On the set-generic multiverse

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Abstract

The forcing method is a powerful tool to prove the consistency of set-theoretic assertions relative to the consistency of the axioms of set theory. Laver's theorem and Bukovský's theorem assert that set generic extensions of a given ground model constitute a quite reasonable and sufficiently general class of standard models of set-theory.

In sections 2 and 3 of this note, we give a proof of Bukovsky's theorem in the modern setting (for another proof of this theorem see [4]). In section 4 we check that the multiverse of set-generic extensions can be treated as a collection of countable transitive models in a conservative extension of ZFC. The last section then deals with the problem of the existence of infinitely-many independent buttons, which arose in the modal-theoretic approach to the set-generic multiverse by J. Hamkins and B. Loewe [11].

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1 The category of forcing extensions as the set-theoretic multiverse

The forcing method is a powerful tool to prove the consistency of set-theoretic (i.e., mathematical) assertions relative to (the consistency of) the axioms of set theory. If a sentence σ in the language $\mathcal{L}_{\rm ZF}$ of set theory is proved to be relatively consistent with the axioms of set theory (ZFC) by some forcing argument then it is so in the sense of the strictly finitist standpoint of Hilbert: the forcing proof can be recast into an algorithm $\mathcal{A}$ such that, if a formal proof $\mathcal{P}$ of a contradiction from ZFC + σ is ever given, then we can transform $\mathcal{P}$ with the help of $\mathcal{A}$ to another proof of a contradiction from ZFC or even ZF alone.

The "working set theorists" however prefer to see their forcing arguments not as mere discussions concerning manipulations of formulas in a formal system but rather concerning the "real" mathematical universe in which they "live". Forcing for them is thus a method of extending the universe of set theory where they originally "live" (the ground model, usually denoted as "V") to many (actually more than class many in the sense of V) different models of set theory called generic extensions of V. Actually, a family of generic extensions is constructed for certain V-definable partial orderings $\mathbb{P}$. Each such generic extension is obtained first by fixing a so-called generic filter G which is a filter over $\mathbb{P}$, sitting outside V with a "generic" sort of transcendence over V, and then by adding G to V to generate a new structure — the generic extension $V[G]$ of V — which is also a model of ZFC. Often this process of taking generic extension over some model of set theory is even repeated transfinitely many times. As a result, a set-theorist performing forcing constructions is seen to live in many different models of set theory simultaneously. This is manifested in many technical expositions of forcing where the reader very often finds narrations beginning with phrases like: "Working in $V[G]$, ...", "Let $\alpha < \kappa$ be such that x is in the α th intermediate model $V[G_\alpha]$ and ...", "Now returning to V, ...", etc., etc.

Although this "multiverse" view of forcing is in a sense merely a modus loquendi, it is worthwhile to study the possible pictures of this multiverse per se. Some initial moves in this direction have been taken e.g. in [1], [2], [5], [6], [7], [8], [10], [11], [16], [18] etc. The term "multiverse" probably originated in work of Woodin in which he considered the "set-generic multiverse", the "class" of set-theoretic universes which forms the closure of the given initial universe V under set-generic extension and set-generic ground models. Sometimes we also have to consider the constellations of the set-generic multiverse where V cannot be reconstructed as a set-generic extension of some of or even any of the proper inner models of V. To deal with such cases it is more convenient to consider the expanded generic multiverse where we also assume that the multiverse is also closed under the construction of definable inner models.

The set-generic universe should be distinguished from the "class-generic multiverse", defined in the same way but with respect to class-forcing extensions and ground models, as well as inner models of class-generic extensions that are not themselves class-generic (see [5]). It is even possible to go beyond class-forcing by considering forcings whose conditions are classes, so-called hyperclass forcings (see [6]). The broadest point of view with regard to the multiverse is expressed in [7], where the "hyperuniverse" is taken to consist of *all* universes which share the same ordinals as the initial universe (which is taken to be countable to facilitate the construction of new universes). The hyperuniverse is closed under all notions of forcing.

In this article we restrict our attention to the set-generic multiverse. The well-posedness of questions regarding the set-generic multiverse is established by the theorems of Laver and Bukovský which we discuss in Section 2. These theorems show that the set-generic extensions and set-generic ground models of a given universe represent a "class" of models with a natural characterization.

The straightforward formulation of the set-generic multiverse requires the notion of "class" of classes which cannot be treated in the usual framework of ZF set theory, but, as emphasized at the beginning, theorems about the set-generic multiverse are actually meta-theorems about ZFC. However we can also consider a theory which is a conservative extension of ZFC in which set-generic extensions and set-generic ground models are real objects in the theory and the set-generic multiverse a definable class. In Section 4, we introduce such a system and show that it is a conservative extension of ZFC.

The multiverse view sometimes highlights problems which would never have been asked in the conventional context of forcing constructions (see [10]). As one such example we consider in Section 5 the problem of the existence of infinitely many independent buttons (in the sense of [11]).

2 Laver's theorem and Bukovský's theorem

In the forcing language, we often have to express that a certain set is already in the ground model, e.g. in a statement like: $p \Vdash_{\mathbb{P}}$ " ... $\dot{x}$ is in V and ... ". In such situations we can always find a large enough ordinal ξ such that the set in question should be found in that level of cumulative hierarchy in the ground model. So we can reformulate a statement like the one above into something like $p \Vdash_{\mathbb{P}}$ " $\ldots$ $\dot{x} \in \check{V}_\xi$ and $\ldots$ " which is a legitimate expression in the forcing language.

This might be one of the reasons why it is proved only quite recently that the ground model is always definable in an arbitrary set-generic extension:

Theorem 2.1 (R. Laver, [14], H. Woodin [17])**.** *There is a formula $\varphi^*(x,y)$ in $\mathcal{L}_{\mathrm{ZF}}$ such that, for any transitive model V of* ZFC *and set-generic extension $V[G]$ of V there is $a \in V$ such that, for any $b \in V[G]$*

$$b \in V \quad \Leftrightarrow \quad V[G] \models \varphi^*(a,b). \qquad \square$$

An important corollary of Laver's theorem is that a countable transitive model of ZFC can have at most countably many ground models for set forcing.

Bukovský's theorem gives a natural characterization of inner models M of V such that V is a set-generic extension of M(1). Note that, by Laver's theorem Theorem 2.1, such M is then definable in V. However the inner model M of V may be introduced as a (definable) class in a model $W \supseteq V$ of ZFC in which V is also a definable inner model, and in such a situation the definability of M in V may not be immediately clear.

