On metrization theorems equivalent to Fodor-type Reflection Principle

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The following theorem (actually, this is merely a direct consequence of some results in [2], [1] and [3]) answers the question posed by Todor Tsankov during my talk given at the logic seminar of Université Paris 7 on 2. December 2013.

Theorem 0.1. *The following assertions are equivalent over* ZFC*:*

(A) *Fodor-type Reflection Principle* (FRP) *holds (for the definition of the principle* FRP *and basic facts about it see [2] and [3]).*

(B) *For every locally compact space X, if all subspaces Y of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable.*

(B') *For every locally compact space X, if all subspaces Y of cardinality $< \max\{|X|, \aleph_2\}$ are metrizable, then X itself is also metrizable.*

(C) *For every locally compact space X, if all closed subspaces Y of density $\leq \aleph_1$ are metrizable, then X itself is also metrizable.*

(C') *For every locally compact space X, if all closed subspaces Y of density $< \max\{|X|, \aleph_2\}$ are metrizable, then X itself is also metrizable.*

(D) *For every locally Lindelöf countably tight space X, if all open supspaces Y of X with Lindelöf number $\leq \aleph_1$ is paracompact then X itself is also paracompact.*

(D') *For every locally Lindelöf countably tight space X, if all open supspaces Y of X with Lindelöf number $< \max\{L(X), \aleph_2\}$ is paracompact then X itself is also paracompact.*

Proof. (A) $\Leftrightarrow$ (B) has been proved in [2] and [3].

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(A) $\Rightarrow$ (D) is Theorem 4.5 in [1].

(B) $\Rightarrow$ (B'), (B) $\Rightarrow$ (C) $\Rightarrow$ (C') and (D) $\Rightarrow$ (D') are trivial.

We show that the proof of (B) $\Rightarrow$ (A) in [3] actually proves (B') $\Rightarrow$ (A), (C') $\Rightarrow$ (A) and (D') $\Rightarrow$ (A).

We prove the implications indirectly: Suppose that FRP does not hold. Then, by Proposition 2.6 in [3],

(0.1) there is a regular cardinal λ^* with an almost essentially disjoint ladder system $g : S \to [\lambda^*]^{\aleph_0}$ for some stationary $S \subseteq E^{\lambda^*}_\omega$.

Here, we call a mapping $g : S \to [\lambda^*]^{\aleph_0}$ a ladder system if $g(\alpha)$ is a countable cofinal subset of α for all $\alpha \in S$. g is almost essentially disjoint if, for any $\beta < \lambda^*$, there is a regressive function $f : S \cap \beta \to \beta$ such that $\{g(\alpha) \setminus f(\alpha) \,:\, \alpha \in S \cap \beta\}$ is pairwise disjoint.

Clearly, for such S and g, we may assume without loss of generality that

(0.2) $\mathrm{otp}(g(\alpha)) = \omega$ and $g(\alpha) \subseteq Succ(\lambda^*)$ for all $\alpha \in S$.

In [2] and [3], the property (0.1) is denoted by $\mathrm{ADS}^-(\lambda^*)$.

Now, let λ^*, S, g be as in (0.1) with (0.2). Let

(0.3) $X = S \,\dot\cup\, \bigcup\{g(\alpha) \,:\, \alpha \in S\}$

be the space with the topology such that all elements of $\bigcup\{g(\alpha) \,:\, \alpha \in S\}$ are isolated and each element of S has the open neighborhood basis consisting of the sets of the form $\{\alpha\} \cup (g(\alpha) \setminus \beta)$ for some $\beta < \alpha$.

We show that X is a counter-example to all of (B'), (C') and (D').

X is locally compact: Elements of S have an open neighborhood homeomorphic to $\omega + 1$ which is compact.

Note that all closed supbsets of X of density $< |\,X\,|$ are bounded in λ^*.

Claim 0.1.1. *$X \cap \beta$ is metrizable for all $\beta < \lambda^*$.*

$\vdash$ By the assumption on the ladder system g, there is a regressive function $f : S \cap \beta \to \beta$ such that $\{g(\alpha) \setminus f(\alpha) \,:\, \alpha \in S \cap \beta\}$ is pairwise disjoint. Thus

$$\{\{\xi\} \,:\, \xi \in g(\alpha) \cap f(\alpha) \text{ for } \alpha \in S \cap \beta\} \cup \{\{\alpha\} \cup (g(\alpha) \setminus f(\alpha)) \,:\, \alpha \in S \cap \beta\}$$

is a clopen partition of $X \cap \beta$ into metrizable subspaces. It follows that $X \cap \beta$ is metrizable. $\dashv$ (Claim 0.1.1)

By Fodor's Lemma (available here since λ^* is regular), the open covering $\{\{\alpha\} \cup g(\alpha) \,:\, \alpha \in S\}$ of X does not have any point countable (or even locally countable) refinement. This shows that X is not paracompact and in particular not metrizable.

X is countably tight since X has an open basis consisting of countable sets. It is also easy to see that $L(Y) = |\,Y\,|$ for all infinite subspace Y of X.

❑ (Theorem 0.1)

In [1] and [2] it is actually shown that the condition "locally compact" in Theorem 0.1 can be weakened to "locally countably compact".

References

[1] S. Fuchino, Fodor-type Reflection Principle and Balogh's reflection theorems, RIMS Kôkyûroku, No.1686, (2010), 41–58.
`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/balogh-x.pdf`

[2] S. Fuchino, I. Juhász, L. Soukup, Z. Szentmiklóssy and T. Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (2010), 1415–1429.
`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/ssmL-erice.pdf`

[3] S. Fuchino, H. Sakai, L. Soukup and T. Usuba, More about Fodor-type Reflection Principle, submitted.
`http://kurt.scitec.kobe-u.ac.jp/~fuchino/papers/moreFRP.pdf`