim}{\mathbb{P}} \restriction \mathbb{P}_\beta$ and $\underset{\sim}{\mathcal{E}} = \underset{\sim}{\mathcal{D}} \restriction \mathbb{P}_\beta$. By Lemma 4.8, $\tilde{i}_{\beta,\kappa}(\underset{\sim}{\mathbb{Q}}) = \underset{\sim}{\mathbb{P}}$ and $\tilde{i}_{\beta,\kappa}(\underset{\sim}{\mathcal{E}}) = \underset{\sim}{\mathcal{D}}$. Hence

(4.47) $\underset{\sim}{\mathbb{Q}}[\mathbb{G}_\beta] = \mathbb{P}$ and fs-24-0

(4.48) $\underset{\sim}{\mathcal{E}}[\mathbb{G}_\beta] = \mathcal{D}$. fs-24-1

[54] Note that the existence of $\mu < \kappa$ as here can be justified by using the ccc-ness of $\mathbb{P}_\kappa$ (see Lemma 1.30).

Let α be such that $\beta \le \alpha < \kappa$ and $f(\alpha) = \langle \beta, \underset{\sim}{\mathbb{Q}} \rangle$. By Lemma 4.8, we have $\tilde{i}_{\beta,\alpha}(\underset{\sim}{\mathbb{Q}}) = \underset{\sim}{\mathbb{P}} \restriction \mathbb{P}_\alpha$. By (4.46) and Lemma 2.24, (5), $\Vdash_{\mathbb{P}_\alpha}$ "$\tilde{i}_{\beta,\alpha}(\underset{\sim}{\mathbb{Q}})$ is a ccc poset". Thus, by the definition (4.45) of the finite support iteration, $\underset{\sim}{\mathbb{Q}}_\alpha = \underset{\sim}{\mathbb{P}} \restriction \mathbb{P}_\alpha$.

In $\mathsf{V}[\mathbb{G}_\kappa]$, let

(4.49) $\mathbb{G} = \{\mathbb{p}(\alpha)[\mathbb{G}_\alpha] : \mathbb{p} \in \mathbb{G}_{\alpha+1}\}$. fs-25

Then $\mathbb{G}$ is a $(\mathsf{V}[\mathbb{G}_\alpha], (\underset{\sim}{\mathbb{P}} \restriction \mathbb{P}_\alpha)[\mathbb{G}_\alpha])$-generic (and hence $(\mathsf{V}[\mathbb{G}_\alpha], \mathbb{P})$-generic) filter by Lemma 4.2, (5) (and its proof). Since $\mathcal{D} \subseteq V[\mathbb{G}_\beta] \subseteq V[\mathbb{G}_\alpha]$ by (4.48), it follows that $\mathbb{G}$ is a $\mathcal{D}$-generic filter on $\mathbb{P}$. ⊣ (Claim 4.16)

❑ (Theorem 4.13)

5 General iterated forcing

5.1 A general framework of iteration gif

The following is a generalization of the notion of finite support iteration discussed in 4.1. ordinary

For an ordinal δ, a sequence $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ is called an *ordinary iteration*[55] if the following (5.1) ∼ (5.7) hold.

(5.1) Each $\mathbb{P}_\alpha$ for $\alpha \le \delta$ is a poset and the underlying set of $\mathbb{P}_\alpha$, which is also denoted by $\mathbb{P}_\alpha$ as before, consists of sequences of length α. In particular $\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{1}_{\mathbb{P}_0} = \emptyset$; ord-0

(5.2) $\underset{\sim}{\mathbb{Q}}_\alpha$ for each $\alpha < \delta$ is a $\mathbb{P}_\alpha$-name and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \langle \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{\le}_{\underset{\sim}{\mathbb{Q}}_\alpha}, \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\alpha} \rangle$ is a poset"; ord-1

(5.3) For $\beta < \alpha \le \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$, $\mathbb{p} \restriction \beta \in \mathbb{P}_\beta$; ord-2

(5.4) For $\alpha < \delta$, ord-3

$\mathbb{P}_{\alpha+1} = \{\mathbb{p}^\frown\langle \underset{\sim}{\mathbb{q}} \rangle : \mathbb{p} \in \mathbb{P}_\alpha, \underset{\sim}{\mathbb{q}}$ is a canonical $\mathbb{P}_\alpha$-name, and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{q}} \in \underset{\sim}{\mathbb{Q}}_\alpha$"$\}$; 定義に変更あり．要チェック．

(5.5) For all limit $\gamma \le \delta$, one of the following holds: ord-4

(5.5a) $\mathbb{P}_\gamma = \{\mathbb{p} : \mathbb{p}$ is a sequence of length γ, $\mathbb{p} \restriction \alpha \in \mathbb{P}_\alpha$ for all $\alpha < \gamma\}$; {ord-4}{a}

(5.5b) $\mathbb{P}_\gamma = \{\mathbb{p} : \mathbb{p}$ is a sequence of length γ, $\mathbb{p} \restriction \alpha \in \mathbb{P}_\alpha$ for all $\alpha < \gamma$, $\{\beta < \gamma : \mathbb{p}(\beta) \ne \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}\} \subseteq \eta$ for some $\eta < \gamma\}$;[56] {ord-4}{b}

(5.6) $\mathbb{1}_{\mathbb{P}_\alpha}(\beta) = \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}$ for all $\beta < \alpha \le \delta$; ord-5

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[55] This is not an ordinary terminology. We introduced the name "ordinary iteration" following "ordinary differential equation". We want to call the following notion of iteration "ordinary" since we would like to consider further generalization of the notion of iteration later and call this a "general iteration".

[56] Just as in the case of finite support iteration, the set $\{\beta < \gamma : \mathbb{p}(\beta) \ne \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}\}$ is called the support of $\mathbb{p}$ and denoted by $\mathrm{supp}(\mathbb{p})$.

(5.7) For $\alpha \le \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\alpha$, $\mathbb{p}' \le_{\mathbb{P}_\alpha} \mathbb{p}$ if and only if $\mathbb{p}' \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\mathbb{p}'(\beta) \le_{\underset{\sim}{\mathbb{Q}}_\beta} \mathbb{p}(\beta)$" for all $\beta < \alpha$. ord-6

In (5.5), if (5.5a) holds, we say that $\mathbb{P}_\gamma$ is the *inverse limit* of $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$. If (5.5b) holds, we say that $\mathbb{P}_\gamma$ is the *direct limit* of $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$.

The following Lemmas for ordinary iterations correspond to Lemmas 4.2, 4.3 for finite support iteration.

Lemma 5.1 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ is an ordinary iteration.* *L-ord-0*

(1) *If $\delta_0 \le \delta$, then $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta_0, \beta < \delta_0 \rangle$ is also an ordinary iteration.*

(2) *$\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{1}_{\mathbb{P}_0} = \emptyset$.*

(3) *For $\beta \le \alpha \le \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\alpha$, if $\mathbb{p}' \le_{\mathbb{P}_\alpha} \mathbb{p}$ then $\mathbb{p}' \restriction \beta \le_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$.*

(4) *For any $\beta < \delta$, $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$.*

(5) *For any limit $\eta < \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\eta$, $\mathbb{p}' \le_{\mathbb{P}_\eta} \mathbb{p}$ if and only if $\mathbb{p}' \restriction \beta \le_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$ for all $\beta < \eta$.*

Proof. (1): If (5.1) $\sim$ (5.7) hold for $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$, then they also hold for $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta_0, \beta < \delta_0 \rangle$.

(2): This follows from (5.1).

(3): This follows from (5.7).

(4): The mapping

(5.8) $i : \mathbb{P}_{\beta+1} \to \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta;\ \mathbb{p} \mapsto \langle \mathbb{p} \restriction \beta, \mathbb{p}(\beta) \rangle$

is an isomorphism: i is 1-1 by definition and i is onto by (5.4). i strictly preserves the ordering by (5.7) and (3).

(5): The direction "$\Rightarrow$" follows from (3).

For "$\Leftarrow$", suppose that $\mathbb{p}' \restriction \beta \le_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$ holds for all $\beta < \eta$. Since η is a limit ordinal, we have $\beta + 1 < \eta$ for all $\beta < \eta$. By the assumption above, we have $\mathbb{p}' \restriction (\beta + 1) \le_{\mathbb{P}_{\beta+1}} \mathbb{p} \restriction (\beta + 1)$. By (5.7), it follows that $\mathbb{p}' \restriction \beta \Vdash_\beta$ "$\mathbb{p}'(\beta) \le_{\underset{\sim}{\mathbb{Q}}_\beta} \mathbb{p}(\beta)$". Since $\beta < \eta$ was arbitrary, it follows that $\mathbb{p}' \le_{\mathbb{P}_\eta} \mathbb{p}$ by (5.7). ❑ (Lemma 5.1)

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ be an ordinary iteration, $\alpha \le \beta \le \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$. Just as for finite support iteration, let $\mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ be the sequence s of length β defined by:

(5.9) $$s(\xi) = \begin{cases} \mathbb{p}(\xi) & \text{if } \xi < \alpha; \\ \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\xi} & \text{if } \xi \in \beta \setminus \alpha. \end{cases}$$ ord-7

Lemma 5.2 *Suppose that* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta \rangle$ *is an ordinary iteration,* $\alpha \le \beta \le \gamma \le \delta$ *and* $\mathbb{p} \in \mathbb{P}_\alpha$. L-ord-1

(1) $\mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} \in \mathbb{P}_\beta$.

(2) *If* $\mathbb{q} \in \mathbb{P}_\beta$ *and* $\mathbb{p} \le_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$, *then* $\mathbb{r} = \mathbb{p} \cup (\mathbb{q} \restriction (\beta \setminus \alpha)) \in \mathbb{P}_\beta$ *and*

(5.10) $\mathbb{r} = \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} \wedge \mathbb{q}$ *holds in* $\mathbb{P}_\beta$.[57] ord-8

(3) $i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$; $\mathbb{p} \mapsto \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ *is a complete embedding.*

(4) $i_{\alpha,\gamma} = i_{\beta,\gamma} \circ i_{\alpha,\beta}$.

(5) $p_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$; $\mathbb{p} \mapsto \mathbb{p} \restriction \alpha$ *is a projection.*

(6) $p_{\gamma,\alpha} = p_{\beta,\alpha} \circ p_{\gamma,\beta}$.

(7) $p_{\beta,\alpha} \circ i_{\alpha,\beta} = \mathrm{id}_{\mathbb{P}_\alpha}$.

Proof. (1): By induction on β (for each fixed α and $\alpha \le \beta \le \gamma$).

(2): By induction on β (for each fixed α and $\alpha \le \beta \le \gamma$). We prove first $\mathbb{r} \in \mathbb{P}_\beta$ and then (5.10).

Case 0. $\beta = \alpha$: $\mathbb{r} = \mathbb{p} \in \mathbb{P}_\alpha = \mathbb{P}_\beta$ and $\mathbb{p} \le_{\mathbb{P}_\beta} \mathbb{q}$.

Case 1. $\beta = \beta_0 + 1$: Suppose that $\mathbb{q} \in \mathbb{P}_\beta$ is such that $\mathbb{p} \le_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. By induction hypothesis, we have $\mathbb{r} \restriction \beta_0 = \mathbb{p} \cup ((\mathbb{q} \restriction \beta_0) \restriction (\beta_0 \setminus \alpha)) \in \mathbb{P}_{\beta_0}$ and

(5.11) $\mathbb{r} \restriction \beta_0 = \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta_0} \wedge (\mathbb{q} \restriction \beta_0)$. ord-9

In particular $(\mathbb{r} \restriction \beta_0) \le_{\mathbb{P}_{\beta_0}} (\mathbb{q} \restriction \beta_0)$. Since $\mathbb{r}(\beta_0) = \mathbb{q}(\beta_0)$, we have $(\mathbb{r} \restriction \beta_0) \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \in \underset{\sim}{\mathbb{Q}}_{\beta_0}$" and $(\mathbb{r} \restriction \beta_0) \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{q}(\beta_0)$". It follows that $\mathbb{r} \in \mathbb{P}_\beta$ by (5.4) and $\mathbb{r} \le_{\mathbb{P}_\beta} \mathbb{q}$ by (5.7). Scan_2023-06-21–6.40 iterated-forcing-katowice-2018 - annotated p.8

By (5.11) and, since $\mathbb{r} \restriction \beta_0 \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{r}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_{\beta_0}}$", we have $\mathbb{r} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$.

wqSuppose that $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}, \mathbb{q}$. Then, $\mathbb{s} \restriction \beta_0 \le_{\mathbb{P}_{\beta_0}} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta_0}, \mathbb{q} \restriction \beta_0$. By (5.11), it follows that $\mathbb{s} \restriction \beta_0 \le_{\mathbb{P}_{\beta_0}} \mathbb{r} \restriction \beta_0$. Since $\mathbb{r}(\beta_0) = \mathbb{q}(\beta_0)$, we have $\mathbb{s} \restriction \beta_0 \Vdash_{\mathbb{P}_{\beta_0}}$ "$\mathbb{s}(\beta_0) \le_{\underset{\sim}{\mathbb{Q}}_{\beta_0}} \mathbb{r}(\beta_0)$". Thus $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{r}$.

