A reflection principle formulated in terms of games *

神戸大学大学院・システム情報学研究科 渕野 昌
Sakaé Fuchino
Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

神戸大学・自然科学系先端融合研究環重点研究部 薄葉 季路
Toshimichi Usuba
Organization of Advanced Science and Technology Core Research Team
Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
usuba@people.kobe-u.ac.jp

Abstract

We introduce a principle formulated in terms of the existence of a winning strategy of a game and prove that this principle is placed between the reflection principle down to internally stationary sets ($\mathsf{RP}_{\mathsf{IS}}$) and the reflection principle down to internally club sets ($\mathsf{RP}_{\mathsf{IC}}$). In particular, under CH this principle gives a new characterization of Fleissner's $\mathsf{Axiom\ R}$.

Date: December 31, 2013 *Last update:* December 19, 2016 (03:13 JST)

2010 Mathematical Subject Classification: 03E05, 03E55

Keywords: reflection principles, winning strategy, elementary substructures, Axiom R, internal stationarity, internal clubness

This is an extended version of the paper with the same title which is to appear in the RIMS Kôkyûroku issue for RIMS set theory workshop 2013. All additional details not in the published version of the paper are typeset in typewriter font. Numbering of the assertions is kept identical with the published version.

1 Introduction

intro

For a game $\mathcal{G}$ played by Players I and II, let $\mathsf{WS}_{II}(\mathcal{G})$ denote the assertion "Player II has a winning strategy in $\mathcal{G}$".

In [9], we introduced a game $G^{\downarrow}_{\omega}(\kappa)$ for uncountable cardinals κ (see Section 3 for the definition of this and other games mentioned here) and proved that the Rado Conjecture (RC, see Section 2 for the definition of this principle) implies the assertion

(G_0) $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ holds for all uncountable κ.

Further, it is proved in [9] that (G_0) implies the Fodor-type Reflection Principle (FRP, see Section 2 for the definition of this principle and, [4] and [5] for basic facts of this principle).

In [1], Philipp Doebler introduced a similar game he called $G_{\omega}([\kappa]^{\omega_1}, \omega_1)$ and proved that the Rado Conjecture also implies the principle

(G_1) $\mathsf{WS}_{II}(G_{\omega}([\kappa]^{\aleph_1}, \omega_1))$ holds for all $\kappa \geq \aleph_2$.

He then proved that the principle (G_1) implies the Semistationary Reflection (SSR).

In this paper, we introduce a game $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$ which generalizes both $G^{\downarrow}(\kappa)$ and $G_{\omega}([\kappa]^{\omega_1}, \omega_1)$. Unfortunately the principle

($\mathsf{G}^{\downarrow\downarrow}$) $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ for all $\kappa \geq \aleph_2$

is not a consequence of the Rado Conjecture: In Section 4, we show that the principle ($\mathsf{G}^{\downarrow\downarrow}$) implies the reflection principle $\mathsf{RP}_{\mathsf{IS}}$. It is known that $\mathsf{RP}_{\mathsf{IS}}$ (or even RP) is not a consequence of RC (see Sakai [14]).

2 Reflection Principles

refl-prin

Let us first review the reflection principles we mentioned in the previous section.

We shall call here a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the initial segment $\{u \in T : u \leq_T t\}$ in T below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

A tree T is *special* if there are $T_i \subseteq T$, $i \in \omega$ such that each of T_i's is pairwise incomparable and $T = \bigcup_{i \in \omega} T_i$.

Rado's Conjecture (RC) is the assertion:

(RC): Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

RC is known to be consistent (modulo a large large cardinal). E.g., Todorčević showed that, if κ is strongly compact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$, then we have

$\Vdash_{\mathbb{P}}$ "Rado's Conjecture".

For a cardinal κ and a regular cardinal $\delta < \kappa$, we denote

$$E^{\kappa}_{\delta} = \{\alpha < \kappa \,:\, cf(\alpha) = \delta\}.$$

a mapping $g : E \to \kappa$ for $E \subseteq E^{\kappa}_{\delta}$ is called a ladder system if $\sup g(\alpha) = \alpha$ and $otp(g(\alpha)) = \delta$ hold for all $\alpha \in E$.

For a regular uncountable cardinal κ, we define the Fodor-type Reflection Principle for κ by

$\mathsf{FRP}(\kappa)$: For all stationary $E \subseteq E^{\kappa}_{\omega}$ and for all ladder system $g : E \to [\kappa]^{\aleph_0}$, there exists $\alpha^* \in E^{\kappa}_{\omega_1}$ such that

$$\{x \in [\alpha^*]^{\aleph_0} \,:\, sup(x) \in E,\ g(sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

The Fodor-type Reflection Principle (FRP) is the assertion:

(FRP): $\mathsf{FRP}(\kappa)$ holds for all regular $\kappa > \aleph_1$.

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [3], [4], [5], [6], [7], see also [8]).

(2.1) Any locally countably compact topological space X is metrizable if and only if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable

is one of such assertions equivalent to FRP over ZFC (see [4] and [5]).

FRP implies Shelah's Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [7]). It also implies the total failure of square principles $\square_\kappa$ for all cardinals $\kappa \geq \aleph_1$.

Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \aleph_2$ and $|\,M\,| = \aleph_1$.

M is said to be *internally cofinal* (abbreviation: IU)[1)] if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. M is *internally stationary* (abbreviation: IS) if $[M]^{\aleph_0} \cap M$ is stationary in $[M]^{\aleph_0}$. M is *internally club* (abbreviation: IC) if $[M]^{\aleph_0} \cap M$ contains a closed unbounded set in $[M]^{\aleph_0}$. Finally, M is *internally approachable* (abbreviation: IA) if M is the union of a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ countable sets such that $\langle M_\alpha : \alpha \leq \delta\rangle \in M_{\delta+1}$ for all $\delta < \omega_1$ [2)].

