数学ノート (2018-)

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1 α-stationarity in $\mathcal{P}_\kappa(\lambda)$

alpha-stat

For a cardinal κ and a set X, we use the expressions $\mathcal{P}_\kappa(X)$ and $[X]^{<\kappa}$ interchangeably to denote $\{a \subseteq X : |a| < \kappa\}$ except that when we write $\mathcal{P}_\kappa(X)$ for a cardinal κ, we always assume that $\kappa \subseteq X$. More generally, for sets X, Y with $Y \subseteq X$ we write $\mathcal{P}_Y(X)$ to denote $[X]^{<|Y|}$.

For $\alpha \in \mathsf{On} \setminus 1$, α-stationarity of subsets of $\mathcal{P}_Y(X)$ is defined by recursion on α as follows: $S \subseteq \mathcal{P}_Y(X)$ is 1-*stationary* if $|Y|$ is regular uncountable and S is stationary in the sense of Jech (i.e. if it intersects with each club subset of $\mathcal{P}_Y(X)$).

For $\alpha \in \mathsf{On} \setminus 2$, S is *α-stationary* in $\mathcal{P}_Y(X)$ if, for any $0 < \beta < \alpha$ and β-stationary $T \subseteq \mathcal{P}_Y(X)$, there is $a \in S$ such that $T \cap \mathcal{P}_{Y\cap a}(a)$ is β-stationary in $\mathcal{P}_{Y\cap a}(a)$.

Similarly we say that $S \subseteq \mathcal{P}_Y(X)$ is *diagonally* 1-*stationary* if $|Y|$ is regular uncountable and S is stationary in the sense of Jech (i.e. if it intersects with each club subset of $\mathcal{P}_Y(X)$).

For $\alpha \in \mathsf{On} \setminus 2$, S is *diagonally α-stationary* in $\mathcal{P}_Y(X)$ if, for any $0 < \beta < \alpha$ and for any family $\langle T_x \subseteq \mathcal{P}_Y(X) : x \in X\rangle$ of sets such that each T_x is β-stationary subset of $\mathcal{P}_Y(X)$, there is $a \in S$ such that $T_x \cap \mathcal{P}_{Y\cap a}(a)$ is β-stationary in $\mathcal{P}_{Y\cap a}(a)$ for all $x \in a$.

P-a-stat-a-0

Lemma 1.1 *For any $\alpha = 1$ or 2, $Y \subseteq X$ with regular uncountable $|Y|$ and $S \subseteq \mathcal{P}_Y(X)$, if S is diagonally α-stationary in $\mathcal{P}_Y(X)$ then S is α-stationary in $\mathcal{P}_Y(X)$.*

Proof. For $\alpha = 1$, 1-stationarity and diagonal 1-stationarity coincide with stationarity by definition. Suppose that $S \subseteq \mathcal{P}_Y(X)$ is diagonally 2-stationary. For 1-stationary $T \subseteq \mathcal{P}_Y(X)$, let $\langle T_x : x \in X\rangle$ be defined by $T_x = T$ for all $x \in X$. Note that all T_x, $x \in X$ are diagonally 1-stationary. By the diagonal 2-stationarity of S, there is $a \in S$ such that $T_x \cap \mathcal{P}_{Y\cap a}(a)$ is diagonally 1-stationary for all $x \in a$. By the definition of T_x, $x \in X$, it follows that $T \cap \mathcal{P}_{Y\cap a}(a)$ is 1-stationary. This shows that S is 2-stationary. □ (Lemma 1.1)

P-a-stat-0

Lemma 1.2 *Suppose that κ is a supercompact cardinal. Then for any $\lambda \geq \kappa$ and $\alpha \in \kappa\setminus 1$, $\mathcal{P}_\kappa(\lambda)$ is α-stationary and diagonally α-stationary (in itself).*

Proof. We prove that $\mathcal{P}_\kappa(\lambda)$ is diagonally α-stationary for all $\alpha \in \kappa \setminus 1$. The proof for α-stationarity is similar.

For $\alpha = 1$, the assertion is trivial. So assume that $\alpha > 1$, $\beta \in \alpha \setminus 1$. Let $\lambda' = \lambda^\kappa$ and let $j : V \overset{\prec}{\rightarrow} M$ be such that $crit(j) = \kappa$, $j(\kappa) > \lambda'$ and

(1.1) $\quad {}^{\lambda'}M \subseteq M.$ a-stat-0

Then $j''\lambda \in M$ by (1.1).

Claim 1.2.1 *If $|x| < \kappa$ then $j(x) = j''x$.*

⊢ Let $f : \delta \to x$ be a surjection for some $\delta < \kappa$. We have $M \models$ "$j(f)''j(\delta) \equiv j(x)$" by elementarity. Since $j(\delta) = \delta$, $M \models j(f)''\delta \equiv j(x)$. For $\alpha < \delta$, $M \models j(f)(\alpha) \equiv j(f(\alpha))$ by elementarity. Thus $j(x) = \{j(y) : y \in x\} = j''x$. ⊣ (Claim 1.2.1)

Suppose that $\vec{T} = \langle T_\xi : \xi < \lambda\rangle$ is a sequence of diagonally β-stationary subsets in $\mathcal{P}_\kappa(\lambda)$. Since

(1.2) $\quad (\mathcal{P}_\kappa(j''\lambda))^M = (\mathcal{P}_\kappa(j''\lambda))^V$ [(1)], a-stat-0-0

we have

(1.3) $\quad M \models j(\vec{T})_{j(\xi)} \cap \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda) = j(T_\xi) \cap \mathcal{P}_\kappa(j''\lambda) = \{j(x) : x \in T_\xi\}$ a-stat-1

for all $\xi \in \lambda$. Since $\mathcal{P}(\mathcal{P}_\kappa(j''\lambda)) \subseteq M$ by (1.2) and (1.1), we have $\mathcal{P}^M(\mathcal{P}_\kappa(j''\lambda)) = \mathcal{P}^V(\mathcal{P}_\kappa(j''\lambda))$ and $\left({}^{\lambda'}\mathcal{P}(\mathcal{P}_\kappa(j''\lambda))\right)^V \in M$ by (1.1).

It follows that

(1.4) $\quad M \models \exists a \in \mathcal{P}_{j(\kappa)}(j(\lambda))\,\forall\xi \in a\,\left(j(\vec{T})_\xi \cap \mathcal{P}_{j(\kappa)\cap a}(a) \text{ is diagonally } \beta\text{-stationary}\right).$

(1) $(\mathcal{P}_\kappa(j''\lambda))^M \subseteq (\mathcal{P}_\kappa(j''\lambda))^V$ is clear. To prove $(\mathcal{P}_\kappa(j''\lambda))^M \supseteq (\mathcal{P}_\kappa(j''\lambda))^V$, let $x \in (\mathcal{P}_\kappa(j''\lambda))^V$ and let $\langle \beta_\xi : \xi < \delta\rangle$ be an emumeration of x for some $\delta < \kappa$. Let $\alpha_\xi = j^{-1}(\beta_\xi)$ for all $\xi < \delta$. $x = j(\{\alpha_\xi : \xi < \delta\}) \in M$ by Claim 1.2.1.

By elementarity and since $\beta < \kappa$

$$V \models \exists a \in \mathcal{P}_\kappa(\lambda)\, \forall \xi \in a\ (T_\xi \cap \mathcal{P}_{\kappa\cap a}(a) \text{ is diagonally } \beta\text{-stationary}). \tag{1.5}$$

This shows that $\mathcal{P}_\kappa(\lambda)$ is diagonally α-stationary. ❑ (Lemma 1.2)

P-a-stat-1

Lemma 1.3 (1) *If $\mathcal{P}_\kappa(\lambda)$ is 2-stationary, then κ is a limit cardinal.*

(2) *If $S \subseteq \mathcal{P}_\kappa(\lambda)$ is 2-stationary then for any stationary $T \subseteq \mathcal{P}_\kappa(\lambda)$ there are stationarily many $r \in S$ such that $T \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*

(3) *If $\mathcal{P}_\kappa(\lambda)$ is 2-stationary, then κ is a weakly Mahlo cardinal.*

Proof. (1): Suppose that $\kappa = \mu^+$ then $C = \{a \in \mathcal{P}_\kappa(\lambda) : |a| = \mu\}$ is a club and hence stationary. But, for any $r \in \mathcal{P}_\kappa(\lambda)$, $|\kappa \cap r| \leq \mu$ and hence $C \cap \mathcal{P}_{\kappa\cap r}(r) = \emptyset$.

(2): Suppose that $S \subseteq \mathcal{P}_\kappa(\lambda)$ is 2-stationary and $T \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Let $C \subseteq \mathcal{P}_\kappa(\lambda)$ be a club. We have to show that there is $r \in S\cap C$ such that $T\cap\mathcal{P}_{\kappa\cap r}(r)$ is stationary.

Let $f : {}^{\omega>}\lambda \to \lambda$ be such that

$$C_f = \{a \in \mathcal{P}_\kappa(\lambda) : \kappa \cap a \in \kappa \text{ and } a \text{ is closed with respect to } f\} \subseteq C. \tag{1.6}$$

Since C_f is a club, $T \cap C_f$ is stationary. Let $r \in S$ be such that $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary in $\mathcal{P}_{\kappa\cap r}(r)$. We have

Cl-a-stat-0

Claim 1.3.1 *$\kappa \cap r \in \kappa$.*

⊢ Otherwise, there is an $\alpha \in \sup(\kappa\cap r)\setminus r$. Let $Y = \{b \in \mathcal{P}_{\kappa\cap r}(r) : \sup(\kappa \cap b) > \alpha\}$. Y is club in $\mathcal{P}_{\kappa\cap r}(r)$ but $Y \cap C_f = \emptyset$. Thus $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r) \cap Y = \emptyset$. This is a contradiction to the stationarity of $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ in $\mathcal{P}_{\kappa\cap r}(r)$. ⊣ (Claim 1.3.1)

Cl-a-stat-1

Claim 1.3.2 *r is closed with respect to f.*

⊢ This is clear since $T \cap C_f$ is cofinal in $\mathcal{P}_{\kappa\cap r}(r)$ (with respect to $\subseteq$) and elements of $T \cap C_f$ are closed with respect to f. ⊣ (Claim 1.3.2)

From the Claims above it follows that $r \in C_f \subseteq C$ is as desired.