Case 2. β is a limit ordinal: If direct limit is taken at β (see (5.5)), since $\mathrm{supp}(\mathbb{r}) \subseteq \mathrm{supp}(\mathbb{p}) \cup \sup(\mathbb{q})$, $\mathrm{supp}(\mathbb{r})$ is bounded in β. Since $\mathbb{r} \restriction \xi = \mathbb{p} \cup ((\mathbb{q} \restriction \xi) \restriction (\xi \setminus \alpha)) \in \mathbb{P}_\xi$ for all $\alpha \le \xi < \beta$, it follows that $\mathbb{r} \in \mathbb{P}_\beta$. Now (5.10) follows from Lemma 5.1, (5) and induction hypothesis.

If inverse limit is taken at β, then $\mathbb{r} \in \mathbb{P}_\beta$ is immediate and the rest of the argument is the same.

(3): $i_{\alpha,\beta}(\mathbb{1}_{\mathbb{P}_\alpha}) = \mathbb{1}_{\mathbb{P}_\alpha}{}^\frown \vec{\mathbb{1}}_{\alpha,\beta} = \mathbb{1}_{\mathbb{P}_\beta}$.

If $\mathbb{p}' \le_{\mathbb{P}_\alpha} \mathbb{p}$ then $i_{\alpha,\beta}(\mathbb{p}') = \mathbb{p}'^\frown \vec{\mathbb{1}}_{\alpha,\beta} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} = i_{\alpha,\beta}(\mathbb{p})$ by (2).

---

[57] With $\mathbb{r} = \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} \wedge \mathbb{q}$, we denote the situation $\mathbb{r} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}, \mathbb{q}$ and $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{r}$ for all $\mathbb{s} \le_{\mathbb{P}_\beta} \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}, \mathbb{q}$.

If $i_{\alpha,\beta}(\mathbb{p}) = \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ and $i_{\alpha,\beta}(\mathbb{p}') = \mathbb{p}'^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ are compatible in $\mathbb{P}_\beta$, say $\mathbb{r} \leq_{\mathbb{P}_\beta} i_{\alpha,\beta}(\mathbb{p})$, $i_{\alpha,\beta}(\mathbb{p}')$, then $\mathbb{r} \restriction \alpha \leq_{\mathbb{P}_\alpha} i_{\alpha,\beta}(\mathbb{p}) \restriction \alpha = \mathbb{p}$, $i_{\alpha,\beta}(\mathbb{p}') \restriction \alpha = \mathbb{p}'$. and hence $\mathbb{p}$ and $\mathbb{p}'$ are compatible in $\mathbb{P}_\alpha$.

Now suppose that $\mathbb{q} \in \mathbb{P}_\beta$. We show that $\mathbb{q} \restriction \alpha$ is a reduction of q for $i_{\alpha,\beta}$ in the sense of (2.34). Let $\mathbb{r} \leq_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$. Then,

$$\mathbb{r} \cup \mathbb{q} \restriction (\beta \setminus \alpha) \leq_{\mathbb{P}_\beta} \underbrace{\mathbb{r}^\frown \vec{\mathbb{1}}_{\alpha,\beta}}_{= i_{\alpha,\beta}(\mathbb{r})}, \mathbb{q} \tag{5.12}$$ ord-10

by (2). Thus $i_{\alpha,\beta}(\mathbb{r})$ and $\mathbb{q}$ are compatible in $\mathbb{P}_\beta$.

(4): Clear by the definition of the complete embeddings in (3).

(5): $p_{\beta,\alpha}$ clearly satisfies (2.42). $p_{\beta,\alpha}$ is order-preserving by Lemma 5.1, (3). (2.44) follows from (2).

(6): Clear by definition. ❑ (Lemma 5.2)

For an ordinary iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$, we call the complete embedding $i_{\alpha,\beta}$ for $\alpha \leq \beta \leq \delta$ defined in Lemma 5.2, (3) the *canonical embedding* and the projection $p_{\beta,\alpha}$ defined in Lemma 5.2, (5) the *canonical projection*.

Lemma 5.3 *A finite support iteration is an ordinary iteration. More precisely, a finite support iteration is an ordinary iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ in which $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma \rangle$ for all limit $\gamma \leq \delta$. Conversely all ordinary iterations such that direct limit is taken at all limit steps are finite support iterations.* L-ord-2

Proof. By cumulative induction on δ, prove

(5.13)$_\delta$ Any sequence $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ of length δ is a finite support iteration if and only if it is an ordinary iteration with ord-11

(5.5′) $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma \rangle$ for all limit $\gamma \leq \delta$. ord-4

❑ (Lemma 5.3)

Suppose that λ is a regular cardinal. A poset $\mathbb{P}$ is *$< \lambda$-directed closed* if, for any downward directed[58] $D \subseteq \mathbb{P}$ of cardinality $< \lambda$, there is $\mathbb{p} \in \mathbb{P}$ such that $\mathbb{p} \leq_{\mathbb{P}} \mathbb{d}$ for all $\mathbb{d} \in D$.

[58] A subset D of a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is downward directed if, for any d, $d' \in D$, there is $d'' \in D$ such that $d'' \leq_{\mathbb{P}} d$, d'.

Lemma 5.4 (Lemma 21.7 in [Jech 2001/2006]) *Suppose that λ is a regular cardinal.* L-ord-2-0

(0) *If a poset $\mathbb{P}$ is $<\lambda$-directed closed, then $\mathbb{P}$ is $<\lambda$-closed. Thus $\mathbb{P}$ preserves cardinals and cofinality $\leq \lambda$.*

(1) *If a poset $\mathbb{P}$ is $<\lambda$-directed closed and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is $<\lambda$-directed closed", then $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $<\lambda$-directed closed.*

(2) *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is an ordinary iteration such that all $\mathbb{P}_\alpha$, $\alpha < \delta$ are $<\lambda$-directed closed. If $cf(\delta) \geq \lambda$ and $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \delta \rangle$, then $\mathbb{P}_\delta$ is $<\lambda$-directed closed.*

(3) *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is an ordinary iteration such that $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is $<\lambda$-directed closed". If $\mathbb{P}_\gamma$ is the inverse limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \delta \rangle$ for all $\gamma \in E^{\delta+1}_{<\lambda} = \{\gamma < \delta + 1 : \gamma$ is a limit and $cf(\gamma) < \lambda\}$, then $\mathbb{P}_\delta$ is $<\lambda$-directed closed.*

Proof. (0): Suppose that $\mathbb{P}$ is a $<\lambda$-directed closed poset. If $\langle \mathbb{p}_\xi : \xi < \delta \rangle$ for some $\delta < \lambda$ is a decreasing sequence in $\mathbb{P}$, then $\{\mathbb{p}_\xi : \xi < \delta\}$ is downward directed. Hence there is $\mathbb{p}^* \in \mathbb{P}$ such that $\mathbb{p}^* \leq_{\mathbb{P}} \mathbb{p}_\xi$ for all $\xi < \delta$. This shows that $\mathbb{P}$ is $<\lambda$-closed. By Corollary 1.42, it follows that $\mathbb{P}$ preserves cardinals and cofinality $\leq \lambda$.

(1): Suppose that $D = \{\langle \mathbb{p}_\xi, \underset{\sim}{\mathbb{q}}_\xi \rangle : \xi < \delta\}$ for a $\delta < \lambda$ is a downward directed subset of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$. We have to show that D has a lower bound in $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$.

$\{\mathbb{p}_\xi : \xi < \delta\}$ is a downward directed subset of $\mathbb{P}$. Since $\mathbb{P}$ is $<\lambda$-directed closed, there is $\mathbb{p}^* \in \mathbb{P}$ such that $\mathbb{p}^* \leq_{\mathbb{P}} \mathbb{p}_\xi$ for all $\xi < \delta$. Let

(5.14) $\underset{\sim}{D} = \{\langle \underset{\sim}{\mathbb{q}}_\xi, \mathbb{1}_{\mathbb{P}} \rangle : \xi < \delta\}$. ord-11-0

Then we have

(5.15) $\mathbb{p}^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{D}$ is a downward directed subset of $\underset{\sim}{\mathbb{Q}}$ of size $< \lambda$". ord-11-1

Since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is $<\lambda$-directed closed", there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{q}}^*$ such that $\mathbb{p}^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}^* \varepsilon \underset{\sim}{\mathbb{Q}}$" and $\mathbb{p}^* \Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}^* \leq_{\underset{\sim}{\mathbb{Q}}} \mathbb{r}$ for all $\mathbb{r} \ \varepsilon \ \underset{\sim}{D}$". By Lemma 3.2, we may choose $\underset{\sim}{\mathbb{q}}^*$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{q}}^* \ \varepsilon \ \underset{\sim}{\mathbb{Q}}$".

It follows that $\langle \mathbb{p}^*, \underset{\sim}{\mathbb{q}}^* \rangle \leq_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}} \langle \mathbb{p}_\xi, \underset{\sim}{\mathbb{q}}_\xi \rangle$ for all $\xi < \delta$.

(2): Suppose that $\{\mathbb{p}_\xi : \xi < \eta\}$ for some $\eta < \lambda$ is a downward directed subset of $\mathbb{P}_\delta$. Since $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \delta \rangle$,

(5.16) $\nu_\xi = \sup(\mathrm{supp}(\mathbb{p}_\xi) \dot{+} 1) < \delta$ for all $\xi < \eta$.[59] ord-11-2

Since $cf(\delta) \geq \lambda$, it follows that $\nu^* = \sup\{\nu_\xi : \xi < \eta\} < \delta$.

Now $\{\mathbb{p}_\xi \restriction \nu^* : \xi < \eta\}$ is downward directed in $\mathbb{P}_{\nu^*}$. Thus, by assumption, there is $\mathbb{p}^* \in \mathbb{P}_{\nu^*}$ such that $\mathbb{p}^* \leq_{\mathbb{P}_{\nu^*}} \mathbb{p}_\xi \restriction \nu^*$ for all $\xi < \eta$.

Let $\mathbb{p}^{**} = \mathbb{p}^* \frown \frown \vec{\mathbb{1}}_{\nu^*, \delta}$. Then we have

[59] For $\alpha \in \mathsf{On}$, we denote with $\alpha \dot{+} 1$ the ordinal α itself if α is a limit ordinal; else the ordinal $\alpha + 1$.

(5.17) $\mathbb{p}^{**} \leq_{\mathbb{P}_\delta} (\mathbb{p}_\xi \restriction \nu^*)^\frown \vec{\mathbb{1}}_{\nu^*,\delta} = \mathbb{p}_\xi$ for all $\xi < \eta$. ord-11-3

(3): Suppose that $\{\mathbb{p}_\xi : \xi < \eta\}$ is a downward directed subset of $\mathbb{P}_\delta$ for some $\eta < \lambda$. By induction on $\alpha \leq \delta$ we construct $\mathbb{q}_\alpha \in \mathbb{P}_\alpha$ such that

(5.18) $\mathbb{q}_\alpha$ is an initial segment of $\mathbb{q}_{\alpha'}$ for all $\alpha < \alpha' \leq \delta$; ord-11-4

(5.19) $\mathbb{q}_\alpha \leq_{\mathbb{P}_\alpha} \mathbb{p}_\xi \restriction \alpha$ for all $\xi < \eta$; and ord-11-5

(5.20) $\mathrm{supp}(\mathbb{q}_\alpha) \subseteq \bigcup_{\xi<\eta}(\mathrm{supp}(\mathbb{p}_\xi) \cap \alpha)$. ord-11-6

For $\alpha = 0$ $\mathbb{q}_0 = \emptyset$ will do.