Let us begin with the following observation concerning κ-c.c. generic extensions. We shall call a partial ordering *atomless* if each element of it has at least two extensions which are incompatible with each other.

Lemma 2.2. *Let κ be a regular uncountable cardinal. If $\mathbb{P}$ is a κ-c.c. atomless partial ordering, then $\mathbb{P}$ adds a new subset of $2^{<\kappa}$.*

Proof. Without loss of generality, we may assume that $\mathbb{P}$ consists of the positive elements of an atomless complete Boolean algebra. Note that $\mathbb{P}$ adds new subset of On since $\mathbb{P}$ adds a new set (e.g. the $(V,\mathbb{P})$-generic set). Suppose that $\dot{S}$ is a $\mathbb{P}$-name of a new subset of On. Let θ be a sufficiently large regular cardinal and let $M \prec \mathcal{H}(\theta)$ be such that

(2.1) $|\, M \,| \leq 2^{<\kappa}$;

(2.2) ${}^{<\kappa}M \subseteq M$ and

(2.3) $\mathbb{P}$, $\dot{S}$, $\kappa \in M$.

Let $\dot{T}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ " $\dot{T} = \dot{S} \cap M$ ". By (2.1), it is enough to show the following, where V denotes the ground model:

(1) In the terminology of [8], M is a ground of V.

Claim 2.2.1. $\Vdash_{\mathbb{P}}$ "$\dot{T} \notin V$".

$\vdash$ Otherwise there would be $p \in \mathbb{P}$ and $T \in V$, $T \subseteq \mathrm{On}$ such that

(2.4) $\quad p \Vdash_{\mathbb{P}}$ "$\dot{T} = \check{T}$".

We show in the following that then we can construct a strictly decreasing sequence $\langle q_\alpha : \alpha < \kappa \rangle$ in $\mathbb{P} \cap M$ such that

(2.5) $\quad p \leq_{\mathbb{P}} q_\alpha$ for all $\alpha < \kappa$.

But since $\{q_\alpha \cdot -q_{\alpha+1} : \alpha < \kappa\}$ is then a pairwise disjoint subset of $\mathbb{P}$, this contradicts the κ-c.c. of $\mathbb{P}$.

Suppose that $\langle q_\alpha : \alpha < \delta \rangle$ for some $\delta < \kappa$ has been constructed. If δ is a limit, let $q_\delta = \prod_{\alpha<\delta} q_\alpha$. Then we have $p \leq_{\mathbb{P}} q_\delta$ and $q_\delta \leq_{\mathbb{P}} q_\alpha$ for all $\alpha < \delta$. Since $\langle q_\alpha : \alpha < \delta \rangle \in M$ by (2.2), we also have $q_\delta \in M$.

If $\delta = \beta + 1$, then, since $M \models$"q_β does not decide $\dot{S}$" by the elementarity of M, there are $\xi \in \mathrm{On} \cap M$ and q, $q' \in \mathbb{P} \cap M$ with q, $q' \leq_{\mathbb{P}} q_\beta$ such that $q \Vdash_{\mathbb{P}}$ "$\xi \in \dot{S}$" and $q' \Vdash_{\mathbb{P}}$ "$\xi \notin \dot{S}$". At least one of them, say q, must be incompatible with p. Then $q_\delta = q_\beta \cdot -q$ is as desired. $\dashv$ (Claim 2.2.1)

$\square$ (Lemma 2.2)

Note that, translated into the language of complete Boolean algebras, the lemma above just asserts that no κ-c.c. atomless Boolean algebra $\mathbb{B}$ is $(2^{<\kappa}, 2)$-distributive.

Suppose now that V is a transitive model of ZFC and M an inner model of ZFC in V (that is M is a transitive class $\subseteq V$ with $(M, \in) \models$ ZFC). For a regular uncountable cardinal κ in M, we say that M *κ-globally covers* V if for every function f (in V) with $\mathrm{dom}(f) \in M$ and $\mathrm{rng}(f) \subseteq M$, there is a function $g \in M$ with $\mathrm{dom}(g) = \mathrm{dom}(f)$ such that $f(i) \in g(i)$ and $M \models |\, g(i)\,| < \kappa$ for all $i \in \mathrm{dom}(f)$.

Theorem 2.3 (L. Bukovský, [3][2]). *Suppose that V is a transitive model of* ZFC, *$M \subseteq V$ an inner model of* ZFC *and κ is a regular uncountable cardinal in M. Then M κ-globally covers V if and only if V is a κ-c.c. set-generic extension of M.*

Proof. If V is a κ-c.c. set generic extension of M, say by a partial ordering $\mathbb{P} \in M$ with $M \models$"$\mathbb{P}$ has the κ-c.c.", then it is clear that M κ-globally covers

(2) Tadatoshi Miyamoto told us that James Baumgartner also proved this theorem in an unpublished note using the infinitary logic.

V (for f as above, let $\dot{f} \in M$ be a $\mathbb{P}$-name of f and g be defined by letting $g(\alpha)$ to be the set of all possible values $\dot{f}(\alpha)$ may take).

The proof of the converse is done via the following Lemma 2.4. Note that, by Grigorieff's theorem (see Corollary 2.6 below), the statement of this Lemma is a consequence of Bukovský's theorem:

Lemma 2.4. *Suppose that M is an inner model of a transitive model V of* ZFC *such that M κ-globally covers V for some κ regular uncountable in M. Then for any $A \in V$, $A \subseteq \mathrm{On}$, $M[A]$ is*[(3)] *a κ-c.c. set generic extension of M.*

We first show that Theorem 2.3 follows from Lemma 2.4. Assume that M κ-globally covers V. We have to show that V is a κ-c.c. set generic extension of M. In V, let λ be a regular cardinal such that $\lambda^{<\kappa} = \lambda$ and $A \subseteq \mathrm{On}$ be a set such that

$$(2.6) \quad (\mathcal{P}(\lambda))^{M[A]} = (\mathcal{P}(\lambda))^V.$$

Then, by Lemma 2.4, $M[A]$ is a κ-c.c. generic extension of M and hence we have $M[A] \models$ "κ is a regular cardinal". Actually we have $M[A] = V$. Otherwise there would be a $B \in V \setminus M[A]$ with $B \subseteq \mathrm{On}$. Since $M[A]$ κ-globally covers $M[A][B]$, we may apply Lemma 2.4 on this pair and conclude that $M[A][B]$ is a (non trivial) κ-c.c. generic extension of $M[A]$. By Lemma 2.2, there is a new element of $\mathcal{P}((2^{<\kappa})^{M[A]}) \subseteq \mathcal{P}(\lambda)$ in $M[A][B]$.But this is a contradiction to (2.6).

