Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum

Sakaé Fuchino (渕野 昌), André Ottenbreit Maschio Rodrigues and Hiroshi Sakai (酒井 拓史)*

Contents

Abstract

Continuing [6], we study the Strong Downward Löwenheim-Skolem Theorems (SDLSs) of the stationary logic and their variations.

---

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp, andreomr@gmail.com, hsakai@people.kobe-u.ac.jp

Date: February 9, 2018 *Last update:* September 17, 2019 (23:03 JST)

2010 Mathematical Subject Classification: 03E35, 03E55, 03E65, 03E75, 05C63

Keywords: Strong Downward Löwenheim Skolem Theorem, stationary logic, generically large cardinals, mixed support iteration, Laver function, superhuge cardinal, continuum problem

The first author would like to thank Joel Hamkins for several valuable comments and suggestions.

This is an extended version of the paper with the same title.
All additional details not to be contained in the submitted version of the paper are either typeset in typewriter font (the font this paragraph is typeset) or put in separate appendices. The numbering of the assertions is kept identical with the submitted version.

In [6] it has been shown that the SDLS for the ordinary stationary logic with weak second-order parameters $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ down to $<\aleph_2$ is equivalent to the conjunction of CH and Cox's Diagonal Reflection Principle for internally clubness.

We show that the SDLS for the stationary logic without weak second-order parameters $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ down to $<2^{\aleph_0}$ implies that the size of the continuum is $\aleph_2$. In contrast, an internal interpretation of the stationary logic can satisfy the SDLS down to $<2^{\aleph_0}$ under the continuum being of size $>\aleph_2$. This SDLS is shown to be equivalent to an internal version of the Diagonal Reflection Principle down to an internally stationary set of size $<2^{\aleph_0}$.

We also consider a $\mathcal{P}_\kappa(\lambda)$ version of the stationary logic and show that the SDLS for this logic in internal interpretation $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <2^{\aleph_0})$ for reflection down to $<2^{\aleph_0}$ is consistent under the assumption of the consistency of ZFC + "the existence of a supercompact cardinal" and this SDLS implies that the continuum is (at least) weakly Mahlo.

These three "axioms" in terms of SDLS are consequences of three instances of a strengthening of generic supercompactness which we call Laver-generic supercompactness. Existence of a Laver-generic supercompact cardinal in each of these three instances also fixes the cardinality of the continuum to be $\aleph_1$ or $\aleph_2$ or very large respectively. We also show that the existence of one of these generic large cardinals implies the "++" version of the corresponding forcing axiom.

1 Introduction

Einf

We use the notation and conventions set up in [6]: We assume that all structures and languages have at most countable signature. $\mathcal{L}^{\aleph_0}$ is the weak second order logic extending the usual first-order logic with monadic second-order variables with the interpretation that they run over countable subsets of the underlying set of the structure and also with the built-in binary predicate symbol ε with which we can build a new type of atomic formulas of the form $x \mathrel{\varepsilon} X$ where x is a first order and X a weak second-order variables. The interpretation of the atomic formula $x \mathrel{\varepsilon} X$ for a structure $\mathfrak{A} = \langle A, ...\rangle$ with $a \in A$ and $U \in [A]^{\aleph_0}$ is, as expected,

(1.1) $$\mathfrak{A} \models x \mathrel{\varepsilon} X\,(a, U) \;\Leftrightarrow a \in U.$$ Einf-0

$\mathcal{L}^{\aleph_0,II}$ is then the logic obtained from $\mathcal{L}^{\aleph_0}$ by adding the weak second order existential quantifier $\exists X$ (and its dual $\forall X$) where for a formula $\varphi(x_0, ..., X_0, ..., X)$ in $\mathcal{L}^{\aleph_0,II}$ and a structure $\mathfrak{A} = \langle A, ...\rangle$ with $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0}$,

(1.2) $\mathfrak{A} \models \exists X\, \varphi(a_0, ..., U_0, ..., X)$ Einf-1
$\Leftrightarrow$ there is $U \in [A]^{\aleph_0}$ such that $\mathfrak{A} \models \varphi(a_0, ..., U_0, ..., U)$.