Suppose that $\mathbb{q}_\alpha$, $\alpha \leq \alpha'$ have been defined according to (5.18), (5.19), and (5.20). If $\alpha' \notin \bigcup_{\xi<\eta} \mathrm{supp}(\mathbb{p}_\xi)$, then let $\mathbb{q}_{\alpha'+1} = \mathbb{q}_{\alpha'} \cup \{\langle \alpha', \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_{\alpha'}} \rangle\}$.

Otherwise, let $\underset{\sim}{D}_{\alpha'} = \{\langle \mathbb{p}_\xi(\alpha'), \mathbb{1}_{\mathbb{P}_{\alpha'}} \rangle : \xi < \eta\}$. By (5.19) for α', and since $\{\mathbb{p}_\xi \restriction \alpha' + 1 : \xi < \eta\}$ is downward directed,

(5.21) $\mathbb{q}_{\alpha'} \Vdash_{\mathbb{P}_{\alpha'}}$ "$\underset{\sim}{D}_{\alpha'}$ is a downward directed subset of $\underset{\sim}{\mathbb{Q}}_{\alpha'}$". ord-11-7

Thus there is a $\mathbb{P}_{\alpha'}$-name $\underset{\sim}{\mathbb{q}}$ such that

(5.22) $\mathbb{q}_{\alpha'} \Vdash_{\mathbb{P}_{\alpha'}}$ "$\underset{\sim}{\mathbb{q}} \leq_{\underset{\sim}{\mathbb{Q}}_{\alpha'}} \mathbb{r}$ for all $\mathbb{r} \in \underset{\sim}{D}_{\alpha'}$". ord-11-8

Let $\mathbb{q}_{\alpha'+1} = \mathbb{q}_{\alpha'} \cup \{\langle \alpha', \underset{\sim}{\mathbb{q}} \rangle\}$.

In both cases, $\mathbb{q}_\alpha$, $\alpha \leq \alpha' + 1$ satisfy (5.18), (5.19), and (5.20).

Suppose now that $\mathbb{q}_\alpha$, $\alpha < \gamma$ have been defined according to (5.18), (5.19), and (5.20) for a limit ordinal $\gamma \leq \delta$.

If $\mathbb{P}_\gamma$ is the inverse limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma \rangle$, then $\mathbb{q}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{q}_\alpha \in \mathbb{P}_\gamma$ and $\mathbb{q}_\alpha$, $\alpha \leq \gamma$ satisfy (5.18), (5.19), and (5.20).

If $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma \rangle$, then, by assumption, $cf(\gamma) \geq \lambda$. Thus, similarly to (2), we have again $\mathbb{q}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{q}_\alpha \in \mathbb{P}_\gamma$ and $\mathbb{q}_\alpha$, $\alpha \leq \gamma$ satisfy (5.18), (5.19), and (5.20). ❑ (Lemma 5.4)

Easton support iteration is an ordinary iteration designed for iteration of length δ which is a Mahlo cardinal.

An iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is an *Easton support iteration* if it satisfies (5.2), (5.3), (5.4), (5.6), (5.7), and,

(5.5″) for any limit $\gamma \leq \delta$, $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \gamma \rangle$ if γ is regular; it is the inverse limit otherwise. ord-4

Compare this definition with the condition of Lemma 5.4.

Lemma 5.5 *If $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta \rangle$ is an Easton support iteration then for any $\mathbb{p} \in \mathbb{P}_\delta$ and regular $\gamma \leq \delta$, $|\mathrm{supp}(\mathbb{p}) \cap \gamma| < \gamma$.* ❑ L-ord-3

Lemma 5.6 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \kappa, \beta < \kappa \rangle$ is an Easton support iteration for a Mahlo cardinal κ.* L-ord-4

(1) *If each $\mathbb{P}_\alpha$, $\alpha < \kappa$ has the κ-cc then $\mathbb{P}_\kappa$ has the κ-cc.*

(2) *If each $\mathbb{P}_\alpha$, $\alpha < \kappa$ has the κ-Knaster property then $\mathbb{P}_\kappa$ has the κ-Knaster property.*

Proof. We prove (1). (2) can be proved similarly. Suppose that $\langle \mathbb{p}_\alpha : \alpha < \kappa \rangle$ is a sequence of elements of $\mathbb{P}_\kappa$. Let $E = \{\alpha < \kappa : \alpha \text{ is regular}\}$. E is stationary by the assumption on κ. For each $\alpha \in E$, let

(5.23) $\nu_\alpha = \sup(\mathrm{supp}(\mathbb{p}_\alpha) \cap \alpha)$. ord-12

Then $\nu_\alpha < \alpha$ by (5.5″). By Fodor's Lemma, there is a stationary $E' \subseteq E$ and $\xi^* < \kappa$ such that $\nu_\alpha = \xi^*$ for all $\alpha \in E'$.

If there is $\xi^{**} < \kappa$ such that $\mathrm{supp}(\mathbb{p}_\alpha) \subseteq \xi^{**}$ for all $\alpha \in E$ then, by assumption of κ-cc of $\mathbb{P}_{\xi^{**}}$, there are distinct $\alpha, \alpha' \in E$ $\mathbb{p}_\alpha \restriction \xi^{**}$ and $\mathbb{p}_{\alpha'} \restriction \xi^{**}$ are compatible in $\mathbb{P}_{\xi^{**}}$. Thus $\mathbb{p}_\alpha = (\mathbb{p}_\alpha \restriction \xi^{**}) \frown \vec{\mathbb{1}}_{\xi^{**},\kappa}$ and $\mathbb{p}_{\alpha'} = (\mathbb{p}_{\alpha'} \restriction \xi^{**}) \frown \vec{\mathbb{1}}_{\xi^{**},\kappa}$ are compatible.

Otherwise, there is a $D \in [E']^\kappa$ such that

(5.24) $sup(\mathrm{supp}(\mathbb{p}_\alpha) \setminus \xi^*) \ < \ \alpha' \ \le \ min(\mathrm{supp}(\mathbb{p}_{\alpha'}) \setminus \xi^*)$ ord-13

for all α, $\alpha' \in D$ with $\alpha < \alpha'$.

Since $\{\mathbb{p}_\alpha \restriction \xi^* : \alpha \in D\} \subseteq \mathbb{P}_{\xi^*}$, there are α, α' with $\alpha < \alpha'$ such that $\mathbb{p}_\alpha \restriction \xi^*$ and $\mathbb{p}_{\alpha'} \restriction \xi^*$ are compatible in $\mathbb{P}_{\xi^*}$. Let $\mathbb{q}^* \in \mathbb{P}_{\xi^*}$ be such that

(5.25) $\mathbb{q}^* \le_{\mathbb{P}_{\xi^*}} \mathbb{p}_\alpha \restriction \xi^*, \mathbb{p}_{\alpha'} \restriction \xi^*$ ord-14

and let

(5.26) $\mathbb{q} = \mathbb{q} \cup \mathbb{p}_\alpha \restriction (\alpha \setminus \xi^*) \cup \mathbb{p}_{\alpha'} \restriction (\kappa \setminus \alpha)$. ord-15

Then $\mathbb{q} \in \mathbb{P}_\kappa$ by Lemma 5.2, (2) and we have $\mathbb{q} \le_{\mathbb{P}_\kappa} \mathbb{p}_\alpha, \mathbb{p}_{\alpha'}$.[60] ❑ (Lemma 5.6)

5.2 Abstract iteration of posets

We need the following notion of abstract iteration for clear formulation of some advanced notions and properties of the iteration of forcing, like the Factor Lemmas which are strengthenings of Lemma 3.14 for ordinary iterations as above, and for generalized iteration in the sense of the next subsection. abst-itr

For an ordinal δ, a sequence $\langle \mathbb{P}_\alpha : \alpha < \delta \rangle$ of posets together with systems of mappings $\langle i_{\alpha,\beta} : \alpha \le \beta < \delta \rangle$ and $\langle p_{\beta,\alpha} : \alpha \le \beta < \delta \rangle$ is said to be an *abstract iteration* if Scan_2023-06-21–6.40

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(5.27) $i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$ is a complete embedding and $i_{\alpha,\alpha} = \mathrm{id}_{\mathbb{P}_\alpha}$ for all $\alpha \le \beta < \delta$; abst-itr-a-0

[60] We can show, by induction on $\xi \le \kappa$, that $\mathbb{q} \restriction \xi \le_{\mathbb{P}_\xi} \mathbb{p}_\alpha \restriction \xi, \mathbb{p}_{\alpha'} \restriction \xi$.

(5.28) $i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma}$ for all $\alpha \le \beta \le \gamma < \delta$; abst-itr-a-2

(5.29) $p_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$ is a projection and $p_{\alpha,\alpha} = \mathrm{id}_{\mathbb{P}_\alpha}$ for all $\alpha \le \beta < \delta$; abst-itr-a-1

(5.30) $p_{\beta,\alpha} \circ p_{\gamma,\beta} \sim_{\mathbb{P}_\alpha} p_{\gamma,\alpha}$ for all $\alpha \le \beta \le \gamma < \delta$; [61] abst-itr-a-3

(5.31) $p_{\beta,\alpha} \circ i_{\alpha,\beta} \sim_{\mathbb{P}_\alpha} \mathrm{id}_{\mathbb{P}_\alpha}$ for all $\alpha \le \beta < \delta$; [61] and abst-itr-a-4

(5.32) $i_{\alpha,\beta}(\mathbb{p}) \wedge \mathbb{q}$ exists for any $\alpha \le \beta < \delta$, $\mathbb{p} \in \mathbb{P}_\alpha$ and $\mathbb{q} \in \mathbb{P}_\beta$ with $\mathbb{p} \le p_{\beta,\alpha}(\mathbb{q})$. abst-itr-a-5

The ordinal δ in an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is called the *length* of $\mathcal{I}$. An abstract iteration of $\mathcal{I}$ length $\delta + 1$ is also denoted it as $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$. It is then also said that $\mathcal{I}$ is of length $\le \delta$. To avoid confusions abstract iterations $\mathcal{I}$ of length δ in the sense above is also called an abstract iterations of length $< \delta$.

Example 5.7 (1) *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta\rangle$ is an ordinary iteration. Then the sequence $\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle$ with its canonical embeddings and canonical projections is an abstract iteration.* Example of abstract iterations

(2) *Suppose that $\langle \mathbb{P}_\alpha : \alpha < \delta\rangle$ is a sequence of cBa posets with a direct system of complete embeddings $i_{\alpha,\beta} : \mathbb{P}_\alpha \overset{\lessdot}{\to} \mathbb{P}_\beta$ for $\alpha \le \beta < \delta$, that is, such a system of complete embeddings that satisfies (5.27) and (5.28). Then the system $\mathcal{I}_0 = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta} : \alpha \le \beta < \delta\rangle\rangle$ can be expanded to an abstract iteration of length δ by letting $p_{\beta,\alpha}$ for $\alpha \le \beta < \delta$ defined by*

(5.33) $p_{\beta,\alpha}(\mathbb{q}) = \min\{\mathbb{p} \in \mathbb{P}_\alpha : \mathbb{q} \le_{\mathbb{P}_\beta} i_{\alpha,\beta}(\mathbb{p})\}$ *abst-itr-a-5-1*

for $\mathbb{q} \in \mathbb{P}_\beta$. And this expansion of $\mathcal{I}_0$ is the unique abstract iteration expanding $\mathcal{I}_0$ with the same length as that of $\mathcal{I}_0$.

Proof. (1); By Lemma 5.2.