It is clear from the definition that, for any $M \prec \mathcal{H}(\lambda)$, we have the implication: M is $\mathsf{IA} \Rightarrow M$ is $\mathsf{IC} \Rightarrow M$ is $\mathsf{IS} \Rightarrow M$ is IU. It is easy to see that all of these notions can be characterized in terms of filtration (see footnote 2)):

[1)] Internally cofinal M is also called *internally unbounded* in the literature (see e.g. Krueger [11]).

[2)] For a structure M of cardinality $\aleph_1$, we shall call a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable subsets of M with $\bigcup_{\alpha<\omega_1} M_\alpha = M$ a *filtration* of M. By thinning out the index set ω_1, we may assume in some cases that the filtration $\langle M_\alpha : \alpha < \omega_1\rangle$ consists of elementary structures.

Lemma 2.1 *Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \omega_2$ and $|M| = \aleph_1$.*

(1) *M is internally cofinal if and only if there is a filtration $\langle a_\alpha : \alpha < \omega_1 \rangle$ of M such that $a_{\alpha+1} \in M$ for every $\alpha < \omega_1$.*

(2) *M is internally stationary if and only if $\{\alpha < \omega_1 : M_\alpha \in M\}$ is stationary for a/any filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M.*

(3) *M is internally club if and only if there is a filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$.* ❑

These notions can be different: e.g. John Krueger proved under PFA, there are stationarily many internally club but not internally approachable $M \prec \mathcal{H}(\lambda)$ for all regular $\lambda > \aleph_1$ (for this and other results of this line see Krueger [11] and [12]). However this is not the case under CH:

L-CH

Lemma 2.2 *Under* CH*, any $M \prec \mathcal{H}(\lambda)$ is* IU *if and only if it is* IS *if and only if it is* IC *if and only if it is* IA*.*

Proof. Assume CH. It is enough to show that for any $M \prec \mathcal{H}(\lambda)$ of cardinality $\aleph_1$ for some regular $\lambda \geq \aleph_2$, if M is IU, then $[M]^{\aleph_0} \subseteq M$.

Note that, if M is IU, we have $\omega_1 \subseteq M$.

Suppose $x \in [M]^{\aleph_0}$ for an IU M. Then there is $y \in [M]^{\aleph_0} \cap M$ such that $x \subseteq y$. By CH and elementarity there is a bijection $g \in M$ from ω_1 to $\mathcal{P}(y)$. Then there is $\alpha \in \omega_1 \subseteq M$ such that $g(\alpha) = x$. Hence $x \in M$. ❑ (Lemma 2.2)

In the following, we shall always denote one of the properties IU, IS, IC or IA with $\mathcal{P}$. "$\sqsubset$" in connection with a cardinal, say λ, denotes a(n arbitrary) well-ordering of the set $\mathcal{H}(\lambda)$ of all sets of hereditarily of cardinality $< \lambda$. If we have to emphasize that the well-ordering $\sqsubset$ refers to $\mathcal{H}(\lambda)$, we write $\sqsubset_{\mathcal{H}(\lambda)}$.

For a cardinal $\lambda > \aleph_1$ let

$\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ there is a $\mathcal{P}$ elementary substructure M of the structure $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ (of cardinality $\aleph_1$) such that

(2.2) $S \cap [M]^{\aleph_0}$ [3] is stationary in $[M]^{\aleph_0}$. refl-0

We define the global version of the reflection principle $\mathsf{RP}_{\mathcal{P}}$ down to a structure with the property $\mathcal{P}$ to be $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ for all cardinal $\lambda > \aleph_1$.

$\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ is equivalent with seemingly stronger variants of the assertion:

L-P-0

Lemma 2.3 *The following are equivalent for any regular cardinal $\lambda > \aleph_1$:*

(a) $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$.

---

[3] That is, S intersection with the set of all countable subsets of the underlying set of the structure M.

(b) *For any stationary* $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ *and any expansion* $\mathcal{M}$ *of the structure* $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ *in an arbitrary countable language, there is a* $\mathcal{P}$ *elementary substructure* M *of* $\mathcal{M}$ *(of cardinality* $\aleph_1$*) with* (2.2).

(c) *For any stationary* $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ *and any expansion* $\mathcal{M}$ *of the structure* $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ *in an arbitrary countable language, there are stationarily many* $\mathcal{P}$ *elementary substructures* M *of* $\mathcal{M}$ *(of cardinality* $\aleph_1$*) with* (2.2).

Proof. Since (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is trivial, it is enough to show (a) $\Rightarrow$ (c).

Assume (a) and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $\mathcal{M}$ is an expansion of the structure $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ into a countable language. Further, let C be a club subset of $[\mathcal{H}(\lambda)]^{\aleph_0}$. We have to show that there is a $\mathcal{P}$ elementary substructure $M \in C$ of $\mathcal{M}$ with (2.2).