❑ (**Theorem 2.3**)

Proof of Lemma 2.4: We work in M and construct a κ-c.c. partial ordering $\mathbb{P}$ such that $M[A]$ is $\mathbb{P}$-generic over M.

Let $\mu \in \mathrm{On}$ be such that $A \subseteq \mu$. For a regular cardinal λ, let $\mathcal{L}_{<\lambda}(\mu)$ be the infinitary sentential logic with atomic sentences

$$(2.7) \quad \text{“}\alpha \in \dot{A}\text{” for } \alpha \in \mu$$

and with sentences closed under $\neg$ and $\bigvee$ such that the latter logical connective may be applied only to a set of sentences of cardinality $< \lambda$. We regard the usual disjunction $\vee$ of two formulas as a special case of $\bigvee$ and other logical connectives like "$\bigwedge$", "$\wedge$", "$\rightarrow$" as being introduced as abbreviations of usual combinations of $\neg$ and $\bigvee$. For a sentence $\varphi \in \mathcal{L}_{<\lambda}(\mu)$ and $B \subseteq \mu$, we write $B \models \varphi$ when φ holds if each atomic sentence of the form "$\alpha \in \dot{A}$" in φ is replaced by "$\alpha \in B$" and logical connectives in φ are interpreted in canonical way. For a set Γ of sentences, we write $B \models \Gamma$ if $B \models \psi$ for all $\psi \in \Gamma$.

(3) $M[A]$ may be defined by $M[A] = \bigcup_{\alpha\in\mathrm{On}} L(V_\alpha^M \cup \{A\})$.

Let $\vdash$ be a notion of provability for $\mathcal{L}_{<\lambda}(\mu)$ in some logical system which is correct (i.e. $\Gamma \vdash \varphi$ always implies $\Gamma \models \varphi$)[4], upward absolute (i.e. $M \subseteq N$ and $M \models$ "$\Gamma \vdash \varphi$" always imply $N \models$ "$\Gamma \vdash \varphi$" for any transitive models M, N of ZFC) and sufficiently strong (so that all the arguments used below work for this $\vdash$). In Section 3 we introduce one such deductive system (as well as an alternative approach without such a deduction system, based on Lévy absoluteness).

Let $f \in V$ be a mapping $f : \big(\mathcal{P}(\mathcal{L}_{<\kappa}(\mu))\big)^M \setminus \{\emptyset\} \to \big(\mathcal{L}_{<\kappa}(\mu)\big)^M$ such that, for any $\Gamma \in \big(\mathcal{P}(\mathcal{L}_{<\kappa}(\mu))\big)^M \setminus \{\emptyset\}$, we have $f(\Gamma) \in \Gamma$ and $A \models f(\Gamma)$ if $A \models \bigvee\!\!\!\bigvee \Gamma$. Since M κ-globally covers V, there is a $g \in M$ with $g : \big(\mathcal{P}(\mathcal{L}_{<\kappa}(\mu))\big)^M \setminus \{\emptyset\} \to \mathcal{P}_{<\kappa}\big(\mathcal{L}_{<\kappa}(\mu)\big)^M$ such that $f(\Gamma) \in g(\Gamma) \subseteq \Gamma$ for all $\Gamma \in (\mathcal{P}(\mathcal{L}_{<\kappa}(\mu)))^M \setminus \{\emptyset\}$.

In M, let

(2.8) $\quad T = \{\bigvee\!\!\!\bigvee \Gamma \to \bigvee\!\!\!\bigvee g(\Gamma) \,:\, \Gamma \in \mathcal{P}(\mathcal{L}_{<\kappa}(\mu)) \setminus \{\emptyset\}\}.$

Note that T is a theory in $\mathcal{L}_{<(\mu^{<\kappa})^+}(\mu)$ and $V \models$ "$A \models T$". It follows that T is consistent with respect to our $\vdash$ (in M). Let

(2.9) $\quad \mathbb{P} = \{\varphi \in \mathcal{L}_{<\kappa}(\mu) \,:\, T \nvdash \neg\varphi\}$

and for φ, $\psi \in \mathbb{P}$, let

(2.10) $\quad \varphi \leq_{\mathbb{P}} \psi \;\Leftrightarrow\; T \vdash \varphi \to \psi.$

Claim 2.4.1. *For $\varphi \in \mathcal{L}_{<\kappa}(\mu)$, if $A \models \varphi$ then we have $\varphi \in \mathbb{P}$. In particular, "$\alpha \in \dot{A}$" $\in \mathbb{P}$ for all $\alpha \in A$ and "$\neg(\alpha \in \dot{A})$" $\in \mathbb{P}$ for all $\alpha \in \mu \setminus A$.*

$\vdash$ Suppose $A \models \varphi$. We have to show $T \nvdash \neg\varphi$: If $T \vdash \neg\varphi$, then we would have $V \models$ "$T \vdash \neg\varphi$". Since $A \models T$ in V, it follows that $A \models \neg\varphi$. This is a contradiction. $\dashv$ (Claim 2.4.1)

Claim 2.4.2. *For φ, $\psi \in \mathbb{P}$, φ and ψ are compatible if and only if*

(2.11) $\quad T \nvdash \neg(\varphi \wedge \psi).$

Note that (2.11) is equivalent to

(2.12) $\quad T \nvdash \neg\varphi \vee \neg\psi \quad \Leftrightarrow \quad T \nvdash \varphi \to \neg\psi.$

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[4] More precisely, we assume that ZFC proves the correctness of $\vdash$.

$\vdash$ Suppose that φ, $\psi \in \mathbb{P}$ are compatible. By the definition of $\leq_{\mathbb{P}}$ this means that there is $\eta \in \mathbb{P}$ such that $T \vdash \eta \to \varphi$ and $T \vdash \eta \to \psi$. For this η we have $T \vdash \eta \to (\varphi \wedge \psi)$. Since $T \nvdash \neg\eta$, it follows that $T \nvdash \neg(\varphi \wedge \psi)$.