(2): The order preserving of $p_{\beta,\alpha}$ for $\alpha \le \beta < \delta$ follows directly from the definition (5.33).

For $\alpha \le \beta \le \gamma$, we have $p_{\beta,\alpha} \circ p_{\gamma,\beta} = p_{\gamma,\alpha}$: Suppose that $\mathbb{q} \in \mathbb{P}_\gamma$ and $\mathbb{p} \in \mathbb{P}_\alpha$. Then

$$\begin{aligned}
\mathbb{p} \ge_{\mathbb{P}} p_{\gamma,\alpha}(\mathbb{q}) \;&\Leftrightarrow\; \underbrace{i_{\alpha,\gamma}(\mathbb{p})}_{= i_{\beta,\gamma}(i_{\alpha,\beta}(\mathbb{p}))} \ge_{\mathbb{P}_\gamma} \mathbb{q} && ;\text{by (5.33) for } p_{\gamma,\alpha} \\
&\qquad\qquad && (\text{ and since } \{...\} \text{ in (5.33) is upward closed}) \\
&\Leftrightarrow\; i_{\alpha,\beta}(\mathbb{p}) \ge_{\mathbb{P}_\beta} p_{\gamma,\beta}(\mathbb{q}) && ;\text{by (5.33) for } p_{\gamma,\beta} \\
&\Leftrightarrow\; \mathbb{p} \ge_{\mathbb{P}_\alpha} p_{\beta,\alpha}(p_{\gamma,\beta}(\mathbb{q})) && ;\text{by (5.33) for } p_{\beta,\alpha}.
\end{aligned}$$

It follows that $p_{\gamma,\alpha}(\mathbb{q}) = \min\{\mathbb{p} \in \mathbb{P}_\alpha : p_{\beta,\alpha} \ge_{\mathbb{P}_\alpha} p_{\beta,\beta}(p_{\gamma,\beta}(\mathbb{q}))\} = \min\{\mathbb{p} \in \mathbb{P}_\alpha : \mathbb{p} \ge_{\mathbb{P}_\alpha} p_{\gamma,\alpha}\} = p_{\beta,\alpha}(p_{\gamma,\beta}(\mathbb{q}))$.

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[61] (5.30) actually means that $p_{\beta,\alpha}(p_{\gamma,\beta}(\mathbb{p})) \sim_{\mathbb{P}_\alpha} p_{\gamma,\alpha}(\mathbb{p})$ for all $\mathbb{p} \in \mathbb{P}_\gamma$.
(5.31) actually means that $p_{\beta,\alpha}(i_{\alpha,\beta}(\mathbb{p})) \sim_{\mathbb{P}_\alpha} \mathbb{p}$ for all $\mathbb{p} \in \mathbb{P}_\alpha$. See the footnote to (3.45).

(5.34) For $\alpha \le \beta < \delta$, we have $p_{\beta,\alpha} \circ i_{\alpha,\beta} = \mathrm{id}_{\mathbb{P}_\alpha}$: abst-itr-a-5-1-0

For $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\alpha$, we have, similarly to the equivalences above,

$$\begin{aligned} \mathbb{p}' \ge_{\mathbb{P}_\alpha} p_{\beta,\alpha}(i_{\alpha,\beta}(\mathbb{p})) &\Leftrightarrow i_{\alpha,\beta}(\mathbb{p}') \ge_{\mathbb{P}_\beta} i_{\alpha,\beta}(\mathbb{p}) \\ &\Leftrightarrow \mathbb{p}' \ge_{\mathbb{P}_\alpha} \mathbb{p} \qquad \text{; by Lemma 2.12, (1).} \end{aligned}$$

Thus, similarly to the argument above, we obtain $p_{\beta,\alpha}(i_{\alpha,\beta}(\mathbb{p})) = \mathbb{p}$.

Suppose now that $\mathbb{r} \in \mathbb{P}_\alpha$, $\mathbb{q} \in \mathbb{P}_\beta$ and

(5.35) $\mathbb{r} \le_{\mathbb{P}_\alpha} p_{\beta,\alpha}(\mathbb{q})$ abst-itr-a-5-2

holds. Then we have $i(\mathbb{r}) \top_{\mathbb{P}_\beta} \mathbb{q}$: Otherwise, we would have $\mathbb{q} \le_{\mathbb{P}_\beta} -i(\mathbb{r}) = i(-\mathbb{r})$. Thus $p_{\beta,\alpha}(\mathbb{q}) \le_{\mathbb{P}_\alpha} \underbrace{p_{\beta,\alpha}(i_{\alpha,\beta}(\mathbb{r})) = \mathbb{r}}_{\text{by (5.34)}}$. But this is a contradiction to (5.35).

Thus, $i(\mathbb{r}) \wedge \mathbb{q} \in \mathbb{P}_\beta$. Scan Jan 9

The uniqueness follows from Lemma 5.8, (6) below. ❑ (Example 5.7)

Since the expansion of a direct system $\mathcal{I}_0 = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta} : \alpha \le \beta < \delta\rangle\rangle$ of cBa posets with complete embeddings to an abstract iteration is uniquely given as in Example 5.7, (2), we shall simply call such $\mathcal{I}_0$ a cBa abstract iteration.

We call an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ *separative* if each of $\mathbb{P}_\alpha$, $\alpha < \delta$ is separative. Similarly, we say that $\mathcal{I}$ is *sub-Boolean* (or *cBa*, respectively) if each of $\mathbb{P}_\alpha$, $\alpha < \delta$ is sub-Boolean (or cBa, respectively).

Note that, if an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is sub-Boolean, then, since all $\le_{\mathbb{P}_\alpha}$, $\alpha < \delta$ are partial orderings, we have

(5.30′) $p_{\beta,\alpha} \circ p_{\gamma,\beta} = p_{\gamma,\alpha}$ for all $\alpha \le \beta \le \gamma < \delta$; and abst-itr-a-3

(5.31′) $p_{\beta,\alpha} \circ i_{\alpha,\beta} = \mathrm{id}_{\mathbb{P}_\alpha}$ for all $\alpha \le \beta < \delta$. abst-itr-a-4

Also in this case, $i_{\alpha,\beta}(\mathbb{p}) \wedge \mathbb{q}$ in (5.32) is a unique object for each $\mathbb{p} \in \mathbb{P}_\alpha$ and $\mathbb{q} \in \mathbb{P}_\beta$.

(4) and (6) of the following Lemma and its corollary support the intuition that the complete embedding $\mathbb{p} \mapsto i_{\alpha,\beta}(\mathbb{p})$ and the projection $\mathbb{q} \mapsto p_{\beta,\alpha}(\mathbb{p})$ in an abstract iteration must correspond to the canonical embedding $\mathbb{p} \mapsto \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ and the canonical projection $\mathbb{q} \mapsto \mathbb{q} \restriction \alpha$ in an ordinary iteration.

Lemma 5.8 *Suppose that the posets $\mathbb{P}$, $\mathbb{Q}$ with the mappings i, p which satisfy the properties in* Lemma 3.14. *That is,* *Abstract Two-step Iteration Lemma*

(3.43) $i : \mathbb{P} \to \mathbb{Q}$ *is a complete embedding;* *two-23-a*

(3.44) $p : \mathbb{Q} \to \mathbb{P}$ *is a projection;* *two-23-a-0*

(3.45) $p \circ i \sim_{\mathbb{P}} \mathrm{id}_{\mathbb{P}}$*; and* *two-23-a-1*

(3.46) *for any* $\mathbb{p} \in \mathbb{P}$ *and* $\mathbb{q} \in \mathbb{Q}$ *with* $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$, $i(\mathbb{p}) \wedge \mathbb{q}$ *exists.* *two-23-a-2*

Then

(1) *For any* $\mathbb{q} \in \mathbb{Q}$, *we have* $\mathbb{q} \top_{\mathbb{Q}} i(p(\mathbb{q}))$.

(2) *For any* $\mathbb{p} \in \mathbb{P}$ *and* $\mathbb{q} \in \mathbb{Q}$, *if* $i(\mathbb{p}) \perp_{\mathbb{Q}} \mathbb{q}$ *then* $\mathbb{p} \perp_{\mathbb{P}} p(\mathbb{q})$.

(3) *If* $\mathbb{Q}$ *is separative, then* $\mathbb{q} \leq_{\mathbb{Q}} i(p(\mathbb{q}))$ *for all* $\mathbb{q} \in \mathbb{Q}$.

(4) *If* $\mathbb{Q}$ *is sub-Boolean, then*

(5.36) $i(\mathbb{p}) = \sup\{\mathbb{q} \in \mathbb{Q} : p(\mathbb{q}) = \mathbb{p}\}$ *for all* $\mathbb{p} \in \mathbb{P}$. *abst-itr-4*

(5) *Suppose that* $\mathbb{P}$ *and* $\mathbb{Q}$ *are separative,* $\mathbb{p} \in \mathbb{P}$, $\mathbb{q} \in \mathbb{Q}$ *and*

(5.37) $\mathbb{r} = i(\mathbb{p}) \wedge_{\mathbb{Q}} \mathbb{q}$ *abst-itr-4-a-0*

for a condition $\mathbb{r} \in \mathbb{P}$.[62] *Then we have* $p(\mathbb{r}) = \mathbb{p} \wedge_{\mathbb{P}} p(\mathbb{q})$.

(6) *If* $\mathbb{P}$ *is sub-Boolean and* $\mathbb{Q}$ *is separative, then*

(5.38) $p(\mathbb{q}) = \inf\{\mathbb{p} \in \mathbb{P}_\alpha : \mathbb{q} \leq_{\mathbb{Q}} i(\mathbb{p})\}$ *for all* $\mathbb{q} \in \mathbb{Q}$. *abst-itr-4-a-1*

Proof. (1): Let $\mathbb{p} = p(\mathbb{q})$. Since $\mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q})$, $\underbrace{i(\mathbb{p})}_{= i(p(\mathbb{q}))} \wedge \mathbb{q}$ exists by (3.46). Thus $i(p(\mathbb{q})) \top_{\mathbb{Q}} \mathbb{q}$.

(2): Let $\mathbb{p} \in \mathbb{P}$ and $\mathbb{q} \in \mathbb{Q}$ and suppose that $\mathbb{p} \top_{\mathbb{P}} p(\mathbb{q})$. We have to show that $i(\mathbb{p}) \top_{\mathbb{Q}} \mathbb{q}$. Let $\mathbb{p}' \leq_{\mathbb{P}} \mathbb{p}$, $p(\mathbb{q})$. Since p is a projection, there is $\mathbb{q}' \in \mathbb{Q}$ such that $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ and $p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{p}'$. By (1), there is $\mathbb{q}'' \leq_{\mathbb{Q}} \mathbb{q}'$, $i(p(\mathbb{q}'))$. Since $i(p(\mathbb{q}')) \leq_{\mathbb{Q}} i(\mathbb{p}') \leq_{\mathbb{Q}} i(\mathbb{p})$, we have $\mathbb{q}'' \leq_{\mathbb{Q}} i(\mathbb{p})$. We also have $\mathbb{q}'' \leq_{\mathbb{Q}} \leq \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$. This shows that $i(\mathbb{p}) \top_{\mathbb{Q}} \mathbb{q}$.

(3): Suppose, toward a contradiction, that $\mathbb{q} \not\leq_{\mathbb{Q}} i(p(\mathbb{q}))$ for some $\mathbb{q} \in \mathbb{Q}$. Since $\mathbb{Q}$ is separative, there is

(5.39) $\mathbb{r} \leq_{\mathbb{Q}} \mathbb{q}$ such that abst-itr-4-0

(5.40) $\mathbb{r} \perp_{\mathbb{Q}} i(p(\mathbb{q}))$. abst-itr-5

By (5.39) and since i and p are order preserving, $i(p(\mathbb{r})) \leq_{\mathbb{Q}} i(p(\mathbb{q}))$. It follows that $\mathbb{r} \perp_{\mathbb{Q}} i(p(\mathbb{r}))$ by (5.40). This is a contradiction to (1).

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[62] See (3.39) for the notation with "$\wedge$".