Without loss of generality, we may assume that $\mathcal{M}$ contains the unary relation C. Let

$$(2.1a) \quad S' = \{x \in S \,:\, sk_{\mathcal{M}}(x) = x\}$$ refl-1

where $sk_{\mathcal{M}}(x)$ denotes the Skolem-hull of x with respect to the structure $\mathcal{M}$. S' is then still stationary. So by $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$, there is a $\mathcal{P}$ elementary substructure of $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ such that $S' \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. In particular $S' \cap [M]^{\aleph_0}$ is cofinal in $[M]^{\aleph_0}$. Hence, by (2.1a), M is closed with respect to the Skolem hull operation. It follows that $M \prec \mathcal{M}$. By elementarity, $C \cap M$ is directed. By the property $\mathcal{P}$ of M we have $\omega_1 \subseteq M$. Hence $\bigcup (C \cap M) = M$ and this shows $M \in C$. ❑ (Lemma 2.3)

Using Lemma 2.3 we can prove the following downward transfer property of $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$:

L-P-1

Lemma 2.4 *For regular cardinals* $\aleph_1 < \lambda' < \lambda$*, if* $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ *holds then* $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda')]^{\aleph_0})$ *also holds.*

Proof. Assume that $\aleph_1 < \lambda' < \lambda$ and $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds and suppose that $S \subseteq [\mathcal{H}(\lambda')]^{\aleph_0}$ is stationary. Then

$$(2.2a) \quad \tilde{S} = \{x \in [\mathcal{H}(\lambda)]^{\aleph_0} \,:\, x \cap \mathcal{H}(\lambda') \in S\}$$

is stationary in $[\mathcal{H}(\lambda)]^{\aleph_0}$. For an arbitrary expansion $\mathcal{M}_{\lambda'} = \langle \mathcal{H}(\lambda'), \in, \sqsubset, ... \rangle$ of $\langle \mathcal{H}(\lambda'), \in, \sqsubset \rangle$ into a countable language, Let $\mathcal{M}_\lambda$ be the structure

$$(2.3a) \quad \mathcal{M}_\lambda = \langle \mathcal{H}(\lambda) \in, \sqsubset, \mathcal{H}(\lambda'), ... \rangle$$

where ... in $\mathcal{M}_\lambda$ is the structure corresponding to the structure of $\mathcal{M}_{\lambda'}$. By the assumption of $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ and Lemma 2.3, (b), there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}_\lambda$ of cardinality $\aleph_1$ such that $\tilde{S} \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a filtration of M witnessing the $\mathcal{P}$ness of M. Let $M' = M \cap \mathcal{H}(\lambda')$ be the structure in the language of $\mathcal{M}_{\lambda'}$ where the structure of M corresponding to ''...'' in the structure $\mathcal{M}_\lambda$ are re-interpreted as structure in the language of $\mathcal{M}_{\lambda'}$. Then, by the elementarity of M and since ''$\mathcal{H}(\lambda')$'' is a unary relation in $\mathcal{M}_\lambda$, we have $M' \prec \mathcal{M}_{\lambda'}$ and $\langle M_\alpha \cap M' : \alpha < \omega_1 \rangle$ witnesses the $\mathcal{P}$ness of M'. Also, by definition of $\tilde{S}$, $S \cap [M']^{\aleph_0}$ is stationary in $[M']^{\aleph_0}$. This shows that $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda')]^{\aleph_0})$ holds. ❑ (Lemma 2.4)

L-P-2

Lemma 2.5 *The following are equivalent:* (a) $\mathsf{RP}_{\mathcal{P}}$.

(b) *For any uncountable X, stationary $S \subseteq [X]^{\aleph_0}$, regular θ with $X \subseteq \mathcal{H}(\theta)$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \sqsubset, X \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [X \cap M]^{\aleph_0}$ is stationary in $[X \cap M]^{\aleph_0}$.*

(c) *For any uncountable cardinal λ, stationary $S \subseteq [\lambda]^{\aleph_0}$, regular $\theta \geq \lambda$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \sqsubset, \lambda \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [\lambda \cap M]^{\aleph_0}$ is stationary in $[\lambda \cap M]^{\aleph_0}$.*

Proof. (a) $\Rightarrow$ (b): Assume that (a) holds and suppose that X is uncountable and $S \subseteq [X]^{\aleph_0}$ stationary. Let θ be such that $X \subseteq \mathcal{H}(\theta)$ and let

(2.4a) $\tilde{S} = \{x \in [\mathcal{H}(\theta)]^{\aleph_0} : x \cap X \in S\}$.

Then $\tilde{S}$ is a stationary subset of $[\mathcal{H}(\theta)]^{\aleph_0}$. Let $\mathcal{M}$ be any expansion of $\langle \mathcal{H}(\theta), \in, \sqsubset, X \rangle$ in a countable language. By the assumption of (a) and by Lemma 2.3, (b), there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ of cardinality $\aleph_1$ such that $\tilde{S} \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

Claim 2.5.1 $S \cap [X \cap M]^{\aleph_0}$ *is stationary in* $[X \cap M]^{\aleph_0}$.

$\vdash$ Suppose that $C \subseteq [X \cap M]^{\aleph_0}$ is a club. We have to show that $C \cap (S \cap [X \cap M]^{\aleph_0}) \neq \emptyset$. Let $\tilde{C} = \{x \in [M]^{\aleph_0} : x \cap (X \cap M) \in C\}$. $\tilde{C}$ is then a club $\subseteq [M]^{\aleph_0}$ and hence there is $a \in (\tilde{S} \cap [M]^{\aleph_0} \cap \tilde{C})$. It follows that $a \cap (X \cap M) \in C$ and $a \cap X \in S$. But since $a \in [M]^{\aleph_0}$, we have $a \subseteq M$ and $a \cap X = a \cap (X \cap M) \in C \cap S \cap [X \cap M]^{\aleph_0}$. $\dashv$ (Claim 2.5.1)

(b) $\Rightarrow$ (c): is trivial.