$\mathcal{L}^{\aleph_0}_{stat}$ is the logic obtained form $\mathcal{L}^{\aleph_0}$ by adding the new weak second-order quantifier $stat\, X$ (and its dual $aa\, X$) where for a formula $\varphi(x_0, ..., X_0, ..., X)$ in $\mathcal{L}^{\aleph_0}_{stat}$ and a structure $\mathfrak{A} = \langle A, ...\rangle$ with $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0}$,

(1.3) $\mathfrak{A} \models stat\, X\, \varphi(a_0, ..., U_0, ..., X)$ Einf-2
$\Leftrightarrow$ $\{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(a_0, ..., U_0, ..., U)\}$ is stationary in $[A]^{\aleph_0}$.

Finally $\mathcal{L}^{\aleph_0,II}_{stat}$ is the logic obtained from $\mathcal{L}^{\aleph_0}$ by adding both types of the weak second-order quantifiers.

For one of the logics $\mathcal{L}$ introduced above, and structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} \subseteq \mathfrak{A}$. We say $\mathfrak{B}$ is $\mathcal{L}$-elementary submodel of $\mathfrak{A}$ (notation: $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$) if, for any formula $\varphi(x_0, ..., X_0, ...)$ in $\mathcal{L}$ of the signature where $x_0, ...$ are first order and $X_0, ...$ weak second-order variables, for any $b_0, ... \in |\mathfrak{B}|$ and for any countable subsets $B_0, ...$ of $|\mathfrak{B}|$, we have

(1.4) $\mathfrak{B} \models \varphi(b_0, ..., B_0, ...)$ holds if and only if $\mathfrak{A} \models \varphi(b_0, ..., B_0, ...)$. Einf-3

$\mathfrak{B}$ is a weakly $\mathcal{L}$-elementary submodel of $\mathfrak{A}$ (notation: $\mathfrak{B} \prec^-_{\mathcal{L}} \mathfrak{A}$), if

(1.5) $\mathfrak{B} \models \varphi(b_0, ..., b_{n-1})$ holds if and only if $\mathfrak{A} \models \varphi(b_0, ..., b_{n-1})$ holds Einf-4

for all formulas $\varphi = \varphi(x_0, ...)$ in $\mathcal{L}$ without free weak second-order variables, and for all $b_0, ..., b_{n-1} \in |B|$.

The Strong Downward Löwenheim-Skolem Theorem [1)] (abbreviated in the following as SDLS) for (elementary substructures with respect to the formulas of a language) $\mathcal{L}$ down to $< \kappa$ is the assertion defined by

$\mathsf{SDLS}(\mathcal{L}, <\kappa)$: *For any structure* $\mathfrak{A}$ *of countable signature there is* $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ *of cardinality* $< \kappa$.

We also consider the SDLS with respect to the weak $\mathcal{L}$-elementary submodel relation:

$\mathsf{SDLS}^-(\mathcal{L}, <\kappa)$: *For any structure* $\mathfrak{A}$ *of countable signature, there is* $\mathfrak{B} \prec^-_{\mathcal{L}} \mathfrak{A}$ *of cardinality* $< \kappa$.

---

1) The adjective "strong" is added to indicate that $\mathfrak{B}$ in the statement of the property is not merely elementarily equivalent to but also elementary submodel of $\mathfrak{A}$.

We shall call the cardinal κ as above the *reflection cardinal* or *Löwenheim-Skolem cardinal* of the respective SDLS.

The SDLSs formulated as above have the following natural strengthenings:

$\mathsf{SDLS}_+(\mathcal{L},<\kappa)$: *For any structure* $\mathfrak{A}=\langle A,...\rangle$ *of countable signature with* $|A|\geq\kappa$*, there are stationarily many* $M\in[A]^{<\kappa}$ *such that* $\mathfrak{A}\restriction M\prec_{\mathcal{L}}\mathfrak{A}$.

$\mathsf{SDLS}^-_+(\mathcal{L},<\kappa)$: *For any structure* $\mathfrak{A}=\langle A,...\rangle$ *of countable signature with* $|A|\geq\kappa$*, there are stationarily many* $M\in[A]^{<\kappa}$ *such that* $\mathfrak{A}\restriction M\prec^-_{\mathcal{L}}\mathfrak{A}$.