(4): Since $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$ is sub-Boolean, we can (injectively) embed $\mathbb{Q}$ into its Boolean completion (see Lemma 2.9) and assume that $\sup\{\mathbb{q} \in \mathbb{Q} : p(\mathbb{q}) = \mathbb{p}\}$ exists.

Since $p(i(\mathbb{p})) = \mathbb{p}$ by (3.45), $i(\mathbb{p})$ is in the set on the right side of the equation (5.36). Thus $i(\mathbb{p}) \leq_{\mathbb{Q}} \sup\{...\}$. For any $\mathbb{q} \in \mathbb{Q}$ with $p(\mathbb{q}) = \mathbb{p}$, $i(\mathbb{p}) = i(p(\mathbb{q})) \geq_{\mathbb{Q}} \mathbb{q}$ by (3). Thus $i(\mathbb{p}) \geq_{\mathbb{Q}} \sup\{...\}$ and hence $i(\mathbb{p}) = \sup\{...\}$.

(5): Suppose that $\mathbb{p}$, $\mathbb{q}$, $\mathbb{r}$ are as in the statement. Since $\mathbb{r} \leq_{\mathbb{Q}} i(\mathbb{p})$, $\mathbb{q}$, we have $p(\mathbb{r}) \leq_{\mathbb{P}} \underbrace{p(i(\mathbb{p}))}_{= \mathbb{p} \text{ by } (3.45)}, p(\mathbb{q})$.

Thus, if $p(\mathbb{r}) = \mathbb{P} \wedge_{\mathbb{P}} p(\mathbb{q})$ does not hold, then There is $\mathbb{r}' \in \mathbb{P}$ such that $\mathbb{r}' \leq_{\mathbb{P}} \mathbb{p}, p(\mathbb{q})$ but $\mathbb{r}' \not\leq_{\mathbb{P}} p(\mathbb{r})$ (see footnote 3.39). By separativity of $\mathbb{P}$, there is $\mathbb{s} \leq_{\mathbb{P}} \mathbb{r}'$ such that

$$\mathbb{s} \perp_{\mathbb{P}} p(\mathbb{r}). \tag{5.41}$$ abst-itr-5-a-0

Since p is a projection, there is $\mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q}$ such that

$$p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{s}. \tag{5.42}$$ abst-itr-5-0

Then

$$p(\mathbb{q}') \perp_{\mathbb{P}} p(\mathbb{r}) \tag{5.43}$$ abst-itr-5-1

by (5.41). Since p is order preserving it follows that $\mathbb{q}' \perp_{\mathbb{Q}} \mathbb{r}$. On the other hand, we also have $\mathbb{q}' \leq_{\mathbb{Q}} i(p(\mathbb{q}')) \leq i(\mathbb{p})$ where the first inequality is by (3) and the second by (5.42) and since i is order preserving. Thus $\mathbb{q}' \leq_{\mathbb{P}} i(\mathbb{p}) \wedge_{\mathbb{Q}} \mathbb{q}$. By (5.43), this is a contradiction to (5.37).

(6): By (3), we have $\mathbb{q} \leq_{\mathbb{Q}} i(p(\mathbb{q}))$. Thus $\mathbb{q} \in \{...\}$ and thus $p(\mathbb{q}) \geq_{\mathbb{P}} \inf\{...\}$.

Suppose, toward a contradiction, that $p(\mathbb{q}) \neq \inf\{...\}$. Since $\leq_{\mathbb{P}}$ is a partial ordering if follows that $p(\mathbb{q}) \not\leq_{\mathbb{P}} \inf\{...\}$. There is a $\mathbb{p}' \in \{...\}$ such that $p(\mathbb{q}) \not\leq_{\mathbb{P}} \mathbb{p}'$. Let

$$\mathbb{r} \leq_{\mathbb{P}} p(\mathbb{q}) \tag{5.44}$$ abst-itr-5-1-0

be such that $\mathbb{r} \perp_{\mathbb{P}} \mathbb{p}'$. Since i is incompatibility preserving, it follows that

$$i(\mathbb{r}) \perp_{\mathbb{Q}} \underbrace{i(\mathbb{p}')}_{= i(p(\mathbb{q})) \geq_{\mathbb{Q}} \mathbb{q} \text{ since } \mathbb{p}' \in \{...\} \text{ and by } (3)}. \tag{5.45}$$ abst-itr-5-2

Thus

$$i(\mathbb{r}) \perp_{\mathbb{Q}} \mathbb{q}. \tag{5.46}$$ abst-itr-5-3

By (2) it follows that $\mathbb{r} \perp_{\mathbb{P}} p(\mathbb{q})$. This is a contradiction to (5.44). ❑ (Lemma 5.8)

Corollary 5.9 (1) *If* $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ *is a separative abstract iteration, then we have* $i_{\alpha,\beta}(p_{\beta,\alpha}(\mathbb{q})) \ge_{\mathbb{P}_\beta} \mathbb{q}$ *for all* $\alpha \le \beta < \delta$. P-abst-itr-1

(2) *If* $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ *is a sub-Boolean abstract iteration, then, for any* $\alpha \le \beta < \gamma$ *and* $\mathbb{p} \in \mathbb{P}_\alpha$, *we have* $i_{\alpha,\beta}(\mathbb{p}) = \sup\{\mathbb{q} \in \mathbb{P}_\beta : p_{\beta,\alpha}(\mathbb{q}) = \mathbb{p}\}$.

(3) *If* $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ *is a sub-Boolean abstract iteration, then, for any* $\alpha \le \beta < \gamma$ *and* $\mathbb{q} \in \mathbb{P}_\beta$, *we have* $p_{\beta,\alpha}(\mathbb{q}) = \inf\{\mathbb{p} \in \mathbb{P}_\alpha : \mathbb{q} \le_{\mathbb{P}_\beta} i_{\alpha,\beta}(\mathbb{p})\}$. ❑

For an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ for a limit ordinal δ, the *inverse limit* $\mathbb{P}^{\mathcal{I},inv}_\delta$ of $\mathcal{I}$ is defined as the the poset with underlying set

(5.47) $\{\mathbb{f} \in \prod_{\alpha<\delta} \mathbb{P}_\alpha : p_{\beta,\alpha}(\mathbb{f}(\beta)) \sim_{\mathbb{P}_\alpha} \mathbb{f}(\alpha) \text{ for all } \alpha < \beta < \delta\}$ abst-itr-6

with the ordering defined by

(5.48) $\mathbb{f}_0 \le_{\mathbb{P}^{\mathcal{I},inv}_\delta} \mathbb{f}_1 \;\Leftrightarrow\; \mathbb{f}_0(\alpha) \le_{\mathbb{P}_\alpha} \mathbb{f}_1(\alpha) \text{ for all } \alpha < \delta$ abst-itr-7

and with the $\mathbb{1}$-element identical with $\mathbb{1}_{\mathbb{P}^{\mathcal{I},inv}_\delta}$, i.e.,

(5.49) $\mathbb{1}_{\mathbb{P}^{\mathcal{I},inv}_\delta}(\alpha) = \mathbb{1}_{\mathbb{P}_\alpha} \text{ for all } \alpha < \delta.$ abst-itr-8

The *direct limit* $\mathbb{P}^{\mathcal{I},dir}_\delta$ of $\mathcal{I}$ is defined as the subordering of $\mathbb{P}^{\mathcal{I},inv}_\delta$ with the underlying set

(5.50) $\{\mathbb{f} \in \prod_{\alpha<\delta} \mathbb{P}_\alpha : \mathbb{f} \in \mathbb{P}^{\mathcal{I},inv}_\delta, \text{ there is a } \beta_0 < \delta \text{ such that } \mathbb{f}(\alpha) \sim_{\mathbb{P}_\alpha} i_{\beta_0,\alpha}(\mathbb{f}(\beta_0)) \text{ for all } \beta_0 < \alpha < \delta\}$ abst-itr-9

with the ordering $\le^{\mathbb{P}^{\mathcal{I},dir}_\delta}$ which is the restriction of $\le^{\mathbb{P}^{\mathcal{I},inv}_\delta}$ to $\mathbb{P}^{\mathcal{I},dir}_\delta$, that is, the ordering defined by

(5.51) $\mathbb{f}_0 \le_{\mathbb{P}^{\mathcal{I},dir}_\delta} \mathbb{f}_1 \;\Leftrightarrow\; \mathbb{f}_0(\alpha) \le_{\mathbb{P}_\alpha} \mathbb{f}_1(\alpha) \text{ for all } \alpha < \delta,$ abst-itr-10

and with the $\mathbb{1}$-element $\mathbb{1}_{\mathbb{P}^{\mathcal{I},inv}_\delta}$ defined by

(5.52) $\mathbb{1}_{\mathbb{P}^{\mathcal{I},dir}_\delta}(\alpha) = \mathbb{1}_{\mathbb{P}_\alpha} \text{ for all } \alpha < \delta.$ abst-itr-11

For an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ for a limit ordinal δ, let $\mathbb{P}_\delta = \mathbb{P}^{\mathcal{I},dir}_\delta$ or $\mathbb{P}^{\mathcal{I},inv}_\delta$ and let $i_{\alpha,\delta} : \mathbb{P}_\alpha \to \mathbb{P}_\delta$ and $p_{\delta,\alpha} : \mathbb{P}_\delta \to \mathbb{P}_\alpha$ for $\alpha < \delta$ be defined by

(5.53) $$i_{\alpha,\delta}(\mathbb{p})(\xi) = \begin{cases} p_{\alpha,\xi}(\mathbb{p}) & \text{if } \xi \le \alpha \qquad \text{... (5.53a)} \\ i_{\alpha,\xi}(\mathbb{p}) & \text{if } \alpha < \xi < \delta \qquad \text{... (5.53b)} \end{cases}$$ abst-itr-12

for $\mathbb{p} \in \mathbb{P}_\alpha$, and

(5.54) $p_{\delta,\alpha}(\mathbb{f}) = \mathbb{f}(\alpha)$ abst-itr-13

for all $\mathbb{f} \in \mathbb{P}_\delta$. We define $i_{\delta,\delta} = \mathrm{id}_{\mathbb{P}_\delta}$ and $p_{\delta,\delta} = \mathrm{id}_{\mathbb{P}_\delta}$.

Lemma 5.10 *Suppose that $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is a separative abstract iteration such that δ is a limit ordinal. Let $\mathbb{P}_\delta = \mathbb{P}_\delta^{\mathcal{I},inv}$ or $\mathbb{P}_\delta^{\mathcal{I},dir}$.* P-abst-itr-1-0

(1) *$i_{\alpha,\delta} : \mathbb{P}_\alpha \to \mathbb{P}_\delta$ for $\alpha < \delta$ given as above is well-defined and it is a complete embedding.*

(2) *$p_{\delta,\alpha} : \mathbb{P}_\delta \to \mathbb{P}_\alpha$ given as above is a projection.*

Proof. (1): For $\mathbb{p} \in \mathbb{P}_\alpha$, we show first that $i_{\alpha,\delta}(\mathbb{p})\mathbb{P}_\delta^{\mathcal{I},inv}$ for all $\mathbb{p} \in \mathbb{P}_\alpha$.

Suppose $\xi < \eta < \delta$. For $\mathbb{p} \in \mathbb{P}_\alpha$, we have to show that $p_{\eta,\xi}(i_{\alpha,\delta}(\mathbb{p})(\eta)) \sim_{\mathbb{P}_\xi} i_{\alpha,\delta}(\mathbb{p})(\xi)$.