(c) $\Rightarrow$ (a): By Lemma 2.4, it is enough to show that $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds for cofinally many λ's in *Card*. For λ with $|\mathcal{H}(\lambda)| = \lambda$ the implication corresponding to (c) $\Rightarrow$ (a) is easy to prove. ❑ (Lemma 2.5)

Fleissner's $\mathsf{Axiom\ R}$ ([2]) is equivalent to $\mathsf{RP}_{\mathsf{IU}}$ in our notation. For a any set X of cardinality $> \aleph_1$, let

($\mathsf{AR}([X]^{\aleph_0})$): For any stationary $S \subseteq [X]^{\aleph_0}$ and ω_1-club[4] $T \subseteq [X]^{\aleph_1}$, there is $U \in T$ such that $S \cap [U]^{\aleph_0}$ is stationary in $[U]^{\aleph_0}$.

Then we define Axiom R to be the assertion that $\mathsf{AR}([\lambda]^{\aleph_0})$ holds for all cardinal $\alpha > \aleph_1$. Since $\mathsf{AR}([\lambda]^{\aleph_0})$, for cardinals $\lambda > \aleph_1$ also satisfy the downward transfer similar to Lemma 2.4 `(see Lemma A 2.1)`, the following Lemma implies the equivalence of $\mathsf{RP}_{\mathsf{IU}}$ and Axiom R:

L-AxR

Lemma 2.6 *For any $\lambda > \aleph_1$, we have $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ if and only if $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$.*

Proof. Note that $|\,\mathcal{H}(\lambda)\,| = 2^{<\lambda}$ and hence $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ is equivalent to $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$.

First, assume $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$. Suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $T \subseteq [\mathcal{H}(\lambda)]^{\aleph_1}$ is ω_1-club.

Let $\mathcal{M} = \langle \mathcal{H}(\lambda), \in, \sqsubset, T\rangle$. By Lemma 2.5, there is $M \prec \mathcal{M}$ such that

(2.3) $|\,M\,| = \aleph_1$; RP-0

(2.4) $M \models \mathsf{IU}$ and RP-1

(2.5) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. RP-2

By (2.3), (2.4) and $M \prec \mathcal{M}$, it is easy to see that M is the union of an ω_1 chain of elements of T. By ω_1-clubness of T it follows that $M \in T$. This shows that $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

Assume now $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary. Let

$$T = \{M \in [\mathcal{H}(\lambda)]^{\aleph_1} \,:\, M \prec \mathcal{H}(\lambda),\ M \models \mathsf{IU}\}.$$

Then T is ω_1-club. By $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ or by its equivalent $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$, there is $M \in T$ such that $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. This shows that $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds. ❑ (Lemma 2.6)

transf

Lemma A 2.1 *If $\aleph_1 < \lambda < \lambda'$ and $\mathsf{AR}([\lambda']^{\aleph_0})$ holds then $\mathsf{AR}([\lambda]^{\aleph_0})$ holds.*

Proof. `Assume that` $\mathsf{AR}([\lambda']^{\aleph_0})$ `holds. Suppose that` $S \subseteq [\lambda]^{\aleph_0}$ `is stationary and` $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$ `is` ω_1`-club. We have to show that there is` $X \in \mathcal{T}$ `such that` $S \cap [X]^{\aleph_0}$ `is stationary in` $[X]^{\aleph_0}$.

`Let` $S' = \{x' \in [\lambda']^{\aleph_0} \,:\, x' \cap \lambda \in S\}$ `and` $\mathcal{T}' = \{X' \in [\lambda']^{\aleph_1} \,:\, X' \cap \lambda \in \mathcal{T}\}$. `Then` S' `is stationary in` $[\lambda']^{\aleph_0}$ `and` $\mathcal{T}'$ `is` ω_1`-club in` $[\lambda']^{\aleph_1}$. `By` $\mathsf{AR}([\lambda']^{\aleph_0})$, `there is` $X' \in \mathcal{T}'$ `such that` $S' \cap [X']^{\aleph_0}$ `is stationary in` $[X']^{\aleph_0}$. `For` $X = X' \cap \lambda$ `we have` $X \in \mathcal{T}$.

Claim A 2.1.1 *$S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.*

---

[4] $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1*-club* (or "tight and unbounded" in Fleissner's terminology in [2]) if T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and for any increasing chain $\langle U_\alpha \,:\, \alpha < \omega_1\rangle$ in T of length ω_1, we have $\bigcup_{\alpha<\omega_1} U_\alpha \in T$.

⊢ Suppose that $C \subseteq [X]^{\aleph_0}$ is a club. then $C' = \{x' \in [X']^{\aleph_0} : x' \cap \lambda \in C\}$ is club. Hence there is $x' \in S' \cap C'$. It follows that $x' \cap \lambda \in S \cap C$.

⊣ (Claim 2.6.0)

□ (Lemma 2.6)

3 Definition of the games

defs

For a cardinal κ, let

(3.1) $\quad {}^{\kappa\downarrow}\kappa = \{f \in {}^{\kappa}\kappa : f \text{ is regressive}\}$. regr-0

The game $G^{\downarrow}_{\omega}(\kappa)$ for Players I and II is defined as follows: A match in $G^{\downarrow}_{\omega}(\kappa)$ is a sequence of the form:

I	$f_0 \in {}^{\kappa\downarrow}\kappa$		$f_1 \in {}^{\kappa\downarrow}\kappa$		$\cdots$	$f_n \in {}^{\kappa\downarrow}\kappa$		$\cdots$
II		$\delta_0 \in \kappa$		$\delta_1 \in \kappa$	$\cdots$		$\delta_n \in \kappa$	$\cdots$

$(n < \omega)$

Player II wins in a match of $G^{\downarrow}_{\omega}(\kappa)$ as above if

(3.2) $\quad \{\alpha \in E^{\kappa}_{\omega_1} : f_n(\alpha) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\}$ is unbounded.

