MAD families over given AD families

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I present in this talk some results from:

S. F., S. Geschke and L. Soukup, How to drive our families MAD ?

and discuss some open problems connected to this paper.

Two countable sets A, B are *almost disjoint* (or A is almost disjoint to B) if $A \cap B$ is finite.

For a given set S, $\mathcal{F} \subseteq [S]^{\aleph_0} = \{X \subseteq S : |X| = \aleph_0\}$ is an *almost disjoint family* (or *ad family*, for short) if any distinct A, $B \in \mathcal{F}$ are almost disjoint. For $\mathcal{F} \subseteq \mathcal{X} \subseteq [S]^{\aleph_0}$, we say that $\mathcal{F}$ is a *maximal almost disjoint family in $\mathcal{X}$* (or *mad in $\mathcal{X}$*, for short) if $\mathcal{F}$ is an ad family and there is no $X \in \mathcal{X}$ almost disjoint from every element of $\mathcal{F}$. S is the *underlying set* of $\mathcal{F}$.

For $\mathcal{X}$, $\mathcal{F} \subseteq [S]^{\aleph_0}$,

$$\mathcal{F}^{\perp} = \{X \in [S]^{\aleph_0} : X \text{ is almost disjoint to every } Y \in \mathcal{F}\}.$$

$$\mathfrak{a}(\mathcal{X}) = \min\{|\mathcal{F}| : |\mathcal{F}| \geq \aleph_0 \text{ and } \mathcal{F} \text{ is mad in } \mathcal{X}\},$$

$$\mathfrak{a}^{+}(\mathcal{F}) = \mathfrak{a}(\mathcal{F}^{\perp}).$$

The usual almost disjoint number $\mathfrak{a}$ is $\mathfrak{a}([S]^{\aleph_0})$ for some/any countable S.

As the underlying set S, we often use the countable set

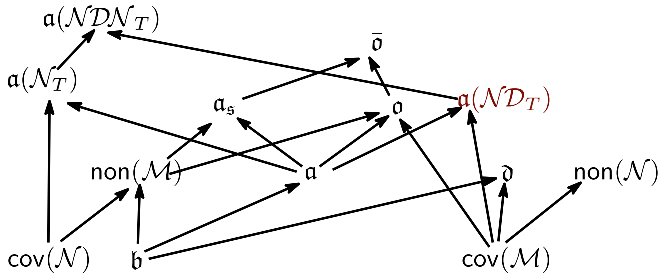
$$T = {}^{\omega>}2 = \{s : s : n \to 2 \text{ for some } n \in \omega\}.$$

For $f \in {}^{\omega}2$, $B(f) = \{f \restriction n : n \in \omega\}$ is the branch in T induced from f. For $X \subseteq T$

$$\lceil X \rceil = \{f \in {}^{\omega}2 : |B(f) \cap X| = \aleph_0\}.$$

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : \lceil X \rceil \text{ is nowhere dense in the Cantor space } {}^{\omega}2\}.$$

Then, we have $\mathsf{cov}(\mathcal{M}), \mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$.



Let $\mathcal{C}_\kappa$ denote the Cohen forcing $\mathrm{Fn}(\kappa, 2)$ for adding κ Cohen reals.

Theorem 1 (CH) *There exists a ad family $\mathcal{F} \subseteq \{X \in [T]^{\aleph_0} : X$ is an antichain in $T\} \subseteq \mathcal{ND}_T$ of size $\aleph_1$ such that for any cardinal κ we have $V^{\mathcal{C}_\kappa} \models \mathcal{F}^\perp \subseteq \mathcal{ND}_T$.*

Theorem 2 *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F}$ in $\mathcal{ND}_T$ of cardinality $\aleph_1$ such that, for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{ND}_T$.*

Theorems 1 and 2 can reformulated in terms of destructibility of madness:

Theorem 3 (1) (CH) *There is an ad family $\mathcal{F}$ which cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family in any generic extension of the ground model of the form $V^{\mathcal{C}_\kappa}$.*

(2) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{ND}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family.*

Analogous "null set" version of the results above are also obtained.

Proof of Theorem 3, (1) from Theorem 1:

The family $\mathcal{F}$ as in Theorem 1 will do. Since we have $\mathcal{F}' \subseteq \mathcal{ND}_T$ for any mad $\mathcal{F}'$ extending $\mathcal{F}$ in $V^{\mathcal{C}_\kappa}$, a further Cohen real over $V^{\mathcal{C}_\kappa}$ introduces a branch almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no more mad in $V^{\mathcal{C}_\kappa * \mathcal{C}_\omega}$.