(3): Let $T = \{a \in \mathcal{P}_\kappa(\lambda) : \kappa \cap a \in \kappa\}$. T is a club and hence stationary. Let $r \in \mathcal{P}_\kappa(\lambda)$ be such that

$$T \cap \mathcal{P}_{\kappa\cap r}(r) \text{ is stationary in } \mathcal{P}_{\kappa\cap r}(r). \tag{1.7}$$ a-stat-2

Similarly to Claim 1.3.1, we can show that $\kappa \cap r \in \kappa$.

Cl-a-stat-2

Claim 1.3.3 *$\kappa \cap r$ is a cardinal.*

⊢ Otherwise there is $\mu < \kappa \cap r$ such that $|\kappa \cap r| = \mu$. But then the set $\{a \in \mathcal{P}_{\kappa\cap r}(r) : \sup(a \cap \kappa) \geq \mu\}$ is a club in $\mathcal{P}_{\kappa\cap r}(r)$ disjoint from T. ⊣ (Claim 1.3.3)

Cl-a-stat-3

Claim 1.3.4 *$\kappa \cap r$ is a regular cardinal.*

$\vdash$ Otherwise there is an $s \subseteq \kappa \cap r$ cofinal in $\kappa \cap r$ with $|s| < \kappa \cap r$. But then the set $\{a \in \mathcal{P}_{\kappa \cap r}(r) : \sup(a \cap \kappa) \supseteq s\}$ is a club in $\mathcal{P}_{\kappa \cap r}(r)$ disjoint from T. $\dashv$ (Claim 1.3.4)

Since there are stationarily many r with (1.7) by (2), it follows from Claim 1.3.4 that κ is weakly Mahlo. ❑ (Lemma 1.3)

A regular cardinal κ is said to be *c.c.c.-generically supercompact* if for every $\lambda \geq \kappa$ there is a c.c.c. poset $\mathbb{P}$ such that for a $(V, \mathbb{P})$-generic $\mathbb{G}$ there is an inner model $M \subseteq V[\mathbb{G}]$ with

(1.8) $\quad V[\mathbb{G}] \models$ "${}^{\lambda}M \subseteq M$" a-stat-2-0

and j with

(1.9) $\quad V[\mathbb{G}] \models$ "$j : V \xrightarrow{\prec} M$, $crit(j) = \kappa$ and $j(\kappa) > \lambda$". a-stat-2-1

This definition is stronger than the generic supercompactness by ccc posets in [5].

P-a-stat-1-a

Lemma 1.4 *Suppose that M is an inner model of V with ${}^{\lambda}M \subseteq M$ and $\mathbb{P} \in M$ a poset. If $\mathbb{G}$ is a $(V, \mathbb{P})$-generic filter then, in $V[\mathbb{G}]$, we have ${}^{\lambda}M[\mathbb{G}] \subseteq M[\mathbb{G}]$.*

Proof. Suppose that $\langle a_\alpha : \alpha < \lambda\rangle \in V[\mathbb{G}]$ is a sequence of elements of $M[\mathbb{G}]$. For each $\alpha < \lambda$, let $\underset{\sim}{a}_\alpha \in M$ be a $\mathbb{P}$-name such that $\underset{\sim}{a}_\alpha[\mathbb{G}] = a_\alpha$.

By the assumption on M, we have $\langle \underset{\sim}{a}_\alpha : \alpha < \lambda\rangle \in M$. Thus

(1.10) $\quad \underset{\sim}{s} = \{\langle op_{\mathbb{P}}(\underset{\sim}{a}_\alpha, \check{\alpha}), \mathbb{1}_{\mathbb{P}}\rangle : \alpha \in \lambda\} \in M.$

Since $\underset{\sim}{s}$ is a $\mathbb{P}$-name and $\underset{\sim}{s}[\mathbb{G}] = \langle \underset{\sim}{a}_\alpha[\mathbb{G}] : \alpha < \lambda\rangle = \langle a_\alpha : \alpha < \lambda\rangle$, it follows that $\langle a_\alpha : \alpha < \lambda\rangle \in M[G]$. ❑ (Lemma 1.4)

P-a-stat-1-a-0

Corollary 1.5 *If the statement "there exists a supercompact cardinal" is consistent (over* ZFC*) then so is the statement "$2^{\aleph_0}$ is c.c.c.-generically supercompact."*

Proof. Let κ be supercompact. Then, letting $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, we have

(1.11) $\quad V[\mathbb{G}] \models$ "$2^{\aleph_0} = \kappa$ and κ is c.c.c.-generically supercompact"

for any $(V, \mathbb{P})$-generic $\mathbb{G}$. ❑ (Corollary 1.5)

P-a-stat-1-0

Lemma 1.6 *Suppose that μ, κ, λ are regular uncountable cardinals with $\mu \leq \kappa \leq \lambda$.*

(1) *If $\mathbb{P}$ is a μ-cc poset and $\underset{\sim}{C}$ is a $\mathbb{P}$-name of a club subset of $\mathcal{P}_\kappa(\lambda)$ (in $V[\mathbb{G}]$), then there is $C \subseteq \mathcal{P}_\kappa(\lambda)$ (in V) such that C is club in $\mathcal{P}_\kappa(\lambda)$ and $\Vdash_{\mathbb{P}}$ "$C \subseteq \underset{\sim}{C}$".*

(2) *If $S \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary and $\mathbb{P}$ is a μ-cc poset, then we have $\Vdash_{\mathbb{P}}$ "$\check{S}$ is stationary in $\mathcal{P}_\kappa(\lambda)$".*

Proof. (1): Let $C = \{x \in \mathcal{P}_\kappa(\lambda) : \Vdash_{\mathbb{P}} \text{“}\check{x} \in \underset{\sim}{C}\text{”}\}$. It is easy to show that C is closed (with respect to $\subseteq$-increasing sequence of length $< \kappa$). To show that C is cofinal in $\mathcal{P}_\kappa(\lambda)$ (with respect to $\subseteq$), suppose $a \in \mathcal{P}_\kappa(\lambda)$. Let $\langle a_n, \underset{\sim}{a}_n : n \in \omega\rangle$ be a sequence such that

(1.12) $a_0 = a$; a-stat-3

(1.13) $\underset{\sim}{a}_n$ is a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}} \text{“}\check{a}_n \subseteq \underset{\sim}{a}_n \in \underset{\sim}{C}\text{”}$; a-stat-4

(1.14) $a_{n+1} \in \mathcal{P}_\kappa(\lambda)$ and $\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{a}_n \subseteq \check{a}_{n+1}\text{”}$. a-stat-5

Note that (1.14) is possible by the μ-cc of $\mathbb{P}$. Let $b = \bigcup_{n\in\omega} a_n$. Then $a \subseteq b$ by (1.12) and $b \in C$ by (1.13) and (1.14).

(2): Suppose that $\nVdash_{\mathbb{P}} \text{“}\check{S}$ is stationary in $\mathcal{P}_\kappa(\lambda)\text{”}$. Then there is $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \text{“}\check{S}$ is non stationary in $\mathcal{P}_\kappa(\lambda)\text{”}$. Let $\underset{\sim}{C}$ be a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{C}$ is a club subset of $\mathcal{P}_\kappa(\lambda)$ and $\underset{\sim}{C} \cap \check{S} = \emptyset\text{”}$. By (1) (applied to $\mathbb{P} \downarrow p$), there is a club $C \subseteq \mathcal{P}_\kappa(\lambda)$ in V such that $p \Vdash_{\mathbb{P}} \text{“}\check{C} \subseteq \underset{\sim}{C}\text{”}$. It follows that $C \cap S = \emptyset$. But this is a contradiction to the stationarity of S. □ (Lemma 1.6)

P-a-stat-2

Lemma 1.7 *If κ is c.c.c.-generically supercompact then $\mathcal{P}_\kappa(\lambda)$ is diagonally 2-stationary for all $\lambda \geq \kappa$.*

Proof. In V, let $\vec{S}\langle S_\xi : \xi \in \lambda\rangle$ be a sequence of stationary subsets of $\mathcal{P}_\kappa(\lambda)$. Let $\mathbb{P}$, $\mathbb{G}$, M, j be as in the definition of the c.c.c.-genric supercomactness.

In $V[\mathbb{G}]$ we have $j''\lambda \in M$ by (1.8). Also by (1.8) and by (1.9), we have

(1.15) $\mathcal{P}_\kappa(j''\lambda)^{V[\mathbb{G}]} = \mathcal{P}_\kappa(j''\lambda)^M = \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda)^M$ and

(1.16) $j(\vec{S})_{j(\xi)} \cap \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda)^M = \{j''x : x \in S_\xi\}$ for all $\xi \in \lambda$.

Since $V[\mathbb{G}] \models \text{“}S_\xi$ is stationary in $\mathcal{P}_\kappa(\lambda)\text{”}$ by Lemma 1.6, (2), we have $M \models \text{“}S_\xi$ is stationary in $\mathcal{P}_\kappa(\lambda)\text{”}$.

Thus we have $M \models \forall\xi \in {}''\lambda\, (j((\vec{S})_\xi \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda))$ is stationary) and

(1.17) $M \models \exists a \in \mathcal{P}_{j(\kappa)}(j(\lambda))\, \forall\xi \in a\, \left(j(\vec{S})_\xi \cap \mathcal{P}_{j(\kappa)\cap a}(a) \text{ is stationary}\right)$.

By elementarity, it follows that

(1.18) $V \models \exists a \in \mathcal{P}_\kappa(\lambda)\, \forall\xi \in a\, (S_\xi \cap \mathcal{P}_{\kappa\cap a}(a) \text{ is stationary})$.

□ (Lemma 1.7)

For regular cardinals κ, λ, λ' with $\kappa \leq \lambda \leq \lambda'$ and $S' \subseteq \mathcal{P}_\kappa(\lambda')$, let

(1.19) $S'_{\cap\lambda} = \{a \cap \lambda : a \in S'\}$. a-stat-6

For $S \subseteq \mathcal{P}_\kappa(\lambda)$, let

(1.20) $S^{\cup\lambda'\setminus\lambda} = \{a \cup b : a \in S,\, b \in \mathcal{P}_\kappa(\lambda' \setminus \lambda)\}$. a-stat-7

P-a-stat-3

Lemma 1.8 (0) *For all $\alpha \in \mathsf{On}$ if $S \subseteq \mathcal{P}_\kappa(\lambda)$ is (diagonally) α-stationary then any $\tilde{S} \subseteq \mathcal{P}_\kappa(\lambda)$ with $S \subseteq \tilde{S}$ is (resp., diagonally) α-stationary.*

(1) *For all $\alpha \in \mathsf{On}$, $S \subseteq \mathcal{P}_\kappa(\lambda)$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$ if and only if $S^{\cup\lambda'\setminus\lambda}$ is (resp., diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$.*

(2) *For all $\alpha \in \mathsf{On}$, if $S' \subseteq \mathcal{P}_\kappa(\lambda')$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$ then $S'_{\cap\lambda}$ is (resp./ diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$.*

Proof. (0): This can be proved by straightforward induction by $\alpha \in \mathsf{On} \setminus 1$.