Conversely if $T \nvdash \neg(\varphi \wedge \psi)$. Then $(\varphi \wedge \psi) \in \mathbb{P}$. Since $T \vdash (\varphi \wedge \psi) \to \varphi$ and $T \vdash (\varphi \wedge \psi) \to \psi$, it follows that $(\varphi \wedge \psi) \leq_{\mathbb{P}} \varphi$ and $(\varphi \wedge \psi) \leq_{\mathbb{P}} \psi$. Thus φ and ψ are compatible with respect to $\leq_{\mathbb{P}}$. $\dashv$ (Claim 2.4.2)

Claim 2.4.3. *$\mathbb{P}$ has the κ-c.c.*

$\vdash$ Suppose that $\Gamma \subseteq \mathbb{P}$ is an antichain. Since $|\, g(\Gamma)\,| < \kappa$, it is enough to show that $g(\Gamma) = \Gamma$. Suppose otherwise and let $\varphi_0 \in \Gamma \setminus g(\Gamma)$. Since "$\bigvee\!\!\!\bigvee\Gamma \to \bigvee\!\!\!\bigvee g(\Gamma)$" $\in T$ and $\vdash \varphi_0 \to \bigvee\!\!\!\bigvee\Gamma$, we have

$$T \vdash \varphi_0 \to \bigvee\!\!\!\bigvee g(\Gamma). \tag{2.13}$$

It follows that there is $\varphi \in g(\Gamma)$ such that φ_0 and φ are compatible. This is because otherwise we would have $T \vdash \varphi_0 \to \neg\varphi$ for all $\varphi \in g(\Gamma)$ by (2.11). Hence $T \vdash \varphi_0 \to \bigwedge\!\!\!\bigwedge\{\neg\varphi \,:\, \varphi \in g(\Gamma)\}$ which is equivalent to $T \vdash \varphi_0 \to \neg\bigvee\!\!\!\bigvee g(\Gamma)$. From this and (2.13), it follows that $T \vdash \neg\varphi_0$. But this is a contradiction to the assumption that $\varphi_0 \in \mathbb{P}$.

Now, since Γ is pairwise incompatible, it follows that $\varphi_0 = \varphi \in g(\Gamma)$. This is a contradiction to the choice of φ_0. $\dashv$ (Claim 2.4.3)

In V, let $G(A) = \{\varphi \in \mathbb{P} \,:\, A \models \varphi\}$. By Claim 2.4.1, we have $G(A) = \{\varphi \in \mathcal{L}_{<\kappa}(\mu) \,:\, A \models \varphi\}$ and A is definable from $G(A)$ over M in V. Thus we have $M[G(A)] = M[A]$.

Hence the following two Claims prove our Claim:

Claim 2.4.4. *$G(A)$ is a filter in $\mathbb{P}$.*

$\vdash$ Suppose that $\varphi \in G(A)$ and $\varphi \leq_{\mathbb{P}} \psi$. Since this means that $A \models \varphi$ and $T \vdash \varphi \to \psi$, it follows that $A \models \psi$. That is, $\psi \in G(A)$.

Suppose now that φ, $\psi \in G(A)$. This means that

$$A \models \varphi \text{ and } A \models \psi. \tag{2.14}$$

Hence we have $A \models \varphi \wedge \psi$. By Claim 2.4.1, it follows that $T \nvdash \neg(\varphi \wedge \psi)$. Thus φ and ψ are compatible by Claim 2.4.2. $\dashv$ (Claim 2.4.4)

Claim 2.4.5. *$G(A)$ is $\mathbb{P}$-generic.*

$\vdash$ Working in M, suppose that Γ is a maximal antichain in $\mathbb{P}$. By Claim 2.4.3, we have $|\,\Gamma\,| < \kappa$ and hence we have $\bigvee\!\!\!\bigvee\Gamma \in \mathbb{P}$. Moreover we have $T \vdash \bigvee\!\!\!\bigvee\Gamma$: Otherwise $\neg\bigvee\!\!\!\bigvee\Gamma$ would be an element of $\mathbb{P}$ incompatible with every $\varphi \in \Gamma$. A contradiction to the maximality of Γ.

Hence $A \models \bigvee\!\!\!\bigvee\Gamma$ and thus there is $\varphi \in \Gamma$ such that $A \models \varphi$. That is, $\varphi \in G(A)$. $\dashv$ (Claim 2.4.5)

❑ (Lemma 2.4)

The proof of Theorem 2.3 from Lemma 2.4 relies on Lemma 2.2 and the Axiom of Choice is involved both in the statement and the proof of this lemma. On the other hand, Lemma 2.4 can be proved without assuming the Axiom of Choice in M: It suffices to eliminate choice from the proof of Claim 2.4.5.

Proof of Claim 2.4.5 without Axiom of Choice in M: Working in M, suppose that D is a dense subset of $\mathbb{P}$. Then $A \models \bigvee\!\!\!\bigvee D$: Otherwise we would have $T \nvdash \bigvee\!\!\!\bigvee D$. Since

(2.15) $T \vdash \bigvee\!\!\!\bigvee D \leftrightarrow \bigvee\!\!\!\bigvee g(D)$,

it follows that $T \nvdash \bigvee\!\!\!\bigvee g(D)$. Thus $\neg\bigvee\!\!\!\bigvee g(D) \in \mathbb{P}$. Since D is dense in $\mathbb{P}$ there is $\varphi_0 \in D$ such that $T \vdash \varphi_0 \rightarrow \neg\bigvee\!\!\!\bigvee g(D)$. By (2.15), it follows that $T \vdash \varphi_0 \rightarrow \neg\bigvee\!\!\!\bigvee D$. On the other hand, since $\varphi_0 \in D$ we have $T \vdash \varphi_0 \rightarrow \bigvee\!\!\!\bigvee D$. Hence we have $T \vdash \neg\varphi_0$ which is a contradiction to $\varphi_0 \in \mathbb{P}$.

Thus there is $\varphi_1 \in D$ such that $A \models \varphi_1$, that is, $\varphi_1 \in G(A)$.

❑ (Claim 2.4.5)

The next corollary follows immediately from this remark:

Corollary 2.5. *Suppose that V is a model of* ZFC *and M is a definable inner model of V (of* ZF*) such that M κ-globally covers V. If $V = M[A]$ for some set $A \subseteq$* On *then V is a κ-c.c. set-generic extension of M.* ❑

Grigorieff's theorem can be also obtained by a modification of the proof of Theorem 2.3.