If $\xi < \eta \le \alpha$, then

$$\begin{aligned} p_{\eta,\xi}(i_{\alpha,\delta}(\mathbb{p})(\eta)) &= p_{\eta,\xi}(p_{\alpha,\eta}(\mathbb{p})) && \text{;by (5.53a)} \\ &\sim_{\mathbb{P}_\xi} p_{\alpha,\xi}(\mathbb{p}) && \text{;by } \mathcal{I} \models (5.30)\ . \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi) && \text{;by (5.53a)} \end{aligned}$$

If $\alpha \le \xi < \eta$, then

$$\begin{aligned} p_{\eta,\xi}(i_{\alpha,\delta}(\mathbb{p})(\eta)) &= p_{\eta,\xi}(i_{\alpha,\eta}(\mathbb{p})) && \text{;by (5.53b)} \\ &= \underbrace{p_{\eta,\xi}(i_{\xi,\eta}}_{\sim_{\mathbb{P}_\alpha} \mathrm{id}_{\mathbb{P}_\xi} \text{ by } \mathcal{I} \models (5.31)} (i_{\alpha,\xi}(\mathbb{p}))) && \text{;by } \mathcal{I} \models (5.28) \\ &\sim_{\mathbb{P}_\xi} i_{\alpha,\xi}(\mathbb{p}) \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi). && \text{;by (5.53b)} \end{aligned}$$

Finally, if $\xi < \alpha$ and $\eta > \alpha$, then

$$\begin{aligned} p_{\eta,\xi}(i_{\alpha,\delta}(\mathbb{p})(\eta)) &= p_{\eta,\xi}(i_{\alpha,\eta}(\mathbb{p})) && \text{;by (5.53b)} \\ &\sim_{\mathbb{P}_\xi} p_{\alpha,\xi}(\underbrace{p_{\eta,\alpha}(i_{\alpha,\eta}}_{\sim_{\mathbb{P}_\alpha} \mathrm{id}_{\mathbb{P}_\alpha} \text{ by } \mathcal{I} \models (5.31)}(\mathbb{p}))) && \text{; by } \mathcal{I} \models\ (5.30) \\ &\sim_{\mathbb{P}_\xi} p_{\alpha,\xi}(\mathbb{p}) && \text{;by Lemma 3.13,(1)} \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi). && \text{;by (5.53a)} \end{aligned}$$

Now that we established $i_{\alpha,\delta}(\mathbb{p}) \in \mathbb{P}_\delta^{\mathcal{I},inv}$, (5.53b) implies that $\imath_{\alpha,\delta}(\mathbb{p}) \in \mathbb{P}_\delta^{\mathcal{I},dir}$.

$i_{\alpha,\delta}$ is clearly order preserving.

$i_{\alpha,\delta}$ is incompatibility preserving: Suppose that $i_{\alpha,\delta}(\mathbb{p}_0) \top_{\mathbb{P}_\delta} i_{\alpha,\delta}(\mathbb{p}_1)$ and $\mathbb{f} \le_{\mathbb{P}_\delta} i_{\alpha,\delta}(\mathbb{p}_0)$, $i_{\alpha,\delta}(\mathbb{p}_1)$. Then $\mathbb{f}(\alpha) \le_{\mathbb{P}_\alpha} \underbrace{i_{\alpha,\delta}(\mathbb{p}_0)(\alpha)}_{=\mathbb{p}_0}, \underbrace{i_{\alpha,\delta}(\mathbb{p}_1)(\alpha)}_{=\mathbb{p}_1}$. Thus $\mathbb{p}_0 \top_{\mathbb{P}_\alpha} \mathbb{p}_1$.

$i_{\alpha,\delta}$ is a complete embedding: We show that $i_{\alpha,\delta}$ satisfies (2.34) in Lemma 2.18. More exactly, we show that, for $\mathbb{f} \in \mathbb{P}_\delta$, $\mathbb{f}(\alpha)$ is a projection of $\mathbb{f}$ for $i_{\alpha,\delta}$. Suppose that

$$\mathbb{r} \le_{\mathbb{P}_\alpha} \mathbb{f}(\alpha). \tag{5.55}$$ abst-itr-13-0

Let $\mathbb{g} \in \prod_{\alpha<\delta} \mathbb{P}_\alpha$ be defined by

$$\mathbb{g}(\xi) = \begin{cases} p_{\alpha,\xi}(\mathbb{r}), & \text{if } \xi \leq \alpha; \\ i_{\alpha,\xi}(\mathbb{r}) \wedge \mathbb{f}(\xi), & \text{if } \alpha < \xi < \delta \end{cases} \tag{5.56}$$ abst-itr-14

for $\xi < \delta$. Note that $i_{\alpha,\xi}(\mathbb{r}) \wedge \mathbb{f}(\xi)$ in (5.56) exists by (5.32).

The following claim show that $\mathbb{g}$ is a common extension of $\mathbb{f}$ and $i_{\alpha,\delta}(\mathbb{r})$ in $\mathbb{P}_\delta$. Since $\mathbb{r}$ was arbitrary, this implies that $\mathbb{f}(\alpha)$ is a projection of $\mathbb{f}$ for $i_{\alpha,\delta}$. Since $\mathbb{f}$ was arbitrary, this proves the completeness of $i_{\alpha,\delta}$.

Claim 5.10.1 *(α) $\mathbb{g} \in \mathbb{P}^{\mathcal{I},inv}_\delta$. (β) If $\mathbb{f} \in \mathbb{P}^{\mathcal{I},dir}_\delta$, then $\mathbb{g} \in \mathbb{P}^{\mathcal{I},dir}_\delta$. Thus $\mathbb{g} \in \mathbb{P}_\delta$.* Cl-abst-itr-0
(γ) $\mathbb{g}$ is a common extension of $i_{\alpha,\delta}(\mathbb{r})$ and $\mathbb{f}$ in $\mathbb{P}_\delta$. Hence $i_{\alpha,\delta}(\mathbb{r}) \top_{\mathbb{P}_\delta} \mathbb{f}$.

$\vdash$ (α): We first show that $\mathbb{g} \in \mathbb{P}^{\mathcal{I},inv}_\delta$. Suppose $\xi < \eta < \delta$.

If $\xi < \eta < \alpha$, then

$$\mathbb{g}(\xi) = p_{\alpha,\xi}(\mathbb{r}) \sim_{\mathbb{P}_\xi} p_{\eta,\xi}(p_{\alpha,\eta}(\mathbb{r})) = p_{\eta,\xi}(\mathbb{g}(\eta)). \tag{5.57}$$ abst-itr-14-0

If $\alpha \leq \xi < \eta$, then, by the definition (5.56) of $\mathbb{g}$,

$$\mathbb{g}(\eta) = i_{\alpha,\eta}(\mathbb{r}) \wedge \mathbb{f}(\eta) = i_{\xi,\eta}(i_{\alpha,\xi}(\mathbb{r})) \wedge \mathbb{f}(\eta). \tag{5.58}$$ abst-itr-15

By Lemma 5.8, (5), it follows that

$$p_{\eta,\xi}(\mathbb{g}(\eta)) = i_{\alpha,\xi}(\mathbb{r}) \wedge p_{\eta,\xi}(\mathbb{f}(\eta)) \sim_{\mathbb{P}_\xi} i_{\alpha,\xi}(\mathbb{r}) \wedge \mathbb{f}(\xi) = \mathbb{g}(\xi) \tag{5.59}$$ abst-itr-16

where the $\sim_{\mathbb{P}_\xi}$ above is because of $\mathbb{f} \in \mathbb{P}^{\mathcal{I},inv}_\delta$ and (5.47). Note that, for $\xi = \alpha$, we have $i_{\alpha,\alpha}(\mathbb{r}) \wedge \mathbb{f}(\alpha) = \mathbb{r} \wedge \mathbb{f}(\alpha) \sim_{\mathbb{P}_\alpha} \mathbb{r}$ by (5.55). Thus,

$$p_{\eta,\alpha}(\mathbb{g}(\eta)) \sim_{\mathbb{P}_\alpha} \underbrace{\mathbb{r}}_{= \mathbb{g}(\alpha)}. \tag{5.60}$$ abst-itr-17

If $\eta > \alpha$ and $\xi < \alpha$, then

$$\mathbb{g}(\xi) = p_{\alpha,\xi}(\mathbb{r}) \underset{\uparrow \text{ by (5.60)}}{\sim_{\mathbb{P}_\xi}} p_{\alpha,\xi}(p_{\eta,\alpha}(\mathbb{g}(\eta))) \sim_{\mathbb{P}_\xi} p_{\eta,\xi}(\mathbb{g}(\eta)). \tag{5.61}$$ abst-itr-18

By (5.57), (5.59) and (5.61), we have $\mathbb{g} \in \mathbb{P}^{\mathcal{I},inv}_\delta$.

(β): Suppose that $\mathbb{f} \in \mathbb{P}^{\mathcal{I},dir}_\delta$ and let $\beta_0 < \delta$ be such that

$$\mathbb{f}(\xi) \sim_{\mathbb{P}_\xi} i_{\beta_0,\xi}(\mathbb{f}(\beta_0)) \text{ for all } \beta_0 < \xi < \delta. \tag{5.62}$$ abst-itr-18-0

Without loss of generality, we may assume that $\beta_0 > \alpha$.

For all $\beta_0 < \xi < \delta$, we have

$$\begin{aligned} \mathbb{g}(\xi) &= i_{\alpha,\xi}(\mathbb{r}) \wedge \mathbb{f}(\xi) && \text{;by (5.56)} \\ &\sim_{\mathbb{P}_\xi} i_{\alpha,\xi}(\mathbb{r}) \wedge i_{\beta_0,\xi}(\mathbb{f}(\beta_0)) && \text{;by (5.62)} \\ &\sim_{\mathbb{P}_\xi} i_{\beta_0,\xi}(i_{\alpha,\beta_0}(\mathbb{r})) \wedge i_{\beta_0,\xi}(\mathbb{f}(\beta_0)) && \text{;by Lemma 3.13, (3)} \\ &\sim_{\mathbb{P}_\xi} i_{\beta_0,\xi}(i_{\alpha,\beta_0}(\mathbb{r}) \wedge \mathbb{f}(\beta_0)) && \text{;by Lemma 3.13, (2)} \\ &= i_{\beta_0,\xi}(\mathbb{g}(\beta_0)) && \text{;by (5.56).} \end{aligned} \tag{5.63}$$ abst-itr-19

This shows that $\mathbb{g} \in \mathbb{P}_\delta^{\mathcal{I},dir}$.

(γ): By the definition of $\mathbb{g}$, and since $\leq_{\mathbb{P}_\delta}$ is defined by coordinate-wise comparison, we obtain $\mathbb{g} \leq_{\mathbb{P}_\delta} i_{\alpha,\delta}(\mathbb{r}), \mathbb{f}$. ⊣ (Claim 5.10.1)

(2): The well-definedness and the order preserving of $p_{\delta,\alpha}$ are clear. Also $p_{\delta,\alpha}(\mathbb{1}_{\mathbb{P}_\delta}) = \mathbb{1}_\alpha$ is immediate by the definition (5.49) or (5.52) of $\mathbb{1}_{\mathbb{P}_\delta}$ and (5.54).

Suppose that $\mathbb{p} \in \mathbb{P}_\alpha$, $\mathbb{f} \in \mathbb{P}_\delta$ and $\mathbb{p} \leq_{\mathbb{P}_\alpha} p_{\delta,\alpha}(\mathbb{f}) = \mathbb{f}(\alpha)$. Let $\mathbb{g} \in \prod_{\xi<\delta} \mathbb{P}_\xi$ be defined by

$$\mathbb{g}(\xi) = \begin{cases} p_{\alpha,\xi}(\mathbb{p}), & \text{if } \xi \leq \alpha; \\ i_{\alpha,\xi}(\mathbb{p}) \wedge \mathbb{f}(\xi), & \text{if } \alpha < \xi < \delta \end{cases} \tag{5.64}$$ abst-itr-20

By Claim 5.10.1, $\mathbb{g} \in \mathbb{P}_\delta$. $\mathbb{g} \leq_{\mathbb{P}_\delta} \mathbb{f}$ by (5.64) and $p_{\delta,\alpha}(\mathbb{g}) = \mathbb{g}(\alpha) = \mathbb{p} \leq_{\mathbb{P}_\alpha} p_{\delta,\alpha}(\mathbb{g})$. This shows that $p_{\delta,\alpha}$ is a projection. ❑ (Lemma 5.10)

Lemma 5.11 *Suppose that $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \leq \beta < \delta\rangle\rangle$ is a sub-Boolean abstract iteration for a limit ordinal δ. If $\mathbb{P}_\delta = \mathbb{P}_\delta^{\mathcal{I},dir}$ or $\mathbb{P}_\delta^{\mathcal{I},inv}$ and, the mappings $i_{\alpha,\delta} : \mathbb{P}_\alpha \to \mathbb{P}_\delta$ and $p_{\delta,\alpha} : \mathbb{P}_\delta \to \mathbb{P}_\alpha$ for $\alpha \leq \delta$ are defined as above. Then $\tilde{\mathcal{I}} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta+1\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \leq \beta < \delta+1\rangle\rangle$ is also a sub-Boolean abstract iteration.* P-abst-itr-2

Proof. We already have shown that $\tilde{\mathcal{I}} \models$ (5.27), (5.29) in Lemma 5.10.