(1): We prove the statement for diagonal stationarity.

(2): The assertion follows from (1) and (2). Suppose that $S' \subseteq \mathcal{P}_\kappa(\lambda')$ is (diagonally) α-stationary. Let $S = S'_{\cap\lambda}$. Then we have $S' \subseteq S^{\cup\lambda'\setminus\lambda}$. By (1), $S^{\cup\lambda'\setminus\lambda}$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$. Thus, by (2), $S = S'_{\cap\lambda}$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$.

❑ (Lemma 1.8)

2 Consistency of theories

consis

The following Theorem 2.1 answers a question Andrjei Kucharski asked during my stay in Katowice in 2018. Since the theorem is about the non-existence of a proof in meta-mathematics, it is a meta-(meta-mathematical) theorem.

The meta-(meta-mathemementical) Theorem must be a well-known fact. I thank Taishi Kurahashi for giving me a hint for the proof of the theorem.

P-consis-0

Theorem 2.1 *Assume that T is a weak set theory in a broad sense (including, e.g., the case "$T = \mathsf{PA}$") in which the Second Incompleteness Theorem can be formulated and proved. Let T' be a theory extending T and such that $T' \vdash consis(\ulcorner\ulcorner T\urcorner\urcorner)$.*

If there is a (meta-mathematical) proof of the consistency of T' from the assumption that T is consistent, then we can obtain a proof of the contradiction from T.

In other words, there is no (meta-mathematical) proof of the consistency of T' from the assumption that T is consistent.

Proof. Assume that there would be

(2.1) a (meta-mathematical) proof of the consistency of T' from the assumption that T is consistent. consis-a

Then we should be able to translate this proof to a proof of $consis(\ulcorner\ulcorner T\urcorner\urcorner) \rightarrow consis(\ulcorner\ulcorner T'\urcorner\urcorner)$ from T. Thus, we have

(2.2) $T \vdash\ consis(\ulcorner\ulcorner T\urcorner\urcorner) \rightarrow consis(\ulcorner\ulcorner T'\urcorner\urcorner)$. consis-1

Note that, since our meta-mathematics should be strictly constructive, the proof (2.1) must give

(2.3) an algorithm $\mathcal{A}$ such that, given a proof $\mathcal{P}$ of the contradiction from T', $\mathcal{A}$ gives us a proof $\mathcal{A}(\mathcal{P})$ of the contradiction from T. consis-1-0

By (2.2), it follows that

(2.4) $T + consis(\ulcorner\ulcorner T \urcorner\urcorner) \vdash consis(\ulcorner\ulcorner T' \urcorner\urcorner)$. consis-2

By the assumption on T' it follows that

(2.5) $T + consis(\ulcorner\ulcorner T \urcorner\urcorner) \vdash consis(\ulcorner\ulcorner T + consis(\ulcorner\ulcorner T \urcorner\urcorner) \urcorner\urcorner)$. consis-2-0

By the Second Incompleteness Theorem (applied to the theory $T + consis(\ulcorner\ulcorner T \urcorner\urcorner)$), it follows that $T + consis(\ulcorner\ulcorner T \urcorner\urcorner)$ is inconsistent. Since $T + consis(\ulcorner\ulcorner T \urcorner\urcorner)$ is a sub-theory of T', T' is also inconsistent. Hence, by the algorithm $\mathcal{A}$ in (2.3), we obtain a proof of the contradiction from T. □ (Theorem 2.1)

The theorem above applies to many situations in set-theory. For example:

There is no meta-mathematical proof of the consistency of ZFC + *"there is an inaccessible cardinal" from the assumption of the consistency of* ZFC *(provided that* ZFC *is consistent);*

There is no meta-mathematical proof of the consistency of ZFC + *"there is a measurable cardinal" from the assumption of the consistency of* ZFC *or even of the consistency of* ZFC + *"there is an inaccessible cardinal" (provided that the theory* ZFC *or* ZFC + *"there is an inaccessible cardinal" respectively is consistent), etc.*

3 σ-linked partial orderings

sigma-l

Proposition 3.1 *Any separative σ-linked partial ordering has cardinality $\leq 2^{\aleph_0}$.*

Proof. Suppose otherwise and assume that $P = \langle P, \leq_P \rangle$ is a separartive partial ordering of size $\geq (2^{\aleph_0})^+$ such that P is partitioned into linked sets P_n, $n \in \omega$. Without loss of generality, we may assume that P is the positive elements of a Boolean algebra B (separativity is needed for this) and $|P| = (2^{\aleph_0})^+$.

Let $\sqsubset$ be a linear ordering on P and let $f : [P]^2 \to \omega^3$; $\{p, q\} \mapsto (k_0, k_1, k_2)$ be such that $p \sqsubset q$; $k_0 = 0$ if p, q are comparable, $k_0 = 1$ otherwise; k_1 is such that $p - q \in P_{k_1}$ if $p - q \neq 0_B$, $k_1 = 0$ otherwise; k_2 is such that $q - p \in P_{k_2}$ if $q - p \neq 0_B$, $k_2 = 0$ otherwise.

By Erdős-Rado Theorem, there is an f-homogeneous $H \in [P]^{\aleph_1}$. Let $f''[H]^2 = \{\langle k_0^*, k_1^*, k_2^* \rangle\}$. $k_0^* = 1$ since P has the c.c.c. (here, it is the actual place where the separativity of P is needed). Thus elements of H are pairwise incomparable. $k_1^* \neq k_2^*$ (since, for distinct p, $q \in H$, $p - q$ and $q - p$ are disjoint). But we can see easily that homogeneous set of this property with 3 elements is inconsistent with the fact that $P_{k_1^*}$ and $P_{k_2^*}$ are linked.

❑ (Proposition 3.1)

4 Generic large cardinals

gen-large

Lemma 4.1 *Any generic measurable cardinal is regular.* L-gen-large-a

Proof. Suppose that $j : \mathsf{V} \overset{\prec}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ is such that $crit(j) = \kappa$.

If κ were singular, there would be a sequence $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \mu \rangle$ in V with $\mu < \kappa$ and $\kappa_\alpha < \kappa$ such that $\kappa = \sup_{\alpha<\mu} \kappa_\alpha$. Since $j(\vec{\kappa}) = \vec{\kappa}$ by elementarity and $\kappa = crit(j)$, it follows that $j(\kappa) = \sup_{\alpha<\mu} \kappa_\alpha$ by elementarity. This is a contradiction to the assumption that $\kappa = crit(j)$.

❑ (Lemma 4.1)

L-gen-large-0

Lemma 4.2 *Suppose that κ is generically measurable for a c.c.c. poset $\mathbb{P}$. Then κ is weakly mahlo.*

Proof. Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, M an inner model in $\mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\prec}{\to} M$ is such that $crit(j) = \kappa$.

Cl-gen-large-0

Claim 4.2.1 $M \models$ "*κ is a regular cardinal*".

⊢ κ is a regular cardinal by Lemma 4.1. By the c.c.c. of $\mathbb{P}$, κ is also a regular cardinal in $\mathsf{V}[\mathbb{G}]$. It follows that κ is a regular cardinal in $M \subseteq \mathsf{V}[\mathbb{G}]$. ⊣ (Claim 4.2.1)

Cl-gen-large-2

Claim 4.2.2 *κ is weakly mahlo.*

⊢ Suppose that $C \subseteq \kappa$ is a club. We have to show that there is an $\alpha \in C$ such that α is a regular cardinal. By elementarity $M \models$ "$j(C)$ is a club in $j(\kappa)$". $j(C) \cap \kappa = C$ and $\kappa = \sup C$. It follows that $M \models \kappa \in j(C)$. By Claim 4.2.1, it follows that

(4.1) $\quad M \models$ "there is an $\alpha \in j(C)$ such that α is a regular cardinal".

By elementarity

(4.2) $\quad \mathsf{V} \models$ "there is an $\alpha \in C$ such that α is a regular cardinal".

⊣ (Claim 4.2.2)

❑ (Lemma 4.2)

The lemma above can be still improved:

L-lt-conti-1-1

Lemma 4.3 (1) *Suppose that κ is generically measurable for a poset $\mathbb{P}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$ $j : V \overset{\prec}{\to} M$, $crit(j) = \kappa$. Then, in $V[\mathbb{G}]$,*

(4.3) $\quad F = \{a \in (\mathcal{P}(\kappa))^{\mathsf{V}} : \kappa \in j(a)\}$ lt-conti-2-2-0 *is a* V*-normal ultrafilter on (the Boolean algebra)* $(\mathcal{P}(\kappa))^V$.

(2) *If* $\mu < \kappa$ *and* κ *is generically measurable for a* μ*-cc poset* $\mathbb{P}$ *then there is a* μ*-saturated normal ideal over* κ *(in* V*). In particular,* κ *is* κ*-weakly Mahlo.*

Proof. See [5] (L-lt-conti-1-1) ❑ (Lemma 4.3)

L-gen-large-1

Lemma 4.4 *Suppose that* κ *is generically measurable by a* $<\mu$*-closed poset for a regular uncountable cardinal* $\mu < \kappa$*. Then* κ *is a regular cardinal and* $2^{<\mu} < \kappa$*.*

Proof. Suppose that κ is generically measurable by a $<\mu$-closed poset $\mathbb{P}$ and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter with transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and elementary embedding $j : \mathsf{V} \overset{\prec}{\to} M$ such that $crit(j) = \kappa$.

κ is regular by Lemma 4.1.