The SDLS theorems for the logics introduced above can be characterized by some (combinations of) known principles:

Theorem 1.1 (Theorem 1.1, (1), Proposition 2.2, Lemma 3.5, (1), Corollary 3.6 in [6]) *Suppose that* κ *is a regular cardinal* $\geq\aleph_2$. P-Einf-0

(1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0},<\kappa)$ *is a theorem in* ZFC.

(2) *Each of* $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0},<\kappa)$, $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0,II},<\kappa)$ *and* $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II},<\kappa)$ *is equivalent to* $\mu^{\aleph_0}<\kappa$ *for all* $\mu<\kappa$.

(3) $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat},<\kappa)$ *is equivalent to* $\mathsf{DRP}(<\kappa,\mathsf{IC}_{\aleph_0})$.

(4) *Each of* $2^{\aleph_0}<\kappa+\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0}_{stat},<\kappa)$, $\mathsf{SDLS}^-_+(\mathcal{L}^{\aleph_0,II}_{stat},<\kappa)$, $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}_{stat},<\kappa)$, $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat},<\kappa)$ *and* $2^{\aleph_0}<\kappa+\mathsf{DRP}(<\kappa,\mathsf{IC}_{\aleph_0})$ *is equivalent to* $\mu^{\aleph_0}<\kappa$ *for all* $\mu<\kappa+\mathsf{DRP}(<\kappa,\mathsf{IC}_{\aleph_0})$. ❑

Note that the parameter "$<\kappa$" in the SDLS statements for the logics $\mathcal{L}$ as above (except for $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0},<\kappa)$) is impossible for $\kappa\leq\aleph_1$. In case of $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat},\kappa)$, this can be seen in the fact that the first order quantifier $Qx\,\varphi$ which states "there are uncountably many x with φ" is expressible with the stationarity quantifier as $stat\,X\,\exists x(x\not\in X\wedge\varphi)$.

For $\kappa=\aleph_2$, the $+$-version of the strong downward Löwenheim-Skolem statements are equivalent with the corresponding statements without $+$:

Lemma 1.2 (Lemma 2.1 in [6]) *Suppose that* $\mathcal{L}$ *is one of the four logics as above. Then* P-Einf-1

(1) $\mathsf{SDLS}_+(\mathcal{L},<\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L},<\aleph_2)$ *are equivalent and*

(2) $\mathsf{SDLS}^-_+(\mathcal{L},<\aleph_2)$ *and* $\mathsf{SDLS}^-(\mathcal{L},<\aleph_2)$ *are equivalent.* ❑

The following is immediate from Theorem 1.1 and Lemma 1.2.

Corollary 1.3 (Theorem 1.1 in [6]) (1) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0},<\aleph_2)$ *is a theorem in* ZFC. P-Einf-2

(2) *Each of* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}, <\aleph_2)$ *is equivalent to* CH.

(3) $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *is equivalent to* $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$.

(4) *Each of* CH + $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ *and* $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$ *is equivalent to* CH + $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$. ❑

$\mathsf{DRP}(<\aleph_2, \mathsf{IC}_{\aleph_0})$ is the Diagonal Reflection Principle for internal clubness introduced in Cox [4] and $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$ is a generalization of it introduced in [6]. We will not repeat the definition of $\mathsf{DRP}(<\kappa, \mathsf{IC}_{\aleph_0})$. Instead, we cite the following combinatorial characterization of this principle given in [6]:

For a cardinal μ, let

(1.6) $\mathsf{IU}_\mu = \{X : [X]^\mu \cap X \text{ is cofinal in } [X]^\mu\}$; Einf-5

(1.7) $\mathsf{IS}_\mu = \{X : [X]^\mu \cap X \text{ is stationary } [X]^\mu\}$; Einf-6

(1.8) $\mathsf{IC}_\mu = \{X : [X]^\mu \cap X \text{ contains a subset which is club in } [X]^\mu\}$. Einf-7

Elements of IU_μ, IS_μ, IC_μ are said to be *internally unbounded*, *internally stationary* and *internally club* (with respect to subsets of cardinality μ) respectively.