The game $G^{\downarrow}_{\omega}(\kappa)$ was introduced in [9]. It is used there to prove the implication of FRP from RC by showing that the assertion (G_0) as in Section 1 defined in terms of this game interpolates the implication.

The following game $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ for Players I and II for a cardinal κ was introduced by Doebler in [1]: A match in $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ is a sequence of the form:

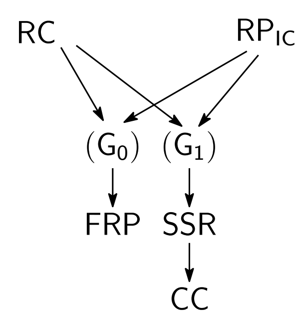
I	$f_0 \in {}^{[\kappa]^{\aleph_1}}\omega_1$		$f_1 \in {}^{[\kappa]^{\aleph_1}}\omega_1$		$\cdots$	$f_n \in {}^{[\kappa]^{\aleph_1}}\omega_1$		$\cdots$
II		$\delta_0 \in \omega_1$		$\delta_1 \in \omega_1$	$\cdots$		$\delta_n \in \omega_1$	$\cdots$

$(n < \omega)$

II wins in a match of $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ as above if

$$\{a \in [\kappa]^{\aleph_1} : f_n(a) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.



Doebler proved that the principle (G_1) as defined in Section 1 in terms of this game follows also from RC and it implies SSR.

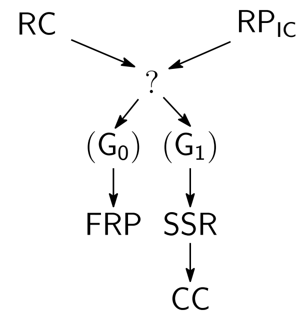
It is easy to see that both of (G_0) and (G_1) are consequences of $\mathsf{RP}_{\mathsf{IC}}$ (this also follows from Corollary 4.4). Hence we have the diagram on the right:

Since FRP and SSR imply almost all known consequences of RC[5], it seems to be an interesting question what is the natural principle which is still a consequence of both RC and $\mathsf{RP}_{\mathsf{IC}}$ while which implies both FRP and SSR.

---

[5] Perhaps with the exception of the negation of Martin's Axiom for $\aleph_1$ dense sets which is a consequence of RC while $\mathsf{RC}_{\mathcal{P}}$'s are consistent with Martin's Axiom since they all follow from $\mathsf{MA}^{+}(\sigma\text{-closed})$.

The assertion of the existence of the winning strategy for player II (the principle $(\mathsf{G}^{\downarrow\downarrow})$ introduced in Section 1) in the following game $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for all $\kappa > \aleph_1$ seemed to be a natural candidate for such an interpolant. Unfortunately, this principle turned out to be too strong to be a consequence of RC while it is still a consequence of $\mathsf{RP}_{\mathsf{IC}}$ as we shall see in Section 4. In [9] we introduce a weakening of $(\mathsf{G}^{\downarrow\downarrow})$ which is an interpolant of RC and $\mathsf{RP}_{\mathsf{IC}}$ on one side and FRP and SSR on the other.



Here is the definition of $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for a cardinal $\kappa > \aleph_1$. We call a function $f : [\kappa]^{\aleph_1} \to \kappa$ regressive if $f(a) \in a$ holds for all $a \in [\kappa]^{\aleph_1}$. Similarly to the definition (3.1), let

$$(3.3) \qquad {}^{[\kappa]^{\aleph_1}\downarrow}\kappa = \{f \in {}^{[\kappa]^{\aleph_1}}\kappa \,:\, f \text{ is regressive}\}.$$ regr-1

A match in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for Players I and II is a sequence of the form:

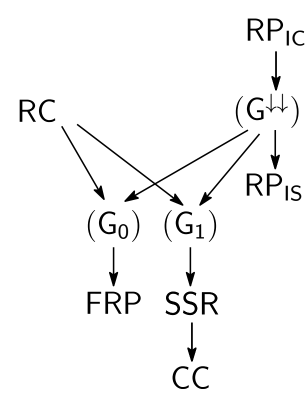
I	$f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa$	$f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa$	$\cdots$	$f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa$	$\cdots$		$(n < \omega)$
II	$d_0 \in [\kappa]^{\aleph_0}$	$d_1 \in [\kappa]^{\aleph_0}$	$\cdots$	$d_n \in [\kappa]^{\aleph_0}$	$\cdots$		

II wins in a match in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ as above if

$$\{a \in [\kappa]^{\aleph_1} \,:\, f_n(a) \in \bigcup\{d_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

Note that by the definition of the games, it is clear that $(\mathsf{G}^{\downarrow\downarrow})$ implies both of (G_0) and (G_1).



4 Characterizations of $(\mathsf{G}^{\downarrow\downarrow})$

char

The following characterization of $(\mathsf{G}^{\downarrow\downarrow})$ can be obtained easily by regarding the moves of Player I in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ as an enumeration of Skolem functions with parameters in some model M and the moves of Player II as the gradual capturing of $\kappa \cap M$:

char-1

Lemma 4.1 *For any cardinal $\kappa > \aleph_1$ the following are equivalent:*

(a) $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$.

(b) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset\rangle$, for any $M \prec \mathcal{M}$ with $|\,M\,| = \aleph_0$ and $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$.*

(c) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset\rangle$, for club many[6] countable $M \prec \mathcal{M}$ with $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$.*

[6] We can also express this "club many ..." in terms of expansion of the structure $\mathcal{M}$ similarly to Lemma 2.3 or Lemma 2.4.