Corollary 2.6 (S. Grigorieff [9])**.** *Suppose that M is an inner model of a model V of* ZFC *and V is a set generic extension of M. Then any inner model N of V (of* ZFC*) with $M \subseteq N$ is a set generic extension of M and hence definable in V. Also, for such N, V is a set generic extension of N.*

If V is κ-c.c. set generic extension of M in addition, then N is a κ-c.c. set generic extension of M and V is a κ-c.c. set generic extension of N. ❑

We do not know if Corollary 2.6 is false without the added assumption that V is $M[A]$ for a set of ordinals A.

More generally, it seems to be open if there is a characterisation of the set-generic extensions of an arbitrary model of ZF; or at least of such extensions given by partial orders which are well-ordered in the ground model.

The condition that N be a model of ZFC in Grigorieff's theorem is necessary: J. Cummings and M. Magidor constructed an inner model of ZF of the Cohen extension of L which is not a set-generic extension of the ground model. Also here seems to be still some open questions about what happens under the absence of the Axiom of Choice.

Similarly to Theorem 2.3, we can also characterize generic extensions obtained via a partial ordering of cardinality $\leq \kappa$.

For M and V as above, we say that V is *κ-decomposable* into M if for any $a \in V$ with $a \subseteq M$, there are $a_i \in M$, $i \in \kappa$ such that $a = \bigcup_{i<\kappa} a_i$.

Theorem 2.7. *Suppose that V is a transitive model of* ZFC *and M an inner model of* ZFC *definable in V and κ is a cardinal in M. Then V is a generic extension of M by a partial ordering in M of size $\leq \kappa$ (in M) if and only if M κ^+-globally covers V and V is κ-decomposable into M.*

Proof. If V is a generic extension of M by a generic filter G over a partial ordering $\mathbb{P} \in M$ of size $\leq \kappa$ (in M) then $\mathbb{P}$ has the κ^+-c.c. and hence M κ^+-globally covers V by Theorem 2.1. V is κ-decomposable into M since, for any $a \in V$ with $a = \dot{a}^G$, we have $a = \bigcup\{\{m \in M : p \Vdash_{\mathbb{P}} \text{“} m \in \dot{a} \text{”}\} : p \in G\}$.

Suppose now that M κ^+-globally covers V and V is κ-decomposable into M. By Theorem 2.3, there is a κ^+-c.c. partial ordering $\mathbb{P}$ in M and a $\mathbb{P}$-generic filter G over M such that $V = M[G]$. Without loss of generality, we may assume that $\mathbb{P}$ consists of positive elements of a complete Boolean algebra $\mathbb{B}$ (in M).

By κ-decomposability, G can be decomposed into κ sets $G_i \in M$, $i < \kappa$. Without loss of generality, we may assume that $\mathbb{1}_{\mathbb{P}}$ forces this fact. So letting $\dot{G}$ be the standard name of G and $\dot{G}_i$, $i < \kappa$ be names of G_i, $i < \kappa$ respectively, we may assume

$$(2.16) \quad \Vdash_{\mathbb{P}} \text{“} \dot{G} = \bigcup_{i<\kappa} \dot{G}_i \text{”}.$$

Working in M, let $X_i \subseteq \mathbb{P}$ be a maximal pairwise incompatible set of conditions p which decide $\dot{G}_i$ to be $G_{i,p} \in M$ for each $i < \kappa$. By the κ^+-c.c. of $\mathbb{P}$, we have $|X_i| \leq \kappa$. Clearly, we have $p \leq_{\mathbb{P}} \prod^{\mathbb{B}} G_{i,p}$ for all $i < \kappa$ and $p \in X_i$. Let $\mathbb{P}' = \bigcup\{X_i : i < \kappa\}$. Then $|\mathbb{P}'| \leq \kappa$.

Claim 2.7.1. $\mathbb{P}'$ *is dense in* $\mathbb{P}$.

$\vdash$ Suppose $p \in \mathbb{P}$. Then there is $q \leq p$ such that q decides some $\dot{G}_i$ to be $G_{i,q}$ and $p \in G_{i,q}$. Let $r \in X_i$ be compatible with q. Then we have $r \leq_{\mathbb{P}} \prod^{\mathbb{B}} G_{i,r} = \prod^{\mathbb{B}} G_{i,q} \leq p$. $\dashv$ (Claim 2.7.1)

Thus V is a $\mathbb{P}'$-generic extension over M. ❑ (Theorem 2.7)

3 A Formal deductive system for $\mathcal{L}_{<\lambda}(\mu)$

In the proof of Lemma 2.4, we used a formal deductive system of $\mathcal{L}_{<\lambda}(\mu)$ without specifying exactly which system we are using. It is enough to consider a system which consists of all logical axioms we used in the course of the proof together with modus ponens and some infinitary deduction rules like:

$$\frac{\varphi_i \rightarrow \psi, \quad i \in I}{\bigvee\!\!\!\bigvee\{\varphi_i \,:\, i \in I\} \rightarrow \psi}$$

What we need for such a system is that its correctness and upward absoluteness hold while we do not make use of any version of completeness of the system.

Nevertheless, to be concrete, we shall introduce below such a deductive system S for $\mathcal{L}_{<\lambda}(\mu)$.

The axioms of S consist of the following formulas:

(A1) $\varphi(\varphi_0, \varphi_1, ..., \varphi_{n-1})$

for each tautology $\varphi(A_0, A_1, ..., A_{n-1})$ of (finitary) propositional logic and φ_0, φ_1,..., $\varphi_{n-1} \in \mathcal{L}_{<\lambda}(\mu)$;

(A2) $\varphi_{i_0} \rightarrow \bigvee\!\!\!\bigvee\{\varphi_i \,:\, i \in I\}$ and $\bigwedge\!\!\!\bigwedge\{\varphi_i \,:\, i \in I\} \rightarrow \varphi_{i_0}$

for any $\varphi_i \in \mathcal{L}_{<\lambda}(\mu)$ for $i \in I$ with $|\, I \,| < \lambda$ and $i_0 \in I$;

(A3) $\neg(\bigwedge\!\!\!\bigwedge\{\varphi_i \,:\, i \in I\}) \leftrightarrow \bigvee\!\!\!\bigvee\{\neg\varphi_i \,:\, i \in I\}$ and
$\neg(\bigvee\!\!\!\bigvee\{\varphi_i \,:\, i \in I\}) \leftrightarrow \bigwedge\!\!\!\bigwedge\{\neg\varphi_i \,:\, i \in I\}$

for any $\varphi_i \in \mathcal{L}_{<\lambda}(\mu)$ for $i \in I$ with $|\, I \,| < \lambda$; and

(A4) $\varphi \wedge (\bigvee\!\!\!\bigvee\{\psi_i \,:\, i \in I\}) \leftrightarrow \bigvee\!\!\!\bigvee\{\varphi \wedge \psi_i \,:\, i \in I\}$ and
$\varphi \vee (\bigwedge\!\!\!\bigwedge\{\psi_i \,:\, i \in I\}) \leftrightarrow \bigwedge\!\!\!\bigwedge\{\varphi \vee \psi_i \,:\, i \in I\}$

for any φ, $\psi_i \in \mathcal{L}_{<\lambda}(\mu)$ for $i \in I$ with $|\, I \,| < \lambda$.