Let $\mathcal{C}$ be one of $\mathsf{IU}_{\aleph_0}$, $\mathsf{IS}_{\aleph_0}$, $\mathsf{IC}_{\aleph_0}$. For $\mathcal{C}$ and regular cardinals κ, λ with $\kappa \le \lambda$,

$(*)^{\mathcal{C}}_{<\kappa,\lambda}$: *For any countable expansion* $\tilde{\mathfrak{A}}$ *of* $\langle \mathcal{H}(\lambda), \in \rangle$ *and sequence* $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ *such that* S_a *is a stationary subset of* $[\mathcal{H}(\lambda)]^{\aleph_0}$ *for all* $a \in \mathcal{H}(\lambda)$, *there is an* $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ *such that*

(1) $M \in \mathcal{C}$;

(2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ *and*

(3) $S_a \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$ *for all* $a \in M$.

Similarly to $\mathsf{SDLS}_+(\cdots)$ and $\mathsf{SDLS}^-_+(\cdots)$ we can also define the following strengthening of the principle $(*)^{\mathcal{C}}_{<\kappa,\lambda}$ above as

$(*)^{+\mathcal{C}}_{<\kappa,\lambda}$: *For any countable expansion* $\tilde{\mathfrak{A}}$ *of* $\langle \mathcal{H}(\lambda), \in \rangle$ *and sequence* $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ *such that* S_a *is a stationary subset of* $[\mathcal{H}(\lambda)]^{\aleph_0}$ *for all* $a \in \mathcal{H}(\lambda)$, *there are stationarily many* $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ *such that*

(1) $M \in \mathcal{C}$;

(2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ *and*

(3) $S_a \cap [M]^{\aleph_0}$ *is stationary in* $[M]^{\aleph_0}$ *for all* $a \in M$. (3) corrected.

We also have the equivalence of this strengthening with the original $(*)^{\mathcal{C}}_{<\kappa,\lambda}$ in case $\kappa = \aleph_2$ (Lemma 3.2 in [6]). The generalizations of Cox's Diagonal Reflection Principle is characterized as the global version of $(*)^{+\mathcal{C}}_{<\kappa,\lambda}$:

Theorem 1.4 (Lemma 3.4 in [6]) *Suppose that* $\mathcal{C}$ *is one of* IU_{ω_0}, IS_{ω_0}, IC_{ω_0}. *For a regular cardinal* $\kappa > \aleph_1$, $\mathsf{DRP}(<\kappa, \mathcal{C})$ *holds if and only if* P-Einf-3

(1.9) $(*)^{+\mathcal{C}}_{<\kappa,\lambda}$ *holds for all cardinal* $\lambda \geq \kappa$. ❑ Einf-8

Below, we shall review some basic facts about the forcing constructions which are used in later sections.

For a poset $\mathbb{P}$ we denote with $\mathrm{ro}(\mathbb{P})$ the complete Boolean algebra $\mathbb{A}$ such that the separative quotient of $\mathbb{P}$ can be densely embedded into $\mathbb{A}^+$. For posets $\mathbb{P}$ and $\mathbb{Q}$ we write $\mathbb{P} \sim \mathbb{Q}$ if $\mathrm{ro}(\mathbb{P}) \cong \mathrm{ro}(\mathbb{Q})$. For a cardinal κ and a set X, $\mathrm{Col}(\kappa, S)$ denotes the poset as is defined in Kanamori [9].

For posets $\mathbb{P}$ and $\mathbb{Q}$, we write $\mathbb{P} \lessdot \mathbb{Q}$ if $\mathrm{ro}(\mathbb{P})$ can be completely embeddable into $\mathrm{ro}(\mathbb{Q})$.

The following Theorem 1.5 is a generalization of Proposition 10.20 in Kanamori [9]. It can be proved similarly to the Proposition.