Suppose now, toward a contradiction, that $2^{<\mu} \geq \kappa$. Since κ is a regular cardinal it follows that there is a $\mu_0 < \mu$ such that $2^{\mu_0} \geq \kappa$. Let $\lambda = 2^{\mu_0}$ and let $f : \lambda \to \mathcal{P}(\mu_0)$ be a bijection. By elementarity,

$$M \models \text{“}j(f) : j(\lambda) \to \mathcal{P}(\mu_0) \text{ and } j \text{ is a bijection”}. \quad (4.4)$$

gen-large-0

Since $\mathbb{P}$ is $<\mu$-closed, we have $\mathcal{P}(\mu_0)^{\mathsf{V}} = \mathcal{P}(\mu_0)^{\mathsf{V}[\mathbb{G}]} \supseteq \mathcal{P}(\mu_0)^M$. By (4.4), it follows that $\mathcal{P}(\mu_0)^{\mathsf{V}} = \mathcal{P}(\mu_0)^M$. Thus

$$M \models \text{“}j(f) : j(\lambda) \to \mathcal{P}(\mu_0)^{\mathsf{V}} \text{ and } j \text{ is a bijection”}. \quad (4.5)$$

gen-large-1

We have $j(f)(j(\alpha)) = j(f(\alpha)) = f(\alpha)$ for each $\alpha \in \lambda$. Thus $j(f)''\lambda = \mathcal{P}(\mu_0)^{\mathsf{V}}$. Since $j(f)$ should be an injection, $\kappa \in j(\lambda) \setminus j''\lambda$ cannot be assigned to any element of $\mathcal{P}(\mu_0)^{\mathsf{V}}$ by $j(f)$. This is a contradiction. ❑ (Lemma 4.4)

5 Axiom of Choice in Zermelo's set theory

AC-Z

When Zermelo proved his „Wohlordnungssatz“ (Well-ordering Theorem), there was not yet the Zermelo-Fraenkel axiom system ZF but at most the axiom system Zermelo introduced in his 1908 paper [13]. The axiom system Z (the system obtained by dropping the Axiom of Replacement and Axiom of Regularity from ZF) corresponds to the axiom system in [13] but it differs from Zermelo's system in the treatment of the axiom of infinity. Since the modern treatment of ordered pairs and functions were not yet introduced in Zermelo's 1908 paper, it is not so clear whether he proved his „Wohlordnungssatz“ really in the framework of the axiomatics of his 1908 paper. Also in all modern textbooks, the equivalence of AC with the „Wohlordnungssatz“ and Zorn's Lemma is proved in the framework of ZF. Thus, it is rather difficult to find an easy to read source to check that this equivalence can be established already in Z. In the following we will check this in a most self-contained manner so that the text will be also accessible for the students who just began to learn the axiomatic set-theory.

In this section we are working in the Zermelo's set theory Z if not mentioned otherwise. For a set X, a binary relation R on X, that is, $R \subseteq X^2$ is said to be a *well-ordering on* X if R is a linear ordering, that is, if it is irreflexive, anti-symmetric and transitive relation satisfying

$$(5.1) \qquad \forall a, b \in X \ (a \mathrel{R} b \ \vee \ a = b \ \vee \ b \mathrel{R} a),$$

such that, reading "$a \mathrel{R} b$" as "a is smaller than b with respect to R", every non empty $Y \subseteq X$ has the minimal element with respect to R.

If R is a linear-ordering (a well-ordering, resp.) on X then we also say that $\langle X, R \rangle$ is a linearly-orderd set (a well-ordered set, resp.), or also simply a linear ordering (a well-ordering, resp.). For such a structure $\langle X, R \rangle$, if it is clear which R is attatched to X, we call $\langle X, R \rangle$ simply as X and, to declare that we call $\langle X, R \rangle$ simply as X, we write: $X = \langle X, R \rangle$.

For a linear ordering $\langle X, R \rangle$, $Y \subseteq X$ is said to be an *initial segment of* X if Y is downward closed in X with respect to R, that is, if for all $a \in Y$ and $b \in X$ with $b \mathrel{R} a$ we always have $b \in Y$. For $Y \subseteq X$ and $R \subseteq X^2$, $R \restriction Y$ denotes the binary relation $R \cap Y^2$ on Y. For linear orderings $X_0 = \langle X_0, R_0 \rangle$ and $X_1 = \langle X_1, R_1 \rangle$, we say that X_1 is a *end-extension* of X_0 if $X_0 \subseteq X_1$, X_0 is an initial segment of X_1 with respect to R_1 and $R_0 = R_1 \restriction X_0$.

L-AC-Z-1

Lemma 5.1 (1) *Suppose that* $\langle u, r \rangle$ *is a linear ordering (a well-orderting, resp.) and* $u_0 \subseteq u$. *Then* $\langle u_0, r \restriction u_0 \rangle$ *is also a linear ordering (a well-ordering, resp.).*

(2) *If* $\langle u, r \rangle$ *is a linear ordering (a well-ordering, resp.) and* $a \notin u$ *then* $\langle u', r' \rangle$ *defined by* $u' = u \cup \{a\}$ *and* $r' = r \cup (u \times \{a\})$ *is also a linear ordering (a well-ordering, resp.) and* $\langle u', r' \rangle$ *is an end-extension of* $\langle u, r \rangle$.

(3) *Suppose that* $\mathcal{F}$ *is a family (set) of linear orderings (of well-orderings, resp.) such that, for any* $\langle u_0, r_0 \rangle$, $\langle u_1, r_1 \rangle \in \mathcal{F}$,

$$(5.2) \qquad \textit{either } \langle u_1, r_1 \rangle \textit{ is an end-extension of } \langle u_0, r_0 \rangle \textit{ or } \langle u_0, r_0 \rangle \textit{ is an end-extension of } \langle u_1, r_1 \rangle.$$ AC-Z-0

Then $\langle U, R \rangle$ *for*

$$(5.3) \qquad U = \bigcup \{u : \langle u, r \rangle \in \mathcal{F} \textit{ for some } r\} \quad \textit{and} \quad R = \bigcup \{r : \langle u, r \rangle \in \mathcal{F} \textit{ for some } u\}$$

is also a linear ordering (a well-ordering, resp.) and, for all $\langle u, r \rangle \in \mathcal{F}$, $\langle U, R \rangle$ *is an end extension of* $\langle u, r \rangle$.

Proof. Exercise. ❑ (Lemma 5.1)

Axiom of Choice (AC) is defined as the following assertion:

AC: For any A with $\emptyset \notin A$, there is a mapping $f : A \to \bigcup A$ such that $f(a) \in a$ for all $a \in A$.

A mapping f as above is called a *choice funcion for A.*

Theorem 5.2 (Zermelo, 1904) *The following are equivalent over* Z*:* T-AC-Z-0
(a) AC. (b) (Well-ordering Theorem) *For any X there is a well-ordering R on X.*

Proof. (b) ⇒ (a): Suppose that (b) holds and let A be a set with $\emptyset \notin A$. Let R be a well-ordering on $\bigcup A$. Then

(5.4) $$f : A \to \bigcup A;\ a \mapsto \text{the minimal element of } a \subseteq \bigcup A \text{ with respect to } R$$ AC-Z-2

is a choice function for A.

(a) ⇒ (b): Suppose that AC holds. Let X be an arbitrary set. If $X = \emptyset$ then $R = \emptyset$ is a well-ordering on X. Hence we may assume that $X \neq \emptyset$.

Let $A = \mathcal{P}(X) \setminus \{\emptyset\}$ and let $f : A \to \bigcup A\ (= X)$ be a choice function on A. We show that there is a well-ordering R on X such that, for any non-empty $Y \subseteq X$, $f(Y)$ is the minimal element of Y with respect to R.

Let

(5.5) $$\mathcal{F} = \{\langle u, r\rangle : u \subseteq X,\ r \subseteq u^2,\ r \text{ is a well-ordering on } u \text{ and, for any non-empty } v \subseteq u, \text{ the minimal element of } v \text{ with respect to } r \text{ is } f(X \setminus (u \setminus v))\}.$$ AC-Z-3

Cl-AC-Z-0

Claim 5.2.1 *$\mathcal{F} \neq \emptyset$.*

⊢ $\langle \emptyset, \emptyset\rangle \in \mathcal{F}$. ⊣ (Claim 5.2.1)

Cl-AC-Z-1

Claim 5.2.2 *For any $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$,*

(5.6) *either $\langle u_1, r_1\rangle$ is an end-extension of $\langle u_0, r_0\rangle$ or $\langle u_0, r_0\rangle$ is an end-extension of $\langle u_1, r_1\rangle$.* AC-Z-4

⊢ Suppose that $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$ are a counter-example to the assertion of the Claim. Let

(5.7) $$\mathcal{F}_0 = \{v : v \subseteq u_0 \cap u_1,\ r_0 \restriction v = r_1 \restriction v \\ v \text{ is an initial segment of } u_0 \text{ with respect to } r_0 \text{ and} \\ v \text{ is an initial segment of } u_1 \text{ with respect to } r_1\}$$ AC-Z-6

Let $v^* = \bigcup \mathcal{F}_0$. Then $v^* \subseteq u_0 \cap u_1$, $r_0 \restriction v^* = r_1 \restriction v^*$ v^* is an initial segment of u_0 with respect to r_0 and v^* is an initial segment of u_1 with respect to r_1. Thus v^* is the maximal element of $\mathcal{F}_0$ with respect to $\subseteq$.

By the choice of $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$, we have $v^* \subsetneqq u_0$ and $v^* \subsetneqq u_1$. By definition of $\mathcal{F}$, we have

(5.8) $$\text{the minimal element of } u_0 \setminus v^* \text{ with respect to } r_0 \\ = f(X \setminus v^*) = \text{the minimal element of } u_1 \setminus v^* \text{ with respect to } r_1.$$ AC-Z-7

Let a be this element. Then $v^* \cup \{a\} \in \mathcal{F}_0$. This is a contradiction to the maximality of v^*. ⊣ (Claim 5.2.2)

Let

$$(5.9)\qquad X_0 = \bigcup\{u \in \mathcal{P}(X) : \langle u, r\rangle \in \mathcal{F} \text{ for some } r \subseteq u\} \text{ and}$$
$$R_0 = \bigcup\{r \in \mathcal{P}(X^2) : \langle u, r\rangle \in \mathcal{F} \text{ for some } u \in \mathcal{P}(X)\}.$$ AC-Z-8

Cl-AC-Z-2

Claim 5.2.3 *R_0 is a well-ordering on X_0.*

⊢ By Claim 5.2.2 and Lemma 5.1, (3). ⊣ (Claim 5.2.3)

Thus, $\langle X_0, R_0\rangle$ is the maximal element of $\mathcal{F}$ with respect to componentwise inclusion. The following Claim finishes the proof:

Cl-AC-Z-3

Claim 5.2.4 $X_0 = X$.