Note that, since $\mathcal{I}$ is sub-Boolean, we have

$$\mathbb{P}_\delta^{\mathcal{I},inv} = \{\mathbb{f} \in \textstyle\prod_{\alpha<\delta} \mathbb{P}_\alpha : \mathbb{f} \in \mathbb{P}_\delta^{\mathcal{I},inv}, \text{ there is a } \beta_0 < \delta \text{ such that } \mathbb{f}(\alpha) = i_{\beta_0,\alpha}(\mathbb{f}(\beta_0)) \text{ for all } \beta_0 < \alpha < \delta\}. \tag{5.50$'$}$$ abst-itr-9

To prove $\tilde{\mathcal{I}} \models$ (5.28), suppose $\alpha < \beta < \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$. We want to prove that $i_{\beta,\delta}(i_{\alpha,\beta}(\mathbb{p}))(\xi) = i_{\alpha,\delta}(\mathbb{p})(\xi)$ holds for all $\xi < \delta$.

If $\xi < \alpha$ then we have

$$\begin{aligned} i_{\beta,\delta}(i_{\alpha,\beta}(\mathbb{p}))(\xi) &= p_{\beta,\xi}(i_{\alpha,\beta}(\mathbb{p})) \\ &= p_{\alpha,\xi}(p_{\beta,\alpha}(i_{\alpha,\beta}(\mathbb{p}))) \\ &= p_{\alpha,\xi}(\mathbb{p}) \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi). \end{aligned} \tag{5.65}$$ abst-itr-21

If $\alpha < \xi < \beta$,

$$\begin{aligned} i_{\beta,\delta}(i_{\alpha,\beta}(\mathbb{p}))(\xi) &= p_{\beta,\xi}(i_{\alpha,\beta}(\mathbb{p})) \\ &= p_{\beta,\xi}(i_{\xi,\eta}(i_{\alpha,\xi}(\mathbb{p}))) \\ &= i_{\alpha,\xi}(\mathbb{p}) \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi). \end{aligned} \tag{5.66}$$ abst-itr-22

If $\beta < \xi$,

$$(5.67)\qquad \begin{aligned} i_{\beta,\delta}(i_{\alpha,\beta}(\mathbb{p}))(\xi) &= i_{\beta,\xi}(i_{\alpha,\beta}(\mathbb{p})) \\ &= i_{\alpha,\xi}(\mathbb{p}) \\ &= i_{\alpha,\delta}(\mathbb{p})(\xi). \end{aligned}$$ abst-itr-23

Thus we have $i_{\beta,\delta} \circ i_{\alpha,\beta} = i_{\alpha,\delta}$.

$\tilde{\mathcal{I}} \models$ (5.30′): Suppose that $\alpha \leq \beta < \delta$ and $\mathbb{f} \in \mathbb{P}_\delta$. Then we have

$$(5.68)\qquad \begin{aligned} p_{\beta,\alpha}(p_{\delta,\beta}(\mathbb{f})) &= p_{\beta,\alpha}(\mathbb{f}(\beta)) \\ &= \mathbb{f}(\alpha) \\ &= p_{\delta,\beta}(\mathbb{f}). \end{aligned}$$ abst-itr-24

Thus $p_{\beta,\alpha} \circ p_{\delta,\beta} = p_{\delta,\alpha}$.

$\tilde{\mathcal{I}} \models$ (5.31): Suppose that $\alpha < \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$. Then

$$(5.69)\qquad p_{\delta,\alpha}(i_{\alpha,\delta}(\mathbb{p})) = (i_{\alpha,\delta}(\mathbb{p}))(\alpha) = \mathbb{p}.$$ abst-itr-25

Thus, $p_{\delta,\alpha} \circ i_{\alpha,\delta} = \mathrm{id}_{\mathbb{P}_\alpha}$.

$\tilde{\mathcal{I}} \models$ (5.32): Suppose that $\alpha < \delta$, $\mathbb{p} \in \mathbb{P}_\alpha$ and $\mathbb{f} \in \mathbb{P}_\delta$ are such that $p_{\delta,\alpha}(\mathbb{f}) = \mathbb{f}(\alpha) \geq_{\mathbb{P}_\alpha} \mathbb{p}$.

Let $\mathbb{g} \in \prod_{\xi<\delta} \mathbb{P}_\xi$ be defined by

$$(5.70)\qquad \mathbb{g}(\xi) = \begin{cases} p_{\alpha,\xi}(\mathbb{p}), & \text{if } \xi < \alpha; \\ i_{\alpha,\xi}(\mathbb{p}) \wedge \mathbb{f}(\xi), & \text{if } \xi \geq \alpha \end{cases}$$ abst-itr-26

for $\xi < \delta$. Then $\mathbb{g} \in \mathbb{P}_\delta$ by Claim 5.10.1. We have $\mathbb{g} = i_{\alpha,\delta}(\mathbb{p}) \wedge \mathbb{f}$ since the corresponding equality holds for each coordinate $\xi < \delta$.

To prove that $\tilde{\mathcal{I}}$ is separative, It is enough to show that $\mathbb{P}_\delta$ is separative. Suppose that $\mathbb{f}$, $\mathbb{g} \in \mathbb{P}_\delta$ and $\mathbb{f} \not\leq_{\mathbb{P}_\delta} \mathbb{g}$. Then there is $\alpha < \delta$ such that $\mathbb{f}(\alpha) \not\leq_{\mathbb{P}_\alpha} \mathbb{g}(\alpha)$.

Since $\mathbb{P}_\alpha$ is separative, there is $\mathbb{r} \in \mathbb{P}_\alpha$ such that $\mathbb{r} \leq_{\mathbb{P}_\alpha} \mathbb{f}(\alpha)$ and $\mathbb{r} \perp_{\mathbb{P}_\alpha} \mathbb{g}(\alpha)$.

Let $\mathbb{h} \in \prod_{\xi<\delta} \mathbb{P}_\xi$ be defined by

$$(5.71)\qquad \mathbb{g}(\xi) = \begin{cases} p_{\alpha,\xi}(\mathbb{r}), & \text{if } \xi < \alpha; \\ i_{\alpha,\xi}(\mathbb{r}) \wedge \mathbb{f}(\xi), & \text{if } \xi \geq \alpha \end{cases}$$ abst-itr-27

for $\xi < \delta$. Then $\mathbb{h} \in \mathbb{P}_\delta$ by Claim 5.10.1. A coordinate-wise comparison gives $\mathbb{h} \leq_{\mathbb{P}_\delta} \mathbb{f}$ and $\mathbb{h} \perp_{\mathbb{P}_\delta} \mathbb{g}$.

To show that $\mathbb{P}_\delta$ is anti-symmetric, suppose that $\mathbb{f}$, $\mathbb{g} \in \mathbb{P}_\delta$ are such that $\mathbb{f} \leq_{\mathbb{P}_\delta} \mathbb{g}$ and $\mathbb{g} \leq_{\mathbb{P}_\delta} \mathbb{f}$. Then, for all $\alpha < \delta$ we have $\mathbb{f}(\alpha) \leq_{\mathbb{P}_\alpha} \mathbb{g}(\alpha)$ and $\mathbb{g}(\alpha) \leq_{\mathbb{P}_\alpha} \mathbb{f}(\alpha)$. Since $\mathbb{P}_\alpha$ is sub-Boolean, it follows that $\mathbb{f}(\alpha) = \mathbb{g}(\alpha)$. If follows that $\mathbb{f} = \mathbb{g}$. ❑ (Lemma 5.11)

For an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \leq \beta < \delta\rangle\rangle$ and $\delta' < \delta$ we denote with $\mathcal{I} \restriction \delta'$ the iteration $\mathcal{I}$ restricted to the index set δ'. That is,

$$(5.72)\qquad \mathcal{I} \restriction \delta' = \langle\langle \mathbb{P}_\alpha : \alpha < \delta'\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \leq \beta < \delta'\rangle\rangle.$$ abst-itr-28

For abstract iterations $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ and $\mathcal{I}' = \langle\langle \mathbb{P}'_\alpha : \alpha < \delta\rangle, \langle i'_{\alpha,\beta}, p'_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$, $\mathcal{I}'$ is a *dense sub-iteration* of $\mathcal{I}$ if $\mathbb{P}'_\alpha$ is a dense sub-poset of $\mathbb{P}_\alpha$ for all α, and $i'_{\alpha,\beta} \subseteq i_{\alpha,\beta}$, $p'_{\beta,\alpha} \subseteq p_{\beta,\alpha}$ for all $\alpha \le \beta < \delta$.

For abstract iterations $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ and $\mathcal{I}' = \langle\langle \mathbb{P}'_\alpha : \alpha < \delta\rangle, \langle i'_{\alpha,\beta}, p'_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$, a sequence of complete embeddings $\Phi = \langle \varphi_\alpha : \alpha < \delta\rangle$ with $\varphi_\alpha : \mathbb{P}_\alpha \overset{\lessdot}{\to} \mathbb{P}'_\alpha$ for $\alpha < \delta$ is called a *complete embedding of $\mathcal{I}$ to $\mathcal{I}'$* if

$$\varphi_\beta \circ i_{\alpha,\beta} = i'_{\alpha,\beta} \circ \varphi_\alpha. \qquad \begin{array}{ccc} \mathbb{P}_\beta & \xrightarrow{\varphi_\beta} & \mathbb{P}'_\beta \\ {\scriptstyle i_{\alpha,\beta}}\uparrow & \circlearrowleft & \uparrow{\scriptstyle i'_{\alpha,\beta}} \\ \mathbb{P}_\alpha & \xrightarrow[\varphi_\alpha]{} & \mathbb{P}'_\alpha \end{array} \tag{5.73}$$

abst-itr-28-0

If φ is a complete embedding of $\mathcal{I}$ into $\mathcal{I}'$ we shall write $\Phi : \mathcal{I} \overset{\lessdot}{\to} \mathcal{I}'$. A complete embedding $\Phi : \mathcal{I} \overset{\lessdot}{\to} \mathcal{I}'$ with $\Phi = \langle \varphi_\alpha : \alpha < \delta\rangle$ is a *dense embedding* if each $\varphi_\alpha : \mathbb{P}_\alpha \to \mathbb{P}'_\alpha$, $\alpha < \delta$ is a dense embedding. $\Phi = \langle \varphi_\alpha : \alpha < \delta\rangle$ is an *isomorphism* if each $\varphi_\alpha : \mathbb{P}_\alpha \to \mathbb{P}'_\alpha$, $\alpha < \delta$ is an isomorphism.

Suppose that δ is a limit ordinal and $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$ is an abstract iteration of length $\le \delta$. $\mathbb{f} \in \prod_{\xi \le \delta} \mathbb{P}_\xi$ is a *thread in $\mathcal{I} \restriction \delta$* if $p_{\beta,\alpha}(\mathbb{f}(\beta)) = \mathbb{f}(\alpha)$ holds for all $\alpha \le \beta < \delta$. $\mathbb{q} \in \mathbb{P}_\delta$ is the *limit of the thread $\mathbb{f}$* if $\mathbb{q} = \prod_{\xi<\delta} i_{\xi,\delta}(\mathbb{f}(\xi))$. We shall write $\prod \mathbb{f}$ to denote $\prod_{\xi<\delta} i_{\xi,\delta}(\mathbb{f}(\xi))$. If there is $\mathbb{q} \in \mathbb{P}_\delta$ such that $\mathbb{q} = \prod_{\xi<\delta} i_{\xi,\delta}(\mathbb{f}(\xi))$ then we shall simply say that $\prod \mathbb{f}$ exists in $\mathbb{P}_\alpha$.