Th-col-0

Theorem 1.5 *Suppose that* κ *is regular and* $\kappa < \lambda$. *If* $\mathbb{P}$ *is a separative poset such that* $|\,\mathbb{P}\,| = \lambda$, $\mathbb{P}$ *is* κ*-closed and*

(1.10) $\Vdash_{\mathbb{P}}$ "*there is a surjection* $\kappa \to \lambda$", col-1

then $\mathrm{ro}(\mathbb{P}) \cong \mathrm{ro}(\mathrm{Col}(\kappa, \{\lambda\}))$.

Proof. Note that (1.10) implies that $\mathbb{P}$ is atomless. Without loss of generality, we assume that $\mathbb{P}$ is a dense subordering of $\mathrm{ro}(\mathbb{P})^+$.

cl-col-0

Claim 1.5.1 *For any* $r \in \mathrm{ro}(\mathbb{P})^+$ *there is a pairwise incompatible* $D \subseteq \mathbb{P} \downarrow r$ *of cardinality* λ.

$\vdash$ By the assumption (1.10), $\mathrm{ro}(\mathbb{P})^+ \downarrow r$ does not have the λ-cc. Hence there is a pairwise disjoint $D' \subseteq \mathrm{ro}(\mathbb{P})^+ \downarrow r$ of cardinality λ. Since $\mathbb{P} \downarrow r$ is dense in $\mathrm{ro}(\mathbb{P})^+ \downarrow r$, D' has a refinement $D \subseteq \mathbb{P} \downarrow r$ of cardinality $\geq \lambda$. On the other hand, we have $|\,D\,| \leq \lambda$ since $|\,\mathbb{P}\,| = \lambda$. $\dashv$ (Claim 1.5.1)

Let $\underset{\sim}{g}$ be a $\mathbb{P}$-name such that

($\aleph$1.1) $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{g} : \kappa \to \underset{\sim}{\mathbb{G}}$ is a bijection" col-1-0

where $\underset{\sim}{\mathbb{G}}$ is the standard $\mathbb{P}$-name of a $(\mathsf{V}, \mathbb{P})$-generic set. Note that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\kappa \leq |\,\mathbb{G}\,|$ by κ-closedness of $\mathbb{P}$ and $|\,\mathbb{G}\,| \leq |\,\lambda\,| = \kappa$ by $|\,\mathbb{P}\,| = \lambda$ in V. Hence we have $|\,\mathbb{G}\,| = \kappa$ in $\mathsf{V}[\mathcal{G}]$.

Let

($\aleph$1.2) $D = \{p \in \mathrm{Col}(\kappa, \{\lambda\}) \,:\, dom(p) = \{\lambda\} \times \alpha$ for some $\alpha < \kappa\}$.

Clearly D is a dense subset of $\mathbb{P}$. It is enough to show that there is an order and incompatibility preserving $e : D \to \mathrm{ro}(\mathbb{P})^+$ such that $e''D$ is dense in $\mathrm{ro}(\mathbb{P})^+$.

We define such e by induction on $\ell(p) < \kappa$ for $p \in D$ where $\ell(p)$ is defined to be the ordinal with $dom(p) = \{\lambda\} \times \ell(p)$.

Let $e(\emptyset) = \mathbb{1}_{\mathbb{P}}$.

Having defined $e(p)$ for $p \in D$ with $\ell(p) = \alpha$, let $\{a^p_\xi : \xi < \lambda\}$ be a maximal antichain in $\mathbb{P} \downarrow e(p)$ such that, for each $\xi < \lambda$, $a^p_\xi \Vdash_{\mathbb{P}}$ "$\underset{\sim}{g}(\alpha) = \check{r}$" for some $r \in \mathbb{P}$. Note that the construction of such a^p_ξ's is possible by Claim 1.5.1. Set $e(p \cup \{\langle\langle\lambda, \alpha\rangle, \xi\rangle\}) = a^p_\xi$. For a limit $\delta < \kappa$ and $p \in D$ with $\ell(p) = \delta$, suppose that $e(p \restriction \beta)$, $\beta < \delta$ has been defined such that $\langle e(p \restriction \beta) : \beta < \delta\rangle$ is a decreasing sequence. Let $e(p) = \prod_{\beta<\delta} e(p \restriction \beta)$. Note that $e(p) > \mathbb{0}_{\mathrm{ro}(\mathbb{P})}$ by κ-closedness of $\mathbb{P}$ and since the elements of $\mathbb{P}$ are cofinal in the sequence $\langle e(p \restriction \beta) : \beta < \delta\rangle$.