Deduction Rules:

(Modus Ponens) $$\frac{\{\varphi, \varphi \to \psi\}}{\psi}$$

(R1) $$\frac{\{\varphi_i \to \psi \,:\, i \in I\}}{\bigvee\!\!\!\bigvee\{\varphi_i \,:\, i \in I\} \to \psi}$$ **(R2)** $$\frac{\{\varphi \to \psi_j \,:\, j \in J\}}{\varphi \to \bigwedge\!\!\!\bigwedge\{\psi_j \,:\, j \in J\}}$$

A proof of $\varphi \in \mathcal{L}_{<\lambda}(\mu)$ from $\Gamma \subseteq \mathcal{L}_{<\lambda}(\mu)$ is a labeled tree $\langle \mathsf{T}, f\rangle$ such that

(3.1) $\mathsf{T} = \langle \mathsf{T}, \leq\rangle$ is a tree growing upwards with its root r_0 and T has no infinite branch;

(3.2) $f : \mathsf{T} \to \mathcal{L}_{<\lambda}(\mu)$;

(3.3) $f(r_0) = \varphi$;

(3.4) if $t \in \mathsf{T}$ is a maximal element then either $f(t) \in \Gamma$ or t is one of the axioms of S;

(3.5) if $t \in \mathsf{T}$ and $I \subseteq \mathsf{T}$ is the set of all immediate successors of t, then

$$\frac{\{f(s) \,:\, s \in I\}}{f(t)}$$

is one of the deduction rules.

Now the proof of the following is an easy exercise:

Proposition 3.1. (1) *For any $B \subseteq \mu$, $T \subseteq \mathcal{L}_{<\lambda}(\mu)$ and $\varphi \in \mathcal{L}_{<\lambda}(\mu)$, if $T \vdash \varphi$ and $B \models T$, then we have $B \models \varphi$.*

(2) *For transitive models M, N of* ZFC *such that M is an inner model of N, if $M \models$ "$\langle \mathsf{T}, f\rangle$ is a proof of φ in $\mathcal{L}_{<\lambda}(\mu)$", then $N \models$ "$\langle \mathsf{T}, f\rangle$ is a proof of φ in $\mathcal{L}_{<\lambda'}(\mu)$" where $\lambda' = \min\{\kappa \in \mathrm{Card}^N \,:\, \lambda \leq \kappa\}$.*

Proof. (1): By induction on cofinal subtrees of a fixed proof $\langle \mathsf{T}, f\rangle$ of φ. (2): Clear by definition. ❑ (Proposition 3.1)

An alternative setting to the argument by means of a deductive system is to make use of the following definition of $\Gamma \vdash \varphi$ in the proof of Lemma 2.4:

> $\Gamma \vdash \varphi$ iff for any $B \subseteq \mu$ in some set-forcing extension $M[G]$ of M, $M[G] \models B \models \psi$ for all $\psi \in \Gamma$ always implies $M[G] \models B \models \varphi$.

Note that this is definable in M using the forcing relation definable on M. It remains to verify that this notion has the desired degree of absoluteness. Actually we can easily prove the full absoluteness, that is, if N is a transitive model containing M with the same ordinals as those of M then, for Γ, $\varphi \in M$ with $M \models \Gamma \subseteq \mathcal{L}_{<\lambda}(\mu)$ and $M \models \varphi \in \mathcal{L}_{<\lambda}(\mu)$ for a regular λ, $\Gamma \vdash \varphi$ holds in M iff $\Gamma \vdash \varphi$ holds in N.

First suppose that $\Gamma \vdash \varphi$ holds in M and let $B \subseteq \mu$ be a set of ordinals in a set-generic extension $N[G]$ of N such that B witnesses the failure of $\Gamma \vdash \varphi$ in N. Let x be a real which is generic over N for the Lévy collapse of a sufficiently large ν to ω such that Γ and μ become countable in the generic extension $N[x]$. Then x is also Lévy generic over M and $M[x]$ is a submodel of $N[x]$. By Lévy Absoluteness Lemma it follows that that there exists $B' \subseteq \mu$ in $M[x]$ which also witnesses the failure of $\Gamma \vdash \varphi$ in M. Conversely, suppose that $\Gamma \vdash \varphi$ holds in N and let $B \subseteq \mu$ be a set of ordinals in a set-generic extension $M[G]$ of M such that B witnesses the failure of $\Gamma \vdash \varphi$ in M. Then B also belongs to an extension of M which is generic for the Lévy collapse of sufficently large ν to ω; choose a condition p in this forcing which forces the existence of such a B. Now if x is Lévy-generic over N and contains the condition p, we see that there is a counterexample to $\Gamma \vdash \varphi$ in N witnessed in $N[x]$, contrary to our assumption.

With both of the interpretations of $\vdash$ we can check that all the arguments in Section 2 go through.

4 An axiomatic framework for the set-generic multiverse

In this section, we introduce a conservative extension MZFC of ZFC in which we can treat the multiverse of set-generic extensions of models of ZFC as a collection of countable transitive models. We hope that this system or some further extension of it (which can possibly also treat tame class forcings) can be used as a basis for direct formulation of statements concerning the multiverse.

The language $\mathcal{L}_{\mathrm{MZF}}$ of the axiom system MZFC consists of the ϵ-relation symbol '$\in$', and a constant symbol '$\mathtt{v}$' which should represent the countable transitive "ground model".