⊢ Suppose not and let $a_0 = f(X \setminus X_0)$. Let $X_1 = X_0 \cup \{a_0\}$ and $R_1 = R_0 \cup (X_0 \times \{a_0\})$. Then $\langle X_1, R_1\rangle \in \mathcal{F}$ by Lemma 5.1, (2). This is a contradiction to the maximality of $\langle X_0, R_0\rangle$. ⊣ (Claim 5.2.4)

□ (Theorem 5.2)

A pair $\langle P, <\rangle$ for a set P and a binary relation $<$ is said to be a *partial ordering* if $<$ is irreflexive, anti-symmetric and transitive relation. A subset C of P for a partial ordering $\langle P, <\rangle$ is a *chain* if $<\restriction C$ is a linear ordering on C. For $X \subseteq P$, $a \in P$ is an *upper-bound of* X (with respect to $<$) if $b = a$ or $b < a$ holds for all $b \in X$. $a \in P$ is a *maximal element* (with respect to $<$) if there is no $b \in P$ such that $a < b$.

Zorn's Lemma is the following assertion:

Zorn's Lemma: Suppose that $\langle P, <\rangle$ is a partial ordering such that

(5.10) any chain has an upper-bound in P. AC-Z-10

Then P has at least one maximal element.

T-AC-Z-1

Theorem 5.3 *The following are equivalent over* Z*:*

(a) AC.

(b) *Zorn's Lemma.*

Proof. By Theorem 5.2, it is enough to show that Zorn's Lemma is equivalent to Well-ordering Theorem over Z.

Assume first that Well-ordering Theorem holds and $\langle P, <\rangle$ is a partial ordering satisfying (5.10). Let R be a well-ordering on the set P and let

(5.11) $\mathcal{P} = \{u \in \mathcal{P}(P) : <\restriction u$ is a well-ordering on u, for any proper initial segment v of u with respect to $<\restriction u$, the minimal element of $u \setminus v$ with respect to $<\restriction u$ is the minimal element of $\{p \in P : p$ is an upper-bound of v with respect to $<\}$ with respect to $R\}$. AC-Z-11

Similarly to the proof of (a) ⇒ (b) of Theorem 5.2, we can prove that elements of $\mathcal{P}$ as linear orderings ordered by $<$ or equivalently R restricted to them are linearly ordered with respect to end-extention. Hence we have $U^* = \cup\mathcal{P} \in \mathcal{P}$ and U^* is the maximal element of $\mathcal{P}$ with respect to end-extension of elements of $\mathcal{P}$. U^* is a chain in P with respect to $<$ by Lemma 5.1, (3). Let $p^* \in P$ be an upper-bound of U^*. Then $p^* \in U^*$, that is p^* must be the maximal element of U^* with respect to $<\restriction U^*$. Also p^* must be a maximal element of P with respect to $<$.

Suppose now that Zorn's Lemma holds and let X be a set. Let

(5.12) $\mathcal{F} = \{\langle u, r\rangle \in \mathcal{P}(X) \times \mathcal{P}(X^2) : r \subseteq u^2$ and r is a well-orderting on $u\}$. AC-Z-12

For $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$, let

(5.13) $\langle u_0, r_0\rangle \sqsubset \langle u_1, r_1\rangle \Leftrightarrow \langle u_1, r_1\rangle$ is an end extension of $\langle u_0, r_0\rangle$. AC-Z-13

Then $\mathcal{F} = \langle \mathcal{F}, \sqsubset\rangle$ is a partial ordering and, by Lemma 5.1, (3), every chain C in $\mathcal{F}$ has the least upper-bound $\langle U, R\rangle$ where

(5.14) $U = \bigcup\{u \in \mathcal{P}(X) : \langle u, r\rangle \in C$ for some $r \subseteq u^2\}$ and
$R = \bigcup\{r \in \mathcal{P}(X^2) : \langle u, r\rangle \in C$ for some $u \in \mathcal{P}(X)\}$.

By Zorn's Lemma, it follows that $\mathcal{F}$ has a maximal element $\langle U_0, R_0\rangle$. The following Claim finishes the proof:

Cl-AC-Z-4

Claim 5.3.1 *$U_0 = X$.*

⊢ Suppose otherwise and let $a \in X \setminus U_0$. Letting $U_1 = U_0 \cup \{a\}$ and $R_1 = R_0 \cup (U_0 \times \{a\})$, we have $\langle U_1, R_1\rangle \in \mathcal{F}$ and $\langle U_0, R_0\rangle$ F $\langle U_1, R_1\rangle$ by Lemma 5.1, (2). This is a contradiction to the assumption that $\langle U_0, R_0\rangle$ is a maximal element. ⊣ (Claim 5.3.1) □ (Theorem 5.3)

Theorem 5.4 (Cantor-Bernstein Theorem) *In* Z*, if there are 1-1 mappings from M to N and also from N to M, then there is a bijection from M to N.*

Proof. See Chapter 19 in [1].
The intuitively elegant form of the proof of theorem 0.20 is due to G. Birkhoff and S. MacLane. JOHN L. KELLEY, GENERAL,TOPOLOGY

□ (Theorem 5.4)

6 Hilbert's Paradox and Grothendieck universe

Gu

The following Theorem 6.1 appears in Hilbert's course note of 1905 (see V. Peckhaus and R. Kahle [12], see also A. Kanamori, [9]):

Theorem 6.1 (in Z, Hilbert's Paradox) *There is no set X such that* P-Gu-a-0

(6.1) *X is closed with respect to powerset operation, that is, for any $a \in X$, $\mathcal{P}(a) \in X$; and* Gu-a-0

(6.2) *For any set $Y \subseteq X$, $\bigcup Y \in X$.* Gu-a-1

Note that there are infinite sets with the property (6.1) alone and infinite sets with the property (6.2) alone. For example: $V_\gamma \models$ (6.1) for any limit ordinal γ. $\mathcal{P}(X) \models$ (6.2) for any set X.(2)

Proof of Theorem 6.1. Suppose that $X \models$ (6.1), (6.2). Then $\bigcup X \in X$ and $\mathcal{P}(\bigcup X) \in X$. It follows that $\mathcal{P}(\bigcup X) \subseteq \bigcup X$: for any $a \in \mathcal{P}(\bigcup X)$, since a is an element of an element of X (namely $\mathcal{P}(\bigcup X)$), we have $a \in \bigcup X$. But this is a contradiction to Cantor's Theorem which states that, for any set a, there is no surjection from a to $\mathcal{P}(a)$. □ (Theorem 6.1)

The class $\mathsf{V} = \{x : x = x\}$ satisfies (6.1) and (6.2). On the other hand, a class X satisfying (6.1) and (6.2) does not need to be equal to V:

Lemma 6.2 (in Z) *There are proper subclasses X of V with $X \models$ (6.1), (6.2).* P-Gu-a-0-0

Proof. Let $X = \{a : |b| \leq |a| \text{ for any } b \in \mathit{trcl}(a)\}$. Clearly $X \neq \mathsf{V}$ but $X \models$ (6.1), (6.2).

$X = \{V_\alpha : \alpha \in \mathsf{On}\}$ is another example. □ (Lemma 6.2)

The second example in the proof of Lemma 6.2 is quite symptomatic (see Corollary 6.4).

The properties of Hilbert's (non-existent) set can be slightly extended to obtain a characterization of V:

P-Gu-a-0-1

Proposition 6.3 *If a class X is transitive and satisfies (6.1) and (6.2), then $X = \mathsf{V}$.*

Proof. We show that for all $\alpha \in X$, $V_\alpha \in X$. Since X is transitive, it follows from this that $X = V$.

Since $\emptyset = \bigcup\emptyset$ and $\emptyset \subseteq X$, we have $V_0 = \emptyset \in X$ by (6.2)

If $V_\alpha \in X$ then $V_{\alpha+1} = \mathcal{P}(V_\alpha) \in X$ by (6.1). If $V_\beta \in X$ for all $\beta < \gamma$ for a limit ordinal then $V_\gamma = \bigcup\{V_\beta : \beta < \gamma\} \in X$ by (6.2). □ (Proposition 6.3)

The proof of the Proposition 6.3 actually shows the following:

P-Gu-a-0-2

Corollary 6.4 *If a class X satisfies (6.1) and (6.2), then $V_\alpha \in X$ for all $\alpha \in \mathsf{On}$.* □

(2) If $a \in V_\gamma$ then $a \in V_\alpha$ for some $\alpha < \gamma$. Since $a \subseteq V_\alpha$ by transitivity of V_α, we have $b \subseteq V_\alpha$ for all $b \subseteq a$. Thus $\mathcal{P}(a) \subseteq V_{\alpha+1}$ and $\mathcal{P}(a) \in V_{\alpha+2} \subseteq V_\gamma$.

If $Y \subseteq \mathcal{P}(X)$ then $\bigcup Y \subseteq X$ and hence $\bigcup Y \in \mathcal{P}(X)$.

By modifying further the properties of the non-existent set in Hilbert's paradox, we obtain the definition of Grothendieck universes (which are sets!). An uncountable Grothendieck universe U exists if and only if an inaccessible cardinal exists. In terms of set theory, an uncountable Grothendieck universe U is simply $\mathcal{H}(\kappa)$ for an inaccessible κ. A weakening of the notion of Grothendieck universe (called here weakly Grothendieck universe) characterizes the families $\mathcal{H}(\kappa)$ of hereditarily of cardinality $<\kappa$ sets for regular κ (see Lemma 6.7, (8) and (9)). Note that such $\mathcal{H}(\kappa)$'s are still "universes" in that they are models of ZFC − Powerset Axiom.