By the induction on $\alpha < \kappa$ we can show that $A_\alpha = \{e(p) : p \in D, \ell(p) = \alpha{+}1\}$ is a maximal antichain. It is also clear that e is order and incompatibility preserving. Thus the following claim implies the forcing equivalence of $\mathbb{P}$ and $\mathrm{Col}(\kappa, \{\lambda\})$.

Claim 1.5.2 *$e''D$ is dense in* $\mathrm{ro}(\mathbb{P})^+$.

$\vdash$ For an arbitrary $r \in \mathbb{P}$ we have $r \Vdash_{\mathbb{P}}$ "$\check{r} \in \underset{\sim}{\mathcal{G}}$". Hence there is $r' \leq_{\mathbb{P}} r$ and $\alpha < \lambda$ such that $r' \Vdash_{\mathbb{P}}$ "$\underset{\sim}{g}(\check{\alpha}) = \check{r}$". Since A_α is a maximal antichain in $\mathbb{P}$, there is a $p \in D$ with $\ell(p) = \alpha{+}1$ such that $e(p) = a^{p\restriction\alpha}_{p(\alpha)}$ is compatible with r'. Since $e(p) = a^{p\restriction\alpha}_{p(\alpha)}$ decides $\underset{\sim}{g}(\check{\alpha})$, we have $e(p) \Vdash_{\mathbb{P}}$ "$\underset{\sim}{g}(\check{\alpha}) = \check{r}$". Thus $e(p) \Vdash_{\mathbb{P}}$ "$\check{r} \in \underset{\sim}{\mathcal{G}}$" by ($\aleph$1.1). Since $\mathbb{P}$ is separative it follows that $e(p) \leq_{\mathbb{P}} r$. $\dashv$ (Claim 1.5.2)

$\square$ (Theorem 1.5)

Cor-col-0

Corollary 1.6 (1) *Suppose that* $\lambda^{<\kappa} = \lambda$. *Then we have*

$$\mathrm{Col}(\kappa, \{\lambda\}) \sim \mathrm{Col}(\kappa, S) \sim \mathrm{Fn}(\kappa, \lambda, <\kappa) \tag{1.11}$$ col-2

for all $S \subseteq \lambda^+$ *with* $\lambda + 1 \leq \sup S < \lambda^+$.

(2) *For any* κ*-closed* $\mathbb{P}$ *with* $|\,\mathbb{P}\,| \leq \mu = \mu^{<\kappa} < \lambda$ *we have* $\mathrm{Col}(\kappa, \lambda) \sim \mathbb{P} \times \mathrm{Col}(\kappa, \lambda) \sim \mathbb{P} * \mathrm{Col}(\kappa, \lambda)^{\mathsf{V}^{\mathbb{P}}}$. *In particular, we have* $\mathbb{P} \lessdot \mathrm{Col}(\kappa, \lambda)$.

Proof. (1): Let $\mathbb{P} = \mathrm{Col}(\kappa, S)$ for S as above. By the assumption we have $|\,\mathbb{P}\,| = \lambda$, $\mathbb{P}$ is separative and κ-closed. Also $\mathbb{P}$ adds a surjection from κ to λ. Hence Theorem 1.5 implies $\mathrm{Col}(\kappa, \{\lambda\}) \sim \mathrm{Col}(\kappa, S)$. $\mathrm{Col}(\kappa, \{\lambda\}) \sim \mathrm{Fn}(\kappa, \lambda, <\kappa)$ also can be shown similarly.