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Sketch of the Proof of Theorem 1:

Let $\langle\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha\rangle : \alpha < \omega_1 \setminus \omega\rangle$ be an enumeration of

$$(0.1)\qquad \mathcal{S} = \{\langle p, \underset{\sim}{B}, t\rangle : p \in \mathcal{C}_\omega,\ \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of a subset of } T,\ t \in T \text{ and } p \Vdash_{\mathcal{C}_\omega} \text{“} \underset{\sim}{B} \text{ is dense below } t\text{”}.\}$$

By induction on $\alpha < \omega_1$, we construct $A_\alpha \subseteq T$, $\alpha < \omega_1$ such that

(0.2) $A_\alpha \in \mathcal{A}_T$ for all $\alpha < \omega_1$,

(0.3) A_n, $n \in \omega$ is a partition of T,

(0.4) $|\, A_\beta \cap A_\alpha \,| < \aleph_0$ for all $\beta < \alpha < \omega_1$, and

(0.5) if $\alpha \in \omega_1 \setminus \omega$, for each $q \leq_{\mathcal{C}_\omega} p_\alpha$ and $n \in \omega$, there are $r \leq_{\mathcal{C}_\omega} q$ and $t \in A_\alpha$ such that $|\, t \,| \geq n$ and $r \Vdash_{\mathcal{C}_\omega}$ “$t \in \underset{\sim}{B}_\alpha$” (in particular, $p_\alpha \Vdash_{\mathcal{C}_\omega}$ “$|\, A_\alpha \cap \underset{\sim}{B}_\alpha \,| = \aleph_0$”).

$\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ with A_α’s as above is as desired. ❑

For a cardinal κ, let

$$\mathfrak{a}^+(\kappa) = \sup\{\mathfrak{a}^+(\mathcal{F}) : \mathcal{F} \text{ is an ad family on } \omega \text{ of cardinality } \leq \kappa\}.$$

Clearly, $\mathfrak{a}^+(\omega) = \mathfrak{a}$ and $\mathfrak{a}^+(\kappa) \leq \mathfrak{a}^+(\lambda) \leq \mathfrak{c}$ for any $\kappa \leq \lambda \leq \mathfrak{c}$.

$\bar{\mathfrak{o}}$ is the cardinal invariant $\geq \mathfrak{a}$ introduced by T. Leathrum. In our terminology, $\bar{\mathfrak{o}} = \mathfrak{a}(\mathcal{O}_T)$ for $\mathcal{O}_T = \{X \in [T]^{\aleph_0} : X \text{ is an antichain in } T\}$. J. Brendle showed $\mathfrak{a}_{\mathfrak{s}}, \mathsf{non}(\mathcal{M}) \leq \bar{\mathfrak{o}}$.

Theorem 4 (K. Kunen) $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$.

Theorem 5 $V^{\mathcal{C}_\kappa} \models \mathfrak{a}^+(\aleph_1) \geq \kappa$ *for all regular* κ.

Corollary 6 *The inequality* $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ *is consistent.*

Theorem 7 *The inequality* $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ *is consistent.*

Almost disjoint families on large underlying sets
(results from another preprint by S.F., S. Geschke and L. Soukup)

$$\mathbb{A}^{\mathbb{P}}(S) \;\Leftrightarrow\; \text{there exists a } \mathbb{P}\text{-indestructible mad family on } S.$$

Theorem 8 (1) *For any cardinal* κ, $\mathbb{A}^{\mathbb{P}}(\kappa)$ *implies* $\mathbb{A}^{\mathbb{P}}(\kappa^+)$.

(2) *If* $\mathrm{cf}(\kappa) > \omega$ *and* $\mathbb{A}^{\mathbb{P}}(\lambda)$ *for all* $\lambda < \kappa$ *then* $\mathbb{A}^{\mathbb{P}}(\kappa)$.

Theorem 9 *Assume that* $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *and* $\Box^{***}_{\aleph_1,\mu}$ *holds for all limit cardinals* $> \omega$ *of countable cofinality. If* $\mathbb{P}$ *is a proper poset, then* $\mathbb{A}^{\mathbb{P}}(\omega)$ *holds if and only if* $\mathbb{A}^{\mathbb{P}}(\kappa)$ *holds for some/any cardinal* κ.

Theorem 10 *Assume that* $\mathrm{cf}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ *and* $\Box^{***}_{\aleph_1,\mu}$ *holds for all limit cardinals* $> \omega$ *of countable cofinality. Then* $\mathfrak{a}([\kappa]^{\aleph_0}) = \max\{\mathfrak{a}, \kappa\}$ *for any cardinal* κ *of cofinality* $> \omega$ *and* $\mathfrak{a}([\mu]^{\aleph_0}) = \max\{\mathfrak{a}, \mu^+\}$ *for any cardinal* μ *of countable cofinality.*

Some Open Problems

- Can we separate cardinal invariants in the diagram shown on the third slide?
- Is $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$ consistent?
- Are the weak square principles in Theorems 9 and 10 really necessary?

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