A set U is said to be a *Grothendieck universe* if

(6.3) U is transitive; Gu-0

(6.4) $\emptyset \in U$; Gu-0-0

(6.5) U is closed with respect to pairing operation, i.e., for any a, $b \in U$, we have $\{a, b\} \in U$; Gu-1

(6.6) U is closed with respect to power set operation, i.e., for any $a \in U$, we have $\mathcal{P}(a) \in U$; and, Gu-2

(6.7) for any $I \in U$ and $f \in {}^{I}U$, we have $\bigcup_{i\in I} f(i) \in U$.[(3)] Gu-3

The definition of Grothendieck universes makes sense under ZF without Axiom of Foundation or as a matter of fact even without Separation and Replacement although the most natural setting for this notion seems to be the full ZFC with Axiom of Foundation (see e.g. Lemma 6.6, (8) and Lemma 6.7, (10) below). The following properties of Grothendieck universes can be proved in such a weak fragrment of ZF:

Let U be a Grothendieck universe. By (6.7), it follows that, $\bigcup a = \bigcup_{b\in a} id_a(b) \in U$ for $a \in U$ and also, for any $a \in [U]^{<\kappa}$, we have $\bigcup a \in U$ where $\kappa = \mathsf{On} \cap U$. Note that $\kappa \in \mathsf{On}$ since U is transitive. We can also show further that κ is a regular cardinal (see Lemma 6.6, (7))

If $a \in [U]^{<\kappa}$, then $\langle \{b\} : b \in a\rangle \in {}^{a}U$ by (6.5). Thus $a = \bigcup\{\{b\} : b \in a\} \in U$ by (6.7).

Under the full axiom system of ZFC with Axiom of Foundation and Axiom of Choice, we can say much more than this (see Lemma 6.6,(7)).

Grothendieck universes obtain certain regularity even under the set theory without the Axiom of Foundation:

Lemma 6.5 (in ZF - Axiom of Foundation) *If U is a Grothendieck universe then $U \notin U$. There is no $\in$-descending sequence starting and ending with U.* P-Gu-a-1

Proof. Let U be a Grothendieck universe. Toward a contradiction, suppose that $U \in U$. Let $\kappa = \mathsf{On} \cap U$.

(3) Actually (6.5) is redundant: If $x \in U$ then $\{x\} \in \mathcal{P}(\mathcal{P}(x)) \in U$. Thus by (6.3), we have $\{x\} \in U$. By (6.4), it follows from this that $2 = \{\emptyset, \{\emptyset\}\} = \mathcal{P}(\{\emptyset\}) \in U$. Thus, for any a, $b \in U$, we have $f = \{\langle 0, \{a\}\rangle, \langle 1, \{b\}\rangle\} \in {}^{2}U$ and $\{a, b\} = \bigcup f \in U$.

Claim 6.5.1 *κ is a limit ordinal.*

$\vdash$ Suppose that $\alpha \in \kappa$. Then $\alpha + 1 = \alpha \cup \{\alpha\} = \bigcup\{\{\alpha\}, \{\{\alpha\}\}\} \in U$ by (6.5) and (6.7). Hence $\alpha + 1 \in \kappa$. $\dashv$ (Claim 6.5.1)

Let $f : U \to \kappa \subseteq U$ be defined by

$$(6.8) \qquad f(x) = \begin{cases} x, & \text{if } x \in \mathsf{On}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\kappa = \bigcup \kappa = \bigcup_{x \in U} f(i) \in U$ by (6.7). (Note that the first equality holds because κ is a limit ordinal.) Thus $\kappa \in \kappa$. This is a contradiction.

Since U is transitive, if there were a $\in$-descending sequence starting and ending with U, then this would imply $U \in U$. ❑ (Lemma 6.5)

P-Gu-0

Lemma 6.6 *Work in* ZF*. Suppose that U is a Grothendieck universe with $\kappa = \mathsf{On} \cap U$.*

(1) *If $a \in U$ and $b \subseteq a$ then $b \in U$.*

(2) *$\omega \subseteq U$ (hence $\kappa \geq \omega$) and $\mathcal{H}(\kappa) \subseteq U$.*

(3) *If $a, b \in U$ then $a \cup b \in U$.*

(4) *If $a, b \in U$ then $a \times b \in U$.*

(5) *If $f \in {}^I U$ for some $I \in U$ then $f''I \in U$.*

(6) *For any $\langle a, r \rangle \in U$, if r is a well ordering on a, then we have $otp(\langle a, r \rangle) \in U$.*

(7) *If $\kappa > \omega$ then κ is a regular cardinal. If we assume* AC*, then κ is an inaccessible cardinal.*

(8) *Under* AC*, we have $U = \mathcal{H}(\kappa)$. If $\kappa > \omega$, $U \models$* ZF *and further, if* AC *holds, then we have $U \models$* ZFC*.*

(9) *Conversely to (7), $\mathcal{H}(\kappa)$ is a Grothendieck universe if either $\kappa = \omega$ or κ is inaccessible.*

Proof. (1): Suppose that $a \in U$ and $b \subseteq a$. By (6.6), we have $b \in \mathcal{P}(a) \in U$. Since U is transitive, it follows that $b \in U$.

(2): We first show that $\omega \subseteq U$.

$\emptyset \in U$ by (6.4). By (6.5), it follows that $\{\emptyset\} \in U$ and $\{\emptyset, \{\emptyset\}\} \in U$. Thus $\kappa \geq 3$. By (6.7), it follows that,

$$(6.9) \qquad \text{for any } a \in U,\ a \cup \{a\} = \bigcup \underbrace{\{\{a\}, \{\{a\}\}\}}_{\in [U]^{<\kappa}} \in U \text{ by (6.7).}$$ Gu-5

Since ω is characterized as the minimal set W (with respect to $\subseteq$) satisfying $\emptyset \in W$ and $a \cup \{a\} \in W$ for all $a \in W$, it follows that $\omega \subseteq U$. In particular, we also have $\kappa \geq \omega$.

To prove $\mathcal{H}(\kappa) \subseteq U$ by induction on the rank of elements of $\mathcal{H}(\kappa)$, it is enough to show: If $a \in \mathcal{H}(\kappa)$ and $a \subseteq U$ then $a \in U$.

Suppose that $a \in \mathcal{H}(\kappa)$ and $a \subseteq U$. then for all $b \in a$, $\{b\} \in U$ by (6.5). Thus

$$a = \bigcup \underbrace{\{\{b\} : b \in a\}}_{\in [U]^{<\kappa}} \in U \tag{6.10}$$ Gu-6

by (6.7).

(3): Suppose that $a,\ b \in U$.

$$a \cup b = \bigcup \underbrace{\{\{a\}, \{b\}\}}_{\in [U]^{<\kappa} \text{ by (2) and (6.5)}} \in U \tag{6.11}$$ Gu-7

by (6.7).

(4): $a \times b \subseteq \underbrace{\mathcal{P}(\mathcal{P}(a \cup b))}_{\in U \text{ by (3) and (6.6)}}$. Thus $a \times b \in U$ by (1).

(5): Suppose $f \in {}^I U$ for some $I \in U$. Let $f^* : I \to U;\ i \mapsto \{f(i)\}$. f^* is well-defined by (6.5). Then $f''I = \bigcup_{i \in I} f^*(i) \in U$ by (6.7).

(6): Suppose, toward a contradiction, that $\langle a, r\rangle \in U$ be such that r is a well-ordering on a but $otp(\langle a, r\rangle) \notin U$. Suppose furhter that $\langle a, r\rangle$ is chosen such that $otp(\langle a, r\rangle)$ is minimal among the order-type of such pairs.

$otp(\langle a, r\rangle)$ is not a successor ordinal: if $otp(\langle a, r\rangle) = \alpha + 1$, then let I be the initial segment of a below the maximal element of a with respect to r. By (6.3), (1) and (6.5), we have $\langle I, r \restriction I\rangle \in U$ and $otp(\langle I, r \restriction I\rangle) = \alpha$. By the minimality of $otp(\langle a, r\rangle)$, it follows that $\alpha \in U$. Hence $otp(\langle a, r\rangle) = \alpha + 1 = \alpha \cup \{\alpha\} \in U$. This is a contradiction to the choice of $\langle a, r\rangle$.

Hence we may assume that $otp(\langle a, r\rangle)$ is a limit ordinal. For each $b \in a$, let I_b be the initial segment of a consisting of all elements of a smaller than b with respect to r and let $\alpha_b = otp(\langle I_b, r \restriction I_b\rangle)$. Since $\langle I_b, r \restriction I_b\rangle \in U$, $\alpha_b \in U$ by the minimality of $otp(\langle a, r\rangle)$. Thus $otp(\langle a, r\rangle) = \bigcup\{\alpha_b : b \in a\} \in U$ by (6.7). This is a contradiction to the choice of $\langle a, r\rangle$.

(7): We first show that κ is a regular cardinal. Suppose otherwise and let $\delta = cf(\kappa) < \kappa$ and $\langle \alpha_\xi : \xi\rangle$ be a sequence of ordinals $< \kappa$ such that $\sup_{\xi<\delta} \alpha_\xi = \kappa$. Then $\kappa = \bigcup\{\{\alpha_\xi\} : \xi \in \delta\} \in U$ by (6.7) and thus $\kappa \in \kappa$. This is a contradiction.

For $\delta < \kappa$, we have $2^\delta = |\mathcal{P}(\delta)| = otp(\langle \mathcal{P}(\delta), r\rangle) \in U$ by (6) where r is the well-ordering on $\mathcal{P}(\delta)$ of order-type $|\mathcal{P}(\delta)|$.

(8): We have shown in (2) that $\mathcal{H}(\kappa) \subseteq U$ holds. To show that $U \subseteq \mathcal{H}(\kappa)$, suppose otherwise and let $a \in U \setminus \mathcal{H}(\kappa)$ be with the minimal possible rank. Thus $b \in \mathcal{H}(\kappa)$ holds for all $b \in a$. By AC, and (6), $\lambda = |a| \in U$. It follows that $\lambda < \kappa$ and hence $a \in \mathcal{H}(\kappa)$. This is a contradiction to the choice of a.

If $\kappa > \omega$ then $\omega \in \kappa \subseteq U$. It follows that U satisfies the Axiom of Infinity. It is also easy to prove that all other Axioms of ZFC hold in U: The Axiom of Extensionality holds in U since U is transitive. The Axiom of Emptyset holds in U by (6.4). $U \models$"The Pairing Axiom" by (6.5). $U \models$"The Axiom of Union" by the remark before Lemma 6.6. The Axiom of Powerset by (6.6). $U \models$"The Axiom of Separation" by (1). $U \models$"The Axiom of Replacement" follows from (5). The Axiom of Foundation holds in any $\in$-model (if we are working in the set theory with the Axiom). If AC holds then for all $a \in U$ with $\emptyset \notin a$ and a choice function $f : a \to \bigcup a$, we have $f \in U$: $a \times (\bigcup a) \in U$ by the remark before Lemma 6.5 and (4). Hence $f \in U$ by $f \subseteq a \times (\bigcup a)$ and (1).