(2): Let $\mu < \lambda$ be such that $|\,\mathbb{P}\,| \leq \mu$ and $\mu^{<\kappa} = \mu$. Then

$$
\begin{aligned}
(1.12)\qquad & \mathbb{P} * \mathrm{Col}(\kappa,\lambda)^{\mathbb{P}} \\
& \sim \mathbb{P} \times \mathrm{Col}(\kappa,\lambda) && ;\text{ by } \kappa\text{-closedness of } \mathbb{P} \\
& \sim (\mathbb{P} \times \mathrm{Col}(\kappa,\mu+1)) \times \mathrm{Col}(\kappa,\lambda \setminus (\mu+1)) \\
& \sim \mathrm{Col}(\kappa,\{\mu\}) \times \mathrm{Col}(\kappa,\lambda \setminus (\mu+1)) && ;\text{ by Theorem 1.5} \\
& \sim \mathrm{Col}(\kappa,\mu+1) \times \mathrm{Col}(\kappa,\lambda \setminus (\mu+1)) && ;\text{ by (1)} \\
& \sim \mathrm{Col}(\kappa,\lambda).
\end{aligned}
$$

❑ (Corollary 1.6)

2 Reflection down to $< 2^{\aleph_0}$

lt-conti

$\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ is consistent under $2^{\aleph_0} = \aleph_2$ (e.g. under the assumption of the existence of a supercompact cardinal): $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ implies $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ and $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ is consistent (under the large cardinal assumption) with $2^{\aleph_0} = \aleph_2$. This is no more the case if the continuum is larger than $\aleph_2$:

P-lt-conti-0

Proposition 2.1 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *for* $\kappa > \aleph_2$ *implies* $\kappa > 2^{\aleph_0}$.

The proof of the Proposition uses the following Theorem 2.2 which is often the main ingredient of a proof showing that a certain principle implies $2^{\aleph_0} \leq \aleph_2$ (see e.g. the proof of Theorem 37.18 in [8]).

Theorem 2.2 (Theorem 3.2 (a) in Baumgartner and Taylor [1]) *If $\mathcal{C}$ is a club subset of $[\omega_2]^{\aleph_0}$, then there is a countable set $A \subseteq \omega_2$ such that $|\,[A]^{\aleph_0} \cap C\,| = 2^{\aleph_0}$.* P-lt-conti-1

❑

Proof of Proposition 2.1: $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies $2^{\aleph_0} \leq \aleph_2$: it is easy to see that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ implies the reflection principle $\mathsf{RP}(\omega_2)$ in [8]. $\mathsf{RP}(\omega_2)$ implies $2^{\aleph_0} \leq \aleph_2$ (a result by Todorčević, see Theorem 37.18 in [8]). We have $\kappa > \aleph_2 \geq 2^{\aleph_0}$.

Thus we may assume that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ does not hold. Hence there is a structure $\mathfrak{A}$ such that, for any $\mathfrak{B} \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$, we have $\|\mathfrak{B}\| \geq \aleph_2$. Let $\lambda = \|\mathfrak{A}\|$. Without loss of generality, we may assume that $|\mathfrak{A}| = \lambda$. Let

$$(2.1)\qquad \mathfrak{A}^* = \langle \mathcal{H}(\lambda^+), \underbrace{\lambda, \ldots,}_{=\mathfrak{A}} \in \rangle.$$ lt-conti-0

Note that we have

$$\text{(2.2)} \qquad \mathfrak{A}^* \models \underbrace{aa\,X\,\exists x \forall y\,(y \mathrel{\varepsilon} X \leftrightarrow y \in x)}_{=\varphi}$$ lt-conti-1

where "$aa\,X$" is the dual quantifier to "$stat\,X$". That is, we treat "$aa\,X\,\psi$" just as an abbreviation of "$\neg stat\,X \neg\psi$". Note that $\mathfrak{A} \models aa\,X\,\varphi(X, \ldots)$ if and only if there are club many $U \in [A]^{\aleph_0}$ with $\mathfrak{A} \models \varphi(U, \ldots)$.

By $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$, there is $M \in [\mathcal{H}(\lambda^+)]^{<\kappa}$ such that $\mathfrak{A}^* \restriction M \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}^*$. It follows that $\mathfrak{A} \restriction (\lambda \cap M) \prec^-_{(\mathcal{L}^{\aleph_0}_{stat})} \mathfrak{A}$. By the choice of $\mathfrak{A}$, we have $|\,M\,| \geq |\,\lambda \cap M\,| \geq \aleph_2$.