Proof. (a) $\Rightarrow$ (b): Suppose that $M \prec \mathcal{M}$ is countable with $\kappa \in M$ and $a \in [\kappa]^{\aleph_1}$. By the assumption of (a) and elementarity, there is a winning strategy $\sigma \in M$ for Player II in $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$. Let

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots \\ \hline II & & d_0 \in [\kappa]^{\aleph_0} & & d_1 \in [\kappa]^{\aleph_0} & \cdots & & d_n \in [\kappa]^{\aleph_0} & \cdots \end{array}$$

be a match in $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$ such that f_n, $d_n \in M$ for all $n \in \omega$, $\{f_n : n \in \omega\} = {}^{[\kappa]^{\aleph_0}\downarrow}\kappa \cap M$ and Player II has played in the match according to σ.

Then II wins and hence there is $b \in [\kappa]^{\aleph_0}$ such that $a \subseteq b$ and $f_n(b) \subseteq \bigcup\{d_n : n \in \omega\} \subseteq \kappa \cap M$ for all n. $N = sk_{\mathcal{M}}(M \cup \{b\})$ is then as desired.

(b) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (a): Suppose that $\mathcal{C} \subseteq [\mathcal{M}]^{\aleph_0}$ is a club set of M's as in (c). In a match of $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots \\ \hline II & & d_0 \in [\kappa]^{\aleph_0} & & d_1 \in [\kappa]^{\aleph_0} & \cdots & & d_n \in [\kappa]^{\aleph_0} & \cdots \end{array}$$

Player II chooses an increasing sequence M_n, $n \in \omega$ in $\mathcal{C}$ such that $f_n \in M_n$ and let $d_n = \kappa \cap M_n$. We claim that Player II wins in all of such matches and hence this is a winning strategy for Player II.

Let $M = \bigcup_{n\in\omega} M_n$. By the clubness of $\mathcal{C}$, we have $M \in \mathcal{C}$. Hence there are cofinally many $b \in [\kappa]^{\aleph_1}$ such that there is countable $N \prec \mathcal{M}$ with $M \subseteq N$, $b \in N$ and $b \cap N = b \cap M$. For such b, we have $b \cap N = b \cap M \subseteq \kappa \cap M = \bigcup\{d_i : i \in \omega\}$. ❑ (Lemma 4.1)

By Lemma 4.1, (c), we see immediately that $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ for cardinals $\kappa > \aleph_1$ also enjoy the downward transfer property:

char-2

Corollary 4.2 *Suppose $\aleph_1 < \kappa' < \kappa$ and $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ holds. Then we also have $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa']^{\aleph_1}))$.*

Proof. Let θ be a sufficiently large regular cardinal and let $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$. Suppose that $M \prec \mathcal{M}$ is countable with κ, $\kappa' \in M$ (note that there are club many such M's).

If $a' \in [\kappa']^{\aleph_1}$ then $a' \in [\kappa]^{\aleph_1}$. Hence, by Lemma 4.1, (b), there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a' \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$. Let $b' = b \cap \kappa$. Then we have $b' \in [\kappa']^{\aleph_1}$, $a' \subseteq b'$, $b' \in N$, $M \subseteq N$ and $b' \cap N = b' \cap M$. This shows that κ' satisfies Lemma 4.1, (c).

❑ (Corollary 4.2)

Thm-char

Theorem 4.3 *The following are equivalent:* (a) $(\mathsf{G}^{\downarrow\downarrow})$.

(b) *For all $\kappa > \aleph_1$, for all sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$,* *there are* $Y \in [\mathcal{H}(\kappa)]^{\aleph_1}$ *and countable* $N \prec \mathcal{M}$ *such that* $X \subseteq Y$, $Y \in N$, $M \subseteq N$ *and* $Y \cap N = Y \cap M$.

(c) *For all* $\kappa > \aleph_1$, *for all sufficiently large regular* θ *with* $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, *there are club many countable* $M \prec \mathcal{M}$ *such that* $\kappa \in M$ *and for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, *there are* $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ *of cardinality* $\aleph_1$ *and countable* $N \prec \mathcal{M}$ *such that* $X \subseteq Z$, $Z \in N$, $M \subseteq N$ *and* $Z \cap N = Z \cap M$.

(d) *For any* $\kappa > \aleph_1$ *and stationary* $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, *for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ *there is a* $Z \prec \mathcal{H}(\kappa)$ *such that* $X \subseteq Z$ $|Z| = \aleph_1$ *and* $S \cap Z$ *is stationary in* $[Z]^{\aleph_0}$.

(e) *For all* $\kappa > \aleph_1$, *for all sufficiently large regular* θ *with* $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, *there are club many countable* $M \prec \mathcal{M}$ *such that* $\kappa \in M$ *and for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, *there are* IS $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ *of cardinality* $\aleph_1$ *and countable* $N \prec \mathcal{H}(\theta)$ *such that* $X \subseteq Z$, $Z \in N$, $M \subseteq N$ *and* $Z \cap N = Z \cap M$.

Proof. (a) $\Rightarrow$ (b): Let $\lambda = 2^{<\kappa} = |\mathcal{H}(\kappa)|$ and let $\varphi : \lambda \to \mathcal{H}(\kappa)$ be a bijection. Then all countable $M \prec \mathcal{M}$ with $\varphi \in M$ satisfies the condition in (b): the situation of Lemma 4.1, (b) (for κ there $= \lambda$) is translated to the desired condition in the present (b) by φ.

(b) $\Rightarrow$ (a): The back-translation by the mapping φ as in the proof of (a) $\Rightarrow$ (b) implies $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([2^{<\kappa}]^{\aleph_1}))$ for all $\kappa > \aleph_1$. By Corollary 4.2, it follows that $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ for all $\kappa > \aleph_1$.