(9): If $\kappa = \omega$ or κ is inaccessible, it is easy to see that $\mathcal{H}(\kappa)$ satisfies the conditions (6.3) $\sim$ (6.7). ❑ (Lemma 6.6)

A set U is said to be a *weakly Grothendieck universe* if

(6.3) U is transitive;

(6.4) $\emptyset \in U$;

(6.5) U is closed with respect to pairing operation, i.e., for any a, $b \in U$, we have $\{a, b\} \in U$;

(6.6)' for all $a \in U$ we have $\mathcal{P}(a) \subseteq U$;[4] and

(6.7) for any $I \in U$ and $f \in {}^{I}U$, $\bigcup_{i \in I} f(i)$. In particular, for any $a \in [U]^{<\kappa}$, we have $\bigcup a \in U$ where $\kappa = \mathsf{On} \cap U$.

With almost the same proof as that of Lemma 6.6, we obtain the following Lemma.

P-Gu-1

Lemma 6.7 *Work in* ZF*. Suppose that* U *is a weakly Grothendieck universe with* $\kappa = \mathsf{On} \cap U$*.*

(1) *If* $a \in U$ *and* $b \subseteq a$ *then* $b \in U$*.*

(2) $\omega \subseteq U$ *(hence* $\kappa \geq \omega$*) and* $\mathcal{H}(\kappa) \subseteq U$*.*

(3) *If* a, $b \in U$ *then* $a \cup b \in U$*.*

(4) *If* a, $b \in U$ *then* $a \times b \in U$*.*

(5) *If* $f \in {}^{I}U$ *for some* $I \in U$ *then* $f''I \in U$*.*

(6) *For any* $\langle a, r \rangle \in U$ *where* r *is a well ordering on* a*,* $otp(\langle a, r \rangle) \in U$*.*

(7) *If* $\kappa > \omega$ *then* κ *is a regular cardinal.*

(8) *Under* AC*, we have* $U = \mathcal{H}(\kappa)$*. If* $\kappa > \omega$*,* $U \models \mathsf{ZF} -$ Axiom of Powerset *and further if* AC *holds we have* $U \models \mathsf{ZFC} -$ Axiom of Powerset*.*

(9) *Conversely to* (8)*,* $\mathcal{H}(\kappa)$ *is a weakly Grothendieck universe if either* $\kappa = \omega$ *or* κ *is a regular uncountable cardinal.*

(10) *Under* AC*, a weakly Grothendieck universe* U *with* $\kappa = \mathsf{On} \cap U$ *is a Grothendieck universe if and only if either* $\kappa = \omega$ *or* κ *is an inaccessible cardinal.*

Proof. We only show the assertions for which we need alternative arguments not in the corresponding proof in Lemma 6.6.

(1): The assertion is simply equivalent to (6.6)'.

---

(4) In [11], a set U with this property is said to be *supertransitive*.

(4): Suppose $a, b \in U$. For each $c \in a$, $\{c\} \times b = \bigcup\{\{\{\{c\},\{c,d\}\}\} : d \in b\} \in U$ by (6.7). Thus $a \times b = \bigcup\{\{\{c\} \times b\} : c \in a\} \in U$.

(10): If $\kappa = \omega$ then $U = \mathcal{H}(\omega)$ by (8) and, by Lemma 6.6, (9), $\mathcal{H}(\omega)$ is a Grothendieck universe. For an uncountable regular cardinal κ, $\mathcal{H}(\kappa) \models$ (6.6) if and only if κ is an inaccessible cardinal. ❑ (Lemma 6.7)

From here on we work in ZFC. Let us call a transitive $\in$-model M of ZFC *full* if M is closed with respect to the powerset operation. By (6.6) and Lemma 6.6, (6), any uncountable Grothendieck universe is a full transitive $\in$-model of ZFC. However the existence of a full transitive $\in$-model of ZFC is consistency-wise strictly weaker than the existence of a Grothendieck universe:

P-Gu-2

Lemma 6.8 *Let us call the assertions "There is a full transitive $\in$-model of* ZFC*" and "There is an uncountable Grothendieck universe" φ_0 and φ_1 respectively. Then we have*

(6.12) $\mathsf{ZFC} + \varphi_1 \vdash \varphi_0$ *and* Gu-8

(6.13) $\mathsf{ZFC} + \varphi_1 \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0)$. Gu-9

Actually we even have

(6.14) $\mathsf{ZFC} + \varphi_1 \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*)$. Gu-9-0

where φ_0^* is the assertion "For any set a there is a full transitive $\in$-model M of ZFC *with $a \in M$".*

Proof. (6.12) is trivial. To show (6.13) we work in $\mathsf{ZFC} + \varphi_1$ and let $U = \mathcal{H}(\kappa)$ be an uncountable Grothendieck universe. By Lemma 6.6, (7), κ is an inaccessible cardinal. Note that we also have $U = V_\kappa$.(5)

For an arbitrary $a \in U$, let $\langle \xi_n : n \in \omega \rangle$ and $\langle M_n : n \in \omega \rangle$ be such that

(6.15) $\langle \xi_n : n \in \omega \rangle$ is a strictly increasing sequence of ordinals below κ; Gu-10

(6.16) $a \in V_{\xi_0}$; Gu-10-0

(6.17) $\langle M_n : n \in \omega \rangle$ is a $\subseteq$-increasing sequence of elements of U; Gu-11

(6.18) $V_{\xi_n} \subseteq M_n \prec U$; and Gu-12

(6.19) $M_n \subseteq V_{\xi_{n+1}}$. Gu-13

Let $M_a = \bigcup_{n\in\omega} M_n = \bigcup_{n\in\omega} V_{\xi_n}$. Then $M_a = V_\xi \in U$ where $\xi = \sup_{n\in\omega} \xi_n < \kappa$. $a \in M_a$ by (6.16) and $M_a \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ since $M_a \prec U$ by (6.17) and (6.18). M_a is full by (6.15). Thus U is a model of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*$. It follows that $consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*)$ holds. ❑ (Lemma 6.8)

For a transitive $\in$-model M, we shall call the ordinal $\mathsf{On} \cap M$ the *height* of M.

P-Gu-3

Theorem 6.9 *If M is a full transitive $\in$-model of* ZFC *with regular height κ then κ is an incaccessible cardinal and $M = \mathcal{H}(\kappa)$. Thus such M is a Grothendieck universe.*

Proof. We first prove that κ is inaccessible. Since the regularity of κ is already assumed, it is enough to show that κ is closed under cardinal exponential. Suppose that $\lambda \in \kappa$. Then $\mathcal{P}(\lambda) \in M$ since M is full. Let $\mu \in \kappa$ be such that $M \models$ "$|\mathcal{P}(\lambda)| \equiv \mu$" Then $2^\lambda = |\mu| \leq \mu < \kappa$.

$M \supseteq \mathcal{H}(\kappa)$: Suppose otherwise and let $a \in \mathcal{H}(\kappa) \setminus M$ be of the minimal rank among elements of $\mathcal{H}(\kappa) \setminus M$. Then $a \subseteq M$. Since κ is regular, there is $\alpha < \kappa$ such that $a \subseteq (V_\alpha)^M$. Since M is full it follows that $a \in \mathcal{P}((V_\alpha)^M) \in M$. Since M is transitive it follows that $a \in M$. This is a contradiction to the choice of a.

$M \subseteq \mathcal{H}(\kappa)$: Suppose otherwise and let $a \in M \setminus \mathcal{H}(\kappa)$ be of the minimal rank among elements of $M \setminus \mathcal{H}(\kappa)$. Then $a \subseteq \mathcal{H}(\kappa)$. Since $a \notin \mathcal{H}(\kappa)$ we should have $|a| \geq \kappa$. But then $\kappa \leq |a| \leq |a|^M < \kappa$. A contradiction. ❑ (Theorem 6.9)

7 Is ω_1 an object in the conventional mathematics?

omega1

Most part of conventional mathematics can be developed in Zermelo's set thoery, or Zermelo's set theory plus Axiom of Choice. The existence of the ordinal number ω_1 as it is defined in the modern set theory cannot be proved in these systems: V_{ω_1} is a model of Z (or ZC if we work in ZFC) and the statement of the non existence of ω_1.

Thus is seems to be a intriguing question whether ω_1 can be considered to be a mathematical object in the scope of the conventional mathematics. The following rather trivial theorem can be seen in connection with this question.

T-omega1-0

Theorem 7.1 (1) (in Z + Countable Choice) *Every countable well-ordered set is order-preservingly and continuously embeddable in $\mathbb{R}$.*

(2) (in Z without Choice) *No uncountable well-ordered set is order-preservingly embeddable into $\mathbb{R}$.*

C-omega1-0

Corollary 7.2 *ω_1 is the least ordinal which is not order-preservingly embeddable into $\mathbb{R}$.*

❑

Of course Corollary 7.2 only makes sense in an axiom system which proves the existence of ω_1.

Proof of Theorem 7.1. (1): By induction on the order-type of countable well-ordered set $w = \langle w, \leq_w \rangle$.

[5] For a regular cardinal κ, $\mathcal{H}(\kappa) = V_\kappa$ if and only if either $\kappa = \omega$ or κ is strongly inaccessible (see e.g. Kunen [10] Ch.IV, Lemma 6.3). The "if" direction of this lemma can be also shown by proving $(V_\alpha)^{\mathcal{H}(\kappa)} = V_\alpha$ for all $\alpha < \kappa$ by induction on $\alpha < \kappa$.

For $w = \emptyset$, the assertion is trivial.

Suppose that w has the maximal element m and $w \setminus \{m\}$ is order-preservingly and continuously embeddale in $\mathbb{R}$. Then, without loss of generality, we may assume that $w \setminus \{m\}$ is embeddable in the open interval $(-\infty, 0)$ with the embedding f. We may also assume that, if $w \setminus \{m\}$ does not have a maximal element, then $f''w \setminus \{m\}$ is cofinal in $(-\infty, 0)$. The mapping $f \cup \{\langle m, 0\rangle\}$ is then an order-preserving continuous embedding of w into $\mathbb{R}$.

Suppose finally that w does not have the maximal element and each proper initial segment of w can be embedded order-preservingly and continuously into $\mathbb{R}$. By countabiliy of w we can find a sequence m_i, $i \in \omega$ of elements in w such that $\langle m_i : i \in \omega\rangle$ is increasing and cofinal in w.