Since $\mathfrak{A}^* \restriction M \models \varphi$ by (2.2), and by elementarity, there is $C \subseteq [M]^{\aleph_0} \cap M$ which is a club in $[M]^{\aleph_0}$. By Theorem 2.2, it follows that $\kappa > |\,M\,| \geq |\,C\,| \geq 2^{\aleph_0}$.

□ (Proposition 2.1)

P-lt-conti-2

Corollary 2.3 $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *implies* $2^{\aleph_0} = \aleph_2$.

Proof. $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ implies $2^{\aleph_0} \leq \aleph_2$ by Proposition 2.1. $2^{\aleph_0} = \aleph_1$ is impossible under $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ since, as we have seen already, $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_1)$ does not hold (in ZFC). □ (Corollary 2.3)

Note that $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds under $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed}) + 2^{\aleph_0} = \aleph_2$. However,

Corollary 2.4 $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ *is inconsistent.*

Proof. Assume that $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds. Then $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds and hence $2^{\aleph_0} = \aleph_2$ by Corollary 2.3. Thus, $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ holds. By Lemma 1.2, (1) and Theorem 1.1, (4), it follows that $2^{\aleph_0} = \aleph_1$. This is a contradiction. □ (Corollary 2.4)

Translated in terms of diagonal reflection, Proposition 2.1 can also be reformulated as:

(2.3) If $\kappa > \aleph_2$, $(*)^{+\mathsf{IC}_{\aleph_0}}_{<\kappa,\lambda}$ for all cardinal $\lambda \geq \kappa$ implies $2^{\aleph_0} < \kappa$. lt-conti-2

However, the following internal variation of $(*)^{+\mathsf{IS}_{\aleph_0}}_{<\kappa,\lambda}$ is compatible with $2^{\aleph_0} \geq \kappa$ (see Theorem 2.10). For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$(*)^{int+}_{<\kappa,\lambda}$: *For any countable expansion* $\tilde{\mathfrak{A}}$ *of* $\langle \mathcal{H}(\lambda), \in \rangle$ *and sequence* $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ *such that* S_a *is a stationary subset of* $[\mathcal{H}(\lambda)]^{\aleph_0}$ *for all* $a \in \mathcal{H}(\lambda)$, *there are stationarily many* $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ *such that*

- (2) $\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$; *and*
- (3′) $S_a \cap M$ *is stationary in* $[M]^{\aleph_0}$ *for all* $a \in M$.

Note that (2) implies that $a \subseteq M$ for all $a \in [M]^{\aleph_0} \cap M$.

By the definition of $(*)^{int+}_{<\kappa,\lambda}$ it is easy to see that we have

$$(2.4) \qquad (*)^{+\mathsf{IC}_{\aleph_o}}_{<\kappa,\lambda} \;\Rightarrow\; (*)^{int+}_{<\kappa,\lambda} \;\Rightarrow\; (*)^{+\mathsf{IS}_{\aleph_0}}_{<\kappa,\lambda}$$

for any regular κ, λ with $\kappa \leq \lambda$.

For a class $\mathcal{P}$ of posets, a cardinal κ is said to be *generically supercompact by* $\mathcal{P}$ if, for any regular $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are transitive $M \subseteq V[\mathbb{G}]$ and $j \subseteq V[\mathbb{G}]$ such that

$$(2.5) \qquad j : \mathsf{V} \overset{\preceq}{\to} M,$$ lt-conti-2-0

$$(2.6) \qquad crit(j) = \kappa,\ j(\kappa) > \lambda,$$ lt-conti-2-1

$$(2.7) \qquad j''\lambda \in M.$$ lt-conti-2-2

Similarly a cardinal κ is said to be *generically measurable by a forcing* $\mathbb{P}$ if, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preceq}{\to} M$ such that $crit(j) = \kappa$ and $j''\kappa \in M$.

Note that the condition (2.7) is weaker than the closure property of M assumed in the usual definition of supercompactness. On the other hand, this condition is often enough to prove strong reflection properties of κ (see the following Lemma 2.5).

By definition, if κ is generically supercompact by $\mathcal{P}$ then κ is generically measurable by some $\mathbb{P} \in \mathcal{P}$.

The following is easy to see:

L-lt-conti-