(b) $\Rightarrow$ (c): Suppose that κ, θ, $\mathcal{M}$, M, X, Y, N are as in (b). Then $Z = sk_{\mathcal{M}}(Y)$ witnesses (c).

(c) $\Rightarrow$ (d): Assume that (c) holds and suppose that $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ is stationary. Let θ $\mathcal{M}$, M be as in (c). Since there are club many M's as in (c), we may assume that

(4.1) $\quad S \in M$ and $\mathcal{H}(\kappa) \cap M \in S$. c-0

Let $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ be defined by

(4.2) $\quad X = \omega_1 \cup (\mathcal{H}(\kappa) \cap M) \cup \{M \cap \mathcal{H}(\kappa)\}$. c-1

Let $Z \prec \mathcal{H}(\kappa)$ and $N \prec \mathcal{H}(\theta)$ be as in (c) for this X. Thus we have N is countable, Z is of cardinality $\aleph_1$, $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and

(4.3) $\quad Z \cap N = Z \cap M$. c-1-0

We are done by showing that $S \cap Z$ is stationary in $[Z]^{\aleph_0}$. Since S, $Z \in N$, it is enough to show that any club $C \subseteq [Z]^{\aleph_0}$ with $C \in N$ intersects with S: Note that we have

(4.4) $\quad Z \cap M = \mathcal{H}(\kappa) \cap M$ c-1-1

by (4.2). For such a club C we have

$$
\begin{aligned}
S \ni M \cap \mathcal{H}(\kappa) & \qquad \text{by (4.1)} \\
= Z \cap M & \qquad \text{by (4.4)} \\
= Z \cap N & \qquad \text{by (4.3)} \\
= \bigcup (C \cap N) \in C. & \qquad \text{by elementarity, } C \in N \text{ and closedness of } C
\end{aligned}
$$

Thus $S \cap C \neq \emptyset$ as desired.

(c) $\Rightarrow$ (e): The proof of (c) $\Rightarrow$ (d) above for $S = [\mathcal{H}(\kappa)]^{\aleph_0}$ shows this.

(e) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (b): trivial.

(d) $\Rightarrow$ (e): Assume that (d) holds. For $\kappa > \aleph_1$, let θ a sufficiently large regular cardinal and $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$. Let

(4.5) $S = \{ M \in [\mathcal{M}]^{\aleph_0} \ : \ M \prec \mathcal{M},\ \kappa \in M,$ there is $X_M \in [\mathcal{H}(\kappa)]^{\aleph_1}$ such that c-2

(4.6) there are no countable $N \prec \mathcal{M}$ and $Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubset_{\mathcal{H}(\kappa)} \rangle$ such that $M \prec N$, $X_M \subseteq Y$, Y is IS and of size $\aleph_1$, $Y \in N$ and $M \cap Y = N \cap Y$ $\}$.

It is enough to show that S is non-stationary. In the following we show this indirectly: We assume that S is stationary and drive a contradiction from this assumption.

For each $M \in S$ we choose $X_M \in [\mathcal{H}]^{\aleph_1}$ such that

(4.7) $X_M \supseteq M \cup \omega_1$, $X_M \prec \mathcal{M}$ and (4.6) holds for M and X_M. c-4

Let $\chi > 2^{<\theta}$ be regular. Note that we have $\mathcal{H}(\theta) \in \mathcal{H}(\chi)$. Let

(4.8) $\tilde{S} = \{ M \in [\mathcal{H}(\chi)]^{\aleph_0} \ : \ M \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle,$ $\kappa, \theta, \cdots \in M,\ M \cap \mathcal{H}(\theta) \in S \ \}.$ c-5

By the assumption of the stationarity of S, $\tilde{S}$ is also stationary. Thus, by (d), there is $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubset \rangle$ such that

(4.9) $|\, Z \,| = \aleph_1$, $\omega_1 \subseteq Z$, z-1

(4.10) κ, θ, S, $\langle X_M \ : \ M \in S \rangle$, $\sqsubset_{\mathcal{H}(\kappa)}$, $\sqsubset_{\mathcal{H}(\theta)}, \cdots \in Z$ and z-2

(4.11) $\tilde{S} \cap Z$ is stationary in $[Z]^{\aleph_0}$. z-3

Let

(4.12) $Y = Z \cap \mathcal{H}(\kappa)$. c-5-0

Then we have $\omega_1 \subseteq Y$ and hence $|\, Y \,| = \aleph_1$ and $Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$.

Y is stationary in $[Y]^{\aleph_0}$: Suppose that $C \subseteq [Y]^{\aleph_0}$ is a club. Let $\tilde{C} = \{ x \in [Z]^{\aleph_0} \ : \ x \cap Y \in C \}$. Then $\tilde{C}$ is a club subset of $[Z]^{\aleph_0}$. By (4.11), $Z \cap [Z]^{\aleph_0}$ is stationary in $[Z]^{\aleph_0}$. Hence there is an $x \in \tilde{C} \cap Z$. By definition of $\tilde{C}$, we have $x \cap \mathcal{H}(\kappa) \in C$.

Since $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle$ and $\mathcal{H}(\kappa) \in Z$ by (4.10), we have $x \cap \mathcal{H}(\kappa) \in Z$. By $x \cap \mathcal{H}(\kappa) \in C$ we have $x \cap \mathcal{H}(\kappa) \in \mathcal{H}(\kappa)$. It follows that $x \cap \mathcal{H}(\kappa) \in Y$. Thus $x \cap \mathcal{H}(\kappa) \in C \cap Y$.