We call $\mathbb{P}_\delta$ a *normal limit of $\mathcal{I} \restriction \delta$* if

$$D^{\mathcal{I}}_\delta = \{\mathbb{q} \in \mathbb{P}_\delta : \mathbb{q} \text{ is a limit of a thread in } \mathcal{I} \restriction \delta\} \tag{5.74}$$

abst-itr-32

is dense in $\mathbb{P}_\delta$.

We say that an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is *normal*, if $\mathbb{P}_\gamma$ is normal limit of $\mathcal{I} \restriction \gamma$ for all limit $\gamma < \delta$.

For a thread $\mathbb{f}$ in an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \kappa\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \kappa\rangle\rangle$, the support of $\mathbb{f}$ is defined by

$$\operatorname{supp}(\mathbb{f}) = \{\alpha < \delta : \mathbb{f}(\alpha+1) = i_{\alpha,\alpha+1}(\mathbb{f}(\alpha))\}. \tag{5.75}$$

abst-itr-33

For an abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$ for a limit ordinal γ, $\mathbb{P}_\delta$ is said to be a *direct normal limit* of $\mathcal{I} \restriction \delta$ if there is a dense sub-iteration $\mathcal{I}^d = \langle\langle \mathbb{P}^d_\alpha : \alpha < \delta\rangle, \langle i^d_{\alpha,\beta}, p^d_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ of $\mathcal{I} \restriction \delta$ such that

$$\begin{aligned} &\{\mathbb{f} : \mathbb{f} \text{ is a thread in } \mathcal{I}^d \text{ and } \textstyle\prod \mathbb{f} \text{ exists in } \mathbb{P}_\delta\} \\ &\quad = \{\mathbb{f} : \mathbb{f} \text{ is a thread in } \mathcal{I}^d \text{ and } \operatorname{supp}(\mathbb{f}) \text{ is bounded in } \delta\} \end{aligned} \tag{5.76}$$

abst-itr-34

and $\{\prod \mathbb{f} : \mathbb{f} \text{ is a thread in } \mathcal{I}^d \text{ and } \prod \mathbb{f} \text{ exists in } \mathbb{P}_\delta\}$ is dense in $\mathbb{P}_\delta$.

$\mathbb{P}_\delta$ is an *inverse normal limit* of $\mathcal{I} \restriction \delta$ if there is a dense sub-iteration $\mathcal{I}^d = \langle\langle \mathbb{P}^d_\alpha : \alpha < \delta\rangle, \langle i^d_{\alpha,\beta}, p^d_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ of $\mathcal{I} \restriction \delta$ such that

(5.77) $\{\mathbb{f} : \mathbb{f}$ is a thread in $\mathcal{I}^d$ and $\prod \mathbb{f}$ exists in $\mathbb{P}_\delta\}$ abst-itr-35
$= \{\mathbb{f} : \mathbb{f}$ is a thread in $\mathcal{I}^d\}$

and $\{\prod \mathbb{f} : \mathbb{f}$ is a thread in $\mathcal{I}^d\}$ is dense in $\mathbb{P}_\delta$.

An abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha < \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta < \delta\rangle\rangle$ is an ordinary normal iteration. If it is normal and, for all limit ordinal $\gamma < \delta$, $\mathbb{P}_\gamma$ is either a direct normal limit or a reverse normal limit of $\mathcal{I} \restriction \delta$.

Example 5.12 *For any ordinary iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta\rangle$, the abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$ constructed in Example 5.7 is an ordinary normal abstract iteration.* Ex-abst-itr-1

For all limit ordinal $\gamma \le \delta$, $\mathbb{P}_\gamma$ is a direct limit of $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha < \gamma, \beta < \gamma\rangle$ in the sense of 5.1 if and only if $\mathbb{P}_\gamma$ is a direct normal limit of the abstract iteration $\mathcal{I} \restriction \gamma$. ❑

Theorem 5.13 *For any ordinary iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \le \delta, \beta < \delta\rangle$, with the associated normal abstract iteration $\mathcal{I} = \langle\langle \mathbb{P}_\alpha : \alpha \le \delta\rangle, \langle i_{\alpha,\beta}, p_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$, there is a normal cBa abstract iteration $\tilde{\mathcal{I}} = \langle\langle \tilde{\mathbb{P}}_\alpha : \alpha \le \delta\rangle, \langle \tilde{i}_{\alpha,\beta}, \tilde{p}_{\beta,\alpha} : \alpha \le \beta \le \delta\rangle\rangle$ with a dense embedding $\Phi : \mathcal{I} \overset{\lessdot}{\to} \tilde{\mathcal{I}}$ with $\Phi = \langle \varphi_\alpha : \alpha \le \delta\rangle$. If $\mathcal{I}$ is ordinary then $\tilde{\mathcal{I}}$ is also ordinary and the set of the indices $\gamma < \delta$ such that $\mathbb{P}_\gamma$ is a direct limit coincides with the set of the indices $\gamma' < \delta$ such that $\tilde{\mathbb{P}}_{\gamma'}$ is a direct ordinary limit.* P-abst-itr-4

Proof.

❑ (Theorem 5.13)

5.3 A yet more general framework of iteration

Suppose that δ is an ordinal and $\langle I_\gamma : \gamma \in Lim(\delta+1)\rangle$ is such that each I_γ for $\gamma \in Lim(\delta+1)$ is an ideal over γ. Such sequence $\langle I_\gamma : \gamma \in Lim(\delta+1)\rangle$ of ideals is said to be *coherent* if generl

(5.78) $[\gamma]^{<\aleph_0} \subseteq I_\gamma$ for all $\gamma \in Lim(\delta+1)$; generl-0

(5.79) $\bigcup_{\gamma \in Lim(\delta+1), \gamma < \gamma_1} \subseteq I_{\gamma_1}$ for all $\gamma_1 \in Lim(\delta+1)$; generl-1

(5.80) for each $\gamma_0, \gamma_1 \in Lim(\delta+1)$ with $\gamma_0 < \gamma_1$, if $s \in I_{\gamma_1}$, then $s \cap \gamma_0 \in I_{\gamma_0}$. generl-2

For a coherent sequence $\vec{I} = \langle I_\gamma : \gamma \in Lim(\delta+1)\rangle$ of ideals, $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \le \delta, \beta < \delta\rangle$ is an $\vec{I}$-support iteration if

(5.1) Each $\mathbb{P}_\alpha$ for $\alpha \le \delta$ is a poset and the underlying set of $\mathbb{P}_\alpha$, which is also denoted by $\mathbb{P}_\alpha$ as before, consists of sequences of length α. In particular $\mathbb{P}_0 = \{\emptyset\}$ and $\mathbb{1}_{\mathbb{P}_0} = \emptyset$; ord-0

(5.2) $\underset{\sim}{\mathbb{Q}}_\alpha$ for each $\alpha < \delta$ is a $\mathbb{P}_\alpha$-name and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha = \langle \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{\le}_{\mathbb{Q}_\alpha}, \underset{\sim}{\mathbb{1}}_{\mathbb{Q}_\alpha}\rangle$ is a poset"; ord-1

(5.3) For $\beta < \alpha \leq \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$, $\mathbb{p} \restriction \beta \in \mathbb{P}_\beta$; ord-2

(5.4) For $\alpha < \delta$, ord-3

$\mathbb{P}_{\alpha+1} = \{\mathbb{p}^\frown\langle \underset{\sim}{\mathbb{q}} \rangle : \mathbb{p} \in \mathbb{P}_\alpha,\ \underset{\sim}{\mathbb{q}}$ is a canonical $\mathbb{P}_\alpha$-name, and $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{q}} \ \varepsilon\ \underset{\sim}{\mathbb{Q}}_\alpha$"$\}$; 定義に変更あり．要チェック．

(5.81) For all limit $\gamma \leq \delta$, $\mathbb{P}_\gamma = \{\mathbb{p} : \mathbb{p} \restriction \alpha \in \mathbb{P}_\alpha$ for all $\alpha < \gamma$ and $\mathrm{supp}(\mathbb{p}) \in I_\gamma\}$; generl-3

(5.6) $\mathbb{1}_{\mathbb{P}_\alpha}(\beta) = \mathbb{1}_{\underset{\sim}{\mathbb{Q}}_\beta}$ for all $\beta < \alpha \leq \delta$; ord-5

(5.7) For $\alpha \leq \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\alpha$, $\mathbb{p}' \leq_{\mathbb{P}_\alpha} \mathbb{p}$ if and only if $\mathbb{p}' \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\mathbb{p}'(\beta) \leq_{\underset{\sim}{\mathbb{Q}}_\beta} \mathbb{p}(\beta)$" for all $\beta < \alpha$. ord-6

Note that the condition (5.80) of the coherent sequence of ideals is necessary to make the combination of (5.3) and (5.81) in the definition of $\vec{I}$-support iteration meaningful.

The following two Lemmas can be proved similarly to Lemma 5.1 and Lemma 5.2. Note that we need the condition (5.79) for the proof of Lemma 5.15, (1) and (2).

Lemma 5.14 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta, \beta < \delta\rangle$ is a $\vec{I}$ support iteration for a coherent sequence $\vec{I} = \langle I_\gamma : \gamma \in Lim(\delta+1)\rangle$ of ideals.* P-generl-a

(1) *If $\delta_0 \leq \delta$, then $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta_0,\ \beta < \delta_0\rangle$ is also a $\vec{I} \restriction \delta_0 + 1$ support iteration.*

(2) $\mathbb{P}_0 = \{\emptyset\}$ *and* $\mathbb{1}_{\mathbb{P}_0} = \emptyset$.

(3) *For $\beta \leq \alpha \leq \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\alpha$, if $\mathbb{p}' \leq_{\mathbb{P}_\alpha} \mathbb{p}$ then $\mathbb{p}' \restriction \beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$.*

(4) *For any $\beta < \delta$, $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta$.*

(5) *For any limit $\eta < \delta$ and $\mathbb{p}$, $\mathbb{p}' \in \mathbb{P}_\eta$, $\mathbb{p}' \leq_{\mathbb{P}_\eta} \mathbb{p}$ if and only if $\mathbb{p}' \restriction \beta \leq_{\mathbb{P}_\beta} \mathbb{p} \restriction \beta$ for all $\beta < \eta$.*

Lemma 5.15 *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \delta,\ \beta < \delta\rangle$ is a $\vec{I}$-support iteration for a coherent sequence $\vec{I} = \langle I_\gamma : \gamma \in Lim(\delta+1)\rangle$ of ideals, $\alpha \leq \beta \leq \gamma \leq \delta$ and $\mathbb{p} \in \mathbb{P}_\alpha$.* P-generl-0

(1) $\mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} \in \mathbb{P}_\beta$.

(2) *If $\mathbb{q} \in \mathbb{P}_\beta$ and $\mathbb{p} \leq_{\mathbb{P}_\alpha} \mathbb{q} \restriction \alpha$, then $\mathbb{r} = \mathbb{p} \cup (\mathbb{q} \restriction (\beta \setminus \alpha)) \in \mathbb{P}_\beta$ and*

(5.82) $\mathbb{r} = \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta} \wedge \mathbb{q}$ *holds in* $\mathbb{P}_\beta$. generl-4

(3) $i_{\alpha,\beta} : \mathbb{P}_\alpha \to \mathbb{P}_\beta$; $\mathbb{p} \mapsto \mathbb{p}^\frown \vec{\mathbb{1}}_{\alpha,\beta}$ *is a complete embedding.*

(4) $i_{\alpha,\gamma} = i_{\beta,\gamma} \circ i_{\alpha,\beta}$.

(5) $p_{\beta,\alpha} : \mathbb{P}_\beta \to \mathbb{P}_\alpha$; $\mathbb{p} \mapsto \mathbb{p} \restriction \alpha$ *is a projection.*

(6) $p_{\beta,\alpha} \circ i_{\alpha,\beta} = \mathrm{id}_{\mathbb{P}_\alpha}$.

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