The axiom system MZFC consists of

(4.1) all axioms of ZFC;

(4.2) "$\mathtt{v}$ is a countable transitive set";

(4.3) "$\mathtt{v} \models \varphi$" for all axioms φ of ZFC;

By (4.1), MZFC proves the (unique) existence of the closure $\mathcal{M}$ of "$\{\mathtt{v}\}$" under forcing extension and definable "inner model" of "ZF" (here 'ZF' is set in quotation marks since we can only argue in metamathematics that such "inner model" satisfies each instance of replacement). Note that $\mathcal{M} \subseteq \mathcal{H}_{\aleph_1}$. Here "inner model" is actually phrased in $\mathcal{L}_{\mathrm{ZF}}$ as "transitive almost universal subset closed under Gödel operations". If we had $\mathtt{v} \models$ ZFC, we would have $w \models$ ZF for any inner model w of $\mathtt{v}$ in this sense by Theorem 13.9 in [12]. In MZFC, however, we have only $\mathtt{v} \models \varphi$ for each axiom φ of ZFC (in the metamathematics). Nevertheless, for all such "inner model" w and hence for all $w \in \mathcal{M}$, we have $w \models \varphi$ for all axiom φ of ZF by the proof of Theorem 13.9 in [12] and the Forcing Theorem. Apparently, this is enough to consider $\mathcal{M}$ in this framework as the set-generic multiverse.

Similarly, we can also start from any extension of ZFC (e.g. with some additional large cardinal axiom) and make $\mathcal{M}$ closed under some more operations such as some well distinguished class of class forcing extensions.

The following theorem shows that we do not increase the consistency strength by moving from ZFC to MZFC.

Theorem 4.1. MZFC *is a conservative extension of* ZFC*: for any sentence* ψ *in* $\mathcal{L}_{\mathrm{ZF}}$*, we have* ZFC $\vdash \psi \;\Leftrightarrow\;$ MZFC $\vdash \psi$*. In particular,* MZFC *is equiconsistent with* ZFC*.*

Proof. "$\Rightarrow$" is trivial.

For "$\Leftarrow$", suppose that MZFC $\vdash \psi$ for a formula ψ in $\mathcal{L}_{\mathrm{ZF}}$. Let $\mathcal{P}$ be a proof of ψ from MZFC and let T be the finite fragment of ZFC consisting of all axioms φ of ZFC such that $\mathtt{v} \models \varphi$ appears in $\mathcal{P}$. Let $\Phi(x)$ be the formula in $\mathcal{L}_{\mathrm{ZF}}$ saying

"x is a countable transitive set and $x \models \bigwedge\!\!\!\bigwedge T$".

By the Deduction Theorem, we can recast $\mathcal{P}$ to a proof of ZFC $\vdash \forall x(\Phi(x) \rightarrow \psi)$. On the other hand we have ZFC $\vdash \exists x\Phi(x)$ (by the Reflection Principle, Downward Löwenheim-Skolem Theorem and Mostowski's Collapsing Theorem). Hence we obtain a proof of ψ from ZFC alone. ❏ (Theorem 4.1)

It may be a little bit disappointing if the set theoretic universe as a whole is a set consisting of countable sets see from the "meta-universe". However if M is an inner model of a model W of ZFC (i.e. a model which is a transitive class $\subseteq W$) there are always partial ordering $\mathbb{P}$ in M for which there is no $(M, \mathbb{P})$-generic set in W (e.g. any partial ordering collapsing a cardinal of W cannot have its generic set in W).

However, if we are content with a meta-universe which is not a model of full ZFC, we can work with the following setting where all models of the set-generic multiverse are inner model of a meta-universe: starting from a model V of ZFC with an inaccessible cardinal κ we generically extend it to $W = V[G]$ by Lévy collapsing κ to ω_1. Letting $M = \mathcal{H}(\kappa)^V$, we have $M \models$ ZFC and M is an inner model of $W = \mathcal{H}(\kappa)^{V[G]} = \mathcal{H}(\omega_1)^{V[G]}$. $W \models$ ZFC $-$ Power Set Axiom and for any partial ordering $\mathbb{P}$ in M there is a $(M, \mathbb{P})$-generic set in W. Thus an NBG-type theory of W with a new unary predicate corresponding to M can be used as a framework of the theory for the set-generic multiverse.

5 Independent buttons

The multiverse view sometimes highlights problems which would be never asked in the conventional context of forcing constructions. The existence of infinitely many independent buttons which arose in connection with the characterization of the modal logic of the set-generic multiverse (see [11]) is one such question.

A sentence φ in $\mathcal{L}_{\mathrm{ZF}}$ is said to be a *button* (for set-genericity) if any set-generic extension $V[G]$ of the ground model V has a further set-generic extension $V[G][H]$ such that φ holds in all set-generic extensions of $V[G][H]$. Let us say that a button φ is pushed in a set-generic extension $V[G]$ if φ holds in all further set-generic extensions $V[G][H]$ of $V[G]$ (including $V[G]$ itself).

Formulas φ_n, $n \in \omega$ are *independent buttons*, if φ_n, $n \in \omega$ are unpushed buttons and for any set-generic extension $V[G]$ of the ground model V and any $X \subseteq \omega$ in $V[G]$,

(5.1) if $\{n \in \omega : V[G] \models \varphi_n \text{ is pushed}\} \subseteq X$ then there is a generic extension $V[G][H]$ such that $\{n \in \omega : V[G][H] \models \varphi_n \text{ is pushed}\} = X$.

In [11], it is claimed that formulas b_n, $n \in \omega$ form an infinite set of independent buttons over $V = L$ where b_n is a formula asserting: "${\omega_n}^L$ is not a cardinal". This is used to prove that the principles of forcing expressible in the modal logic of the set-theoretic multiverse as a Kripke frame where modal operator $\Box$ is interpreted as:

(5.2) $M \models \Box\varphi \Leftrightarrow$ in all generic extensions $M[G]$ of M we have $M[G] \models \varphi$

coincides with the modal theory $\mathsf{S4.2}$ (Main Theorem 6 in [11]).

Unfortunately, it seems that there is no guarantee that (5.1) holds in an arbitrary generic extension $V[G]$ for these b_n, $n \in \omega$.

In the following, we introduce an alternative set of infinitely many formulas which are actually independent buttons for any ground model of ZFC+ "GCH below $\aleph_\omega$" + "$\aleph_n = \aleph_n^L$ for all $n \in \omega$" which can be used as b_n, $n \in \omega$ in [11].