For each $M \in \tilde{S} \cap Z$ we have $M \cap \mathcal{H}(\theta) \in S \cap Z$ as we just saw and, by (4.10), $X_{M \cap \mathcal{H}(\theta)} \in Z \cap \mathcal{H}(\kappa) = Y$. Since $\omega_1 \subseteq Y$, it follows that

$$X_{M \cap \mathcal{H}(\theta)} \subseteq Y. \tag{4.13}$$ c-6

By (4.11), there is countable $N^* \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle$ such that

$$N^* \cap Z \in \tilde{S} \cap Z \text{ and} \tag{4.14}$$ c-7

$$X, Y, Z, \dots \in N^*. \tag{4.15}$$ c-8

Let $M^* = (N^* \cap Z) \cap \mathcal{H}(\theta)$. Then we have $M^* \in S$ by (4.14). $M^* \subseteq N^* \cap \mathcal{H}(\theta)$ by the definition of M^* and $X_{M^*} \subseteq Y$ by (4.13). $Y \in N^* \cap \mathcal{H}(\theta)$ by (4.12) and (4.15). We also have

$$M^* \cap Y = ((N^* \cap Z) \cap \mathcal{H}(\theta)) \cap (Z \cap \mathcal{H}(\kappa)) = M^* \cap \mathcal{H}(\kappa) = (N^* \cap \mathcal{H}(\theta)) \cap Y. \tag{4.16}$$

Thus $N^* \cap \mathcal{H}(\theta)$ and Y contradict to the choice of X_{M^*}. ❑ (Theorem 4.3)

cor-1

Corollary 4.4 *The following implications hold:*

$$\mathsf{RP}_{\mathsf{IC}} \ \Rightarrow\ (\mathtt{G}^{\downarrow\downarrow})\ \Rightarrow\ \mathsf{RP}_{\mathsf{IS}}.$$ ❑

Proof. By Theorem 4.3, (d). The implication "$\mathsf{RP}_{\mathsf{IC}} \Rightarrow (\mathtt{G}^{\downarrow\downarrow})$" follows from the following trivial observation. ❑ (Corollary 4.4)

Lemma 4.5 *If $M \prec \mathcal{H}(\theta)$ is* IC *and $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, then $S \cap (M \cap [M]^{\aleph_0})$ is stationary in $[M]^{\aleph_0}$ as well.* ❑

Corollary 4.6 *Under the* CH*, we have:*

$$\text{Axiom R} \ \Leftrightarrow\ \mathsf{RP}_{\mathsf{IU}} \ \Leftrightarrow\ \mathsf{RP}_{\mathsf{IS}} \ \Leftrightarrow\ (\mathtt{G}^{\downarrow\downarrow}) \ \Leftrightarrow\ \mathsf{RP}_{\mathsf{IC}} \ \Leftrightarrow\ \mathsf{RP}_{\mathsf{IA}}.$$

Proof. By Lemma 2.2, Lemma 2.6 and Corollary 4.4. ❑ (Corollary 4.6)

References

[1] Philipp Doebler, Rado's Conjecture implies that all stationary set preserving forcing are semiproper, Journal of Mathematical Logic Vol.13, 1 (2013).

[2] W. Fleissner, Left-separated spaces with point-countable bases, Transactions of American Mathematical Society, 294, No.2, (1986), 665–677.

[3] Sakaé Fuchino, Left-separated topological spaces under Fodor-type Reflection Principle, (RIMS Kôkyûroku) No.1619, (2008), 32–42.

[4] Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltán Szentmiklóssy and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications Vol.157, 8 (2010), 1415–1429.

[5] Sakaé Fuchino, Lajos Soukup, Hiroshi Sakai and Toshimichi Usuba, More about Fodor-type Reflection Principle, submitted.

[6] Sakaé Fuchino, Fodor-type Reflection Principle and Balogh's reflection theorems, RIMS Kôkyûroku No.1686, (2010), 41–58.

[7] Sakaé Fuchino and Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261-283.

[8] Sakaé Fuchino, Topological Reflection Theorems, RIMS Kôkyûroku No.1833, (2013), 5–26.

[9] Sakaé Fuchino, Hiroshi Sakai, Victor Torres-Perez and Toshimichi Usuba, Rado's Conjecture and the Fodor-type Reflection Principle, in preparation.

[10] T. Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[11] John Krueger, Internally club and approachable, Advances in Mathematics, Vol.213 (2007), 734–740.

[12] John Krueger, Internally approachability and reflection, Journal of Mathematical Logic Vol.8, No.1 (2008), 23–29.

[13] Richard Rado, Theorems on intervals of ordered sets, Discrete Mathematics 35 (1981), 199–201.

[14] Hiroshi Sakai, Semistationary and stationary reflection. Journal of Symbolic Logic, 73 (1), (2008), 181–192.

[15] Stevo Todorčević, Stationary sets, trees and continua, Publ. Inst. Math. Beograd 43 (1981), 249–262.

[16] Stevo Todorčević, On a conjecture of Rado, Journal London Mathematical Society Vol.s2-27, (1) (1983), 1–8.

[17] Stevo Todorčević, Real functions on the family of all well-ordered subsets of a partially ordered set, the Joiurnal of Symbolic Logic, Vol.48, No.1 (1983), 91–96.

[18] Stevo Todorčević, Conjectures of Rado and Chang and Cardinal Arithmetic, in: Finite and Infinite Sets in Combinatorics (N. W. Sauer et al., eds), Kluwer Acad. Publ. (1993) 385-398.

[19] Stevo Todorcevic, Combinatorial dichotomies in set theory, the Bulletin of Symbolic Logic, Vol.17, No.1 (2011), 1–72.