Let $w_0 = \{n \in w : n <_w m_0\}$ and $w_{i+1} = \{n \in w : m_i \leq_w n < m_{i+1}\}$ for $i \in \omega$. By the assumption there is a sequence f_i, $i \in \omega$ of order-preserving and continuous embeddings of w_0 to $(-\infty, 0)$ and w_{i+1} to $[i, i+1)$ such that

(7.1) f_{i+1} sends the minimal element of w_{i+1} to i; omega1-0

(7.2) if w_i is cofinal in $\{n \in w : n <_w m_i\}$ (that is, if m_i is a limit in w) then $f''w_i$ is cofinal in $(-\infty, i)$. omega1-1

Note that we need the Countable Choice to find such sequence $\langle f_i : i \in \omega\rangle$.

Let $f = \bigcup_{i\in\omega} f_i$. Then f is a order-preserving continuous embedding of w into $\mathbb{R}$.

(2): Toward a conrtadiction, suppose that $w = \langle w, <_w\rangle$ is an uncountable well-ordered set and $f : w \to \mathbb{R}$ is an order-preserving embedding. For each $m \in w$ and the successor m' of m, let $q_m \in \mathbb{Q}$ be such that

(7.3) $f(m) < q_m < f(m')$ omega1-2

(we do not need the Axiom of Choice here to choose q_m's since $\mathbb{Q}$ is well-orderable). Since $\mathbb{Q}$ is countable, there are m, $n \in w$ such that $m \neq n$ but $q_m = q_n$. By the choice (7.3) of q_m and q_n this is a contradiction.(6) ❑ (**Theorem 7.1**)

8 Absolutensess over $\mathcal{H}(\kappa)$

abs

Theorem 8.1 (Lévy, see p.299 in Kanamori [8]) *For any regular $\kappa > \omega$, $\mathcal{H}(\kappa) \prec_{\Sigma_1} \mathsf{V}$.*

Proof. Suppose that $a \in \mathcal{H}(\kappa)$ and $\varphi = \varphi(x)$ is is a formula of the form $\exists y \psi(x, y)$ where ψ is Δ_0.(7)

(6) Here, again, we do not need Axiom of Choice since, if there were no m, $n \in w$ with $m \neq n$ and $q_m = q_n$, $w \ni m \mapsto q_m \in \mathbb{Q}$ would be a one to one mapping which would be a contradiction to the assumption of non countability of w.

(7) Note that, since $\mathcal{H}(\kappa)$ satisfies the Pairing Axiom, and "$x \equiv \langle y, z\rangle$ is Δ_0, this case imples the general case with a block of existential quantification.

If $\mathcal{H}(\kappa) \models \varphi(a)$, that is, if $\mathcal{H}(\kappa) \models \exists y \psi(a, y)$, Let $b \in \mathcal{H}(\kappa)$ be such that $\mathcal{H}(\kappa) \models \psi(a, b)$. It follows $\mathsf{V} \models \psi(a, b)$. Thus $\mathsf{V} \models \exists y \psi(a, y)$.

Suppose now $\mathsf{V} \models \varphi(a)$ and let $b \in \mathsf{V}$ be such that $\mathsf{V} \models \psi(a, b)$. By Lévy-Montague Reflection Theorem, there is $\alpha \in \mathsf{On}$ such that $a, b \in \mathsf{V}$ and $V_\alpha \models \psi(a, b)$. Let $M \prec V_\alpha$ be such that $trcl(a) \subseteq M$, $b \in M$ and $|M| < \kappa$. Let $f : M \xrightarrow{\cong} trcol(M)$ be the transitive collapse. Then $f(a) = a$ and hence we have $trcol(M) \models \psi(a, f(b))$. Since $trcol(M) \subseteq \mathcal{H}(\kappa)$ it follows $\mathcal{H}(\kappa) \models \psi(a, f(b))$. Thus $\mathcal{H}(\kappa) \models \varphi(a)$. □ (Theorem 8.1)

[8] 22.3 Proposition. If κ is supercompact, then $V_\kappa \prec_2 \mathsf{V}$.
Cantor's Atic: Reflecting cardinals

For a regular cardinal κ, the following are equivalent:

(1) $V_\kappa \models \mathsf{ZFC}$ (2) $V_\kappa \prec_{\Sigma_1} V$ i.e., V_κ reflects all Σ_1 sentences with parameters.

J. Bagaria, Natural axioms of set theory and the continuum problem, In: Proceedings of the 12-th International Congress of Logic, Methodology, and Philosophy of Science, King' s College London (2005), 43-64.

9 Generic elementary embedding

gen

Lemma 9.1 *Suppse that M and N are innermodels in V and $j \subseteq \mathsf{V}$ be such that $j : M \xrightarrow{\preceq} N$. Suppose further that $\mathbb{P} \in M$ is a poset in M, $\mathbb{Q} = j(\mathbb{P})$, $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter and $\mathbb{H}$ is an $(N, \mathbb{Q})$-generic filter such that $j''\mathbb{G} \subseteq \mathbb{H}$.* L-gen-0

Then the class mapping introduced by

$$\text{“} \tilde{j} : M[\mathbb{G}] \to N[\mathbb{H}];\ \underset{\sim}{a}[\mathbb{G}] \mapsto j(\underset{\sim}{a})[\mathbb{H}] \quad \text{for } \mathbb{P}\text{-name } \underset{\sim}{a} \text{ in } M\text{”} \tag{9.1}$$ gen-a

is well defined, $j \subseteq \tilde{j}$ and $\tilde{j} : M[\mathbb{G}] \xrightarrow{\preceq} N[\mathbb{H}]$.

Proof. (ア) Well-definedness: Suppose that $\underset{\sim}{a}$, $\underset{\sim}{a}' \in M$ are $\mathbb{P}$-names such that $M[\mathbb{G}] \models$ “$\underset{\sim}{a}[\mathbb{G}] \equiv \underset{\sim}{a}'[\mathbb{G}]$”. Then there is a $\mathbb{p} \in \mathbb{G}$ such that $M \models$ “$\mathbb{p} \Vdash_{\mathbb{P}}$ “$\underset{\sim}{a} \equiv \underset{\sim}{a}'$””. By elementarity, it follows that $N \models$ “$j(\mathbb{p}) \Vdash_{\mathbb{Q}}$ “$j(\underset{\sim}{a}) \equiv j(\underset{\sim}{a}')$””. Since $j(\mathbb{p}) \in \mathbb{H}$. It follows that $N[\mathbb{H}] \models$ “$j(\underset{\sim}{a})[\mathbb{H}] \equiv j(\underset{\sim}{a}')[\mathbb{H}]$”.

(イ) $j \subseteq \tilde{j}$: Suppose that $j(a) = b$ for $a \in M$ and $b \in N$. Let $\check{(a)} \in M$ be such that $M \models$ “$\check{a}$ is a standard $\mathbb{P}$-name for a”. By elementarity, $N \models$ “$j(\check{a})$ is a standard $\mathbb{Q}$-name for $j(a)$”. Thus

$$\tilde{j}(a) = j(\check{a})[\mathbb{H}] = j(a) = b. \tag{9.2}$$ gen-0

(ウ) $\tilde{j} : M[\mathbb{G}] \xrightarrow{\preceq} N[\mathbb{H}]$: Suppose that $M[\mathbb{G}] \models \varphi(a_0, ..., a_{n-1})$ for an $\mathcal{L}_\in$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $a_0, ..., a_{n-1} \in M[\mathbb{G}]$. Then there are $\mathbb{p} \in \mathbb{G}$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ of $a_0, ..., a_{n-1}$ such that

(9.3) $M \models$ " $\underset{\sim}{p} \Vdash_{\mathbb{P}}$ " $\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$ "". gen-1

By elementarity,

(9.4) $N \models$ " $j(\underset{\sim}{p}) \Vdash_{\mathbb{Q}}$ " $\varphi(j(\underset{\sim}{a}_0), ..., j(\underset{\sim}{a}_{n-1}))$ "". gen-2

Since $j(\underset{\sim}{p}) \in \mathbb{H}$, it follows form the definition (9.1) of $\tilde{j}$ that

(9.5) $N[\mathbb{H}] \models$ " $\varphi(\tilde{j}(a_0), ..., \tilde{j}(a_{n-1}))$". ❑ (Lemma 9.1) gen-3

L-gen-1

Lemma 9.2 *Suppose that $N \subseteq \mathsf{V}$ is an inner model and $\mathbb{Q} \in N$ a poset. If*

(9.6) *${}^{\lambda}N \subseteq N$ for some cardinal λ* gen-a-0

and $\mathbb{H}$ is $(\mathsf{V}, \mathbb{Q})$-generic, then $({}^{\lambda}N[\mathbb{H}])^{\mathsf{V}[\mathbb{H}]} \subseteq N[\mathbb{H}]$.

Proof. Suppose that $\langle f_\alpha : \alpha < \lambda \rangle \in ({}^{\lambda}N[\mathbb{H}])^{\mathsf{V}[\mathbb{H}]}$. Then there is $\langle \underset{\sim}{b}_\alpha : \alpha < \lambda \rangle \in ({}^{\lambda}N)^{\mathsf{V}}$ such that, for each $\alpha < \lambda$, $\underset{\sim}{b}_\alpha$ is a $\mathbb{Q}$-name in N such that $\underset{\sim}{b}_\alpha[\mathbb{H}] = b_\alpha$. By (9.6), $\langle \underset{\sim}{b}_\alpha : \alpha < \lambda \rangle \in N$. Let

(9.7) $\underset{\sim}{f} = \{\langle op_{\mathbb{Q}}(\check{\alpha}_{\mathbb{Q}}, \underset{\sim}{b}_\alpha), \mathbb{1}_{\mathbb{P}} \rangle : \alpha \in \lambda\}$. gen-4

Then $\underset{\sim}{f}$ is a $\mathbb{Q}$-name. $\underset{\sim}{f} \in N$ by (9.6). Thus

(9.8) $N[\mathbb{H}] \ni \underset{\sim}{f}[\mathbb{H}] = \langle \underset{\sim}{b}_\alpha[\mathbb{H}] : \alpha < \lambda \rangle = \langle b_\alpha : \alpha < \lambda \rangle$. ❑ (Lemma 9.2) gen-5

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