We first note that, for Main Theorem 6 in [11] we actually need only the existence of an arbitrary finite number of independent buttons. In the case of $V = L$ the following formulas can be used for this: Let ψ_n be the statement that $\aleph_n^L$ is a cardinal and the L-least $\aleph_n^L$-Suslin tree T_n^L in L (i.e., the L-least normal tree of height $\aleph_n^L$ with no antichain of size $\aleph_n^L$ in L) is still $\aleph_n^L$-Suslin. If M is a set-generic (or arbitrary) extension of L in which the button $\neg\psi_n$ has not been pushed, then by forcing with T_n^L over M we push this button and do not affect any of the other unpushed buttons $\neg\psi_m$, $m \neq n$, as this forcing is $\aleph_n$-distributive and has size $\aleph_n$. Rittberg [15] also found independent buttons under $V = L$.

Now we turn to a construction of infinitely many independent buttons for which we even do not need the existence of Suslin trees. For $n \in \omega$, let φ_n be the statement:

(5.3) there is an injection from ${\aleph_{n+2}}^L$ to $\mathcal{P}({\aleph_n}^L)$.

Note that φ_n is pushed in a generic extension $V[G]$ if and only if it holds in $V[G]$. Thus φ_n for each $n \in \omega$ is a button provided that φ_n does not hold in the ground model. We show that these φ_n, $n \in \omega$ are independent buttons (over any ground model where they are unpushed — e.g., when $V = L$).

Suppose that we are working in some model W of ZFC. In W, let $A = \{n \in \omega : \Box\varphi_n \text{ holds}\}$ and $B \subseteq \omega$ be arbitrary with $A \subseteq B$. It is enough to prove the following

Proposition 5.1. *We can force (over W) that φ_n holds for all $n \in B$ and $\neg\varphi_n$ for all $n \in \omega \setminus B$.*

Proof. In W, let $\kappa_n = |\,{\aleph_n}^L\,|$ for $n \in \omega$. We use the notation of [13] on the partial orderings with partial functions and denote with $\mathrm{Fn}(\kappa, \lambda, \mu)$ the set of all partial functions from κ to λ with cardinality $< \mu$ ordered by reverse inclusion. By Δ-System Lemma, it is easy to see that $\mathrm{Fn}(\kappa, \lambda, \mu)$ has the $(\lambda^{<\mu})^+$-c.c. Let

$$(5.4) \quad \mathbb{P}_n = \begin{cases} \mathrm{Fn}(\kappa_{n+2}, 2, \kappa_n) & \text{if } n \in B \setminus A \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Let $\mathbb{P} = \prod_{n\in\omega} \mathbb{P}_n$ be the full support product of $\mathbb{P}_n$, $n \in \omega$. Then we clearly have $\Vdash_{\mathbb{P}}$ "φ_n" for all $n \in B$. Thus to show that $\mathbb{P}$ creates a generic extension as desired, it is enough to show that $\Vdash_{\mathbb{P}}$ "$\neg\varphi_n$" for all $n \in \omega \setminus B$.

Suppose that

(5.5) $\quad n \in \omega \setminus B$.

Then we have

(5.6) $\quad \mathbb{P}_n = \mathbb{1}$.

Since φ_n does not hold in W, we have $\kappa_n < \kappa_{n+1} < \kappa_{n+2}$ and $2^{\kappa_n} = \kappa_{n+1}$ in W. By (5.6), $\mathbb{P}$ factors as $\mathbb{P} \sim \mathbb{P}(< n) \times \mathbb{P}(> n)$ where $\mathbb{P}(< n) = \prod_{k<n} \mathbb{P}_k$ and $\mathbb{P}(> n) = \prod_{k>n} \mathbb{P}_k$.

We show that both $\mathbb{P}(> n)$ and $\mathbb{P}(< n)$ over $\mathbb{P}(> n)$ do not add any injection from κ_{n+2} into $\mathcal{P}(\kappa_n)$.

$\mathbb{P}(> n)$ is κ_{n+1}-closed. Thus it does not add any new subsets of κ_n. So if it added an injection from κ_{n+2} into $\mathcal{P}(\kappa_n)$ then it would collapse the cardinal κ_{n+2}. Since $\mathbb{P}(> n)$ further factors as $\mathbb{P}(> n) \sim \mathbb{P}_{n+1} \times \prod_{k>n+1} \mathbb{P}_k$ and $\prod_{k>n+1} \mathbb{P}_k$ is κ_{n+2}-closed the only way $\mathbb{P}(> n)$ could collapse κ_{n+2} would be if $\mathbb{P}_{n+1}$ did so. But then, since $\mathbb{P}_{n+1}$ has the $(2^{<\kappa_{n+1}})^+$-c.c. with $(2^{<\kappa_{n+1}})^+ = (2^{\kappa_n})^+$, we would have $2^{\kappa_n} \geq \kappa_{n+2}$. This is a contradiction to the choice (5.5) of n. So $\mathbb{P}(> n)$ forces φ_n to fail.

In the rest of the proof, we work in $W^{\mathbb{P}(>n)}$ and show that $\mathbb{P}(< n)$ does not add any injection from κ_{n+2} into $\mathcal{P}(\kappa_n)$. Note that, by κ_{n+1}-closedness of $\mathbb{P}(> n)$, we have $\mathrm{Fn}(\kappa_{m+2}, 2, \kappa_m)^W = \mathrm{Fn}(\kappa_{m+2}, 2, \kappa_m)^{W^{\mathbb{P}(>n)}}$ for $m < n$.

We have the following two cases:

Case I. $n - 1 \in A \cup (\omega \setminus B)$. Then $\mathbb{P}(< n) \sim \mathbb{P}(< m)$ for some $m < n$ and $\mathbb{P}(< m)$ has the $(2^{\kappa_{m-1}})^+$-c.c. with $(2^{\kappa_{m-1}})^+ \leq \kappa_n$.

Case II. $n - 1 \in B \setminus A$. Then $2^{<\kappa_{n-1}} = \kappa_n$ and $\mathbb{P}(< n)$ has the κ_{n+1}-c.c.

In both cases the partial ordering $\mathbb{P}(< n)$ has κ_{n+1}-c.c. and hence the cardinals κ_{n+1} and κ_{n+2} are preserved. Since $\mathbb{P}(< n)$ has at most cardinality $2^{\kappa_{n-1}} \cdot \kappa_{n+1} = \kappa_{n+1}$, it adds at most ${\kappa_{n+1}}^{\kappa_n} = \kappa_{n+1}$ new subsets of κ_n and thus the size of $\mathcal{P}(\kappa_n)$ remains unchanged. This shows that $\Vdash_{\mathbb{P}}$ "$\neg\varphi_n$".

❑ (Proposition 5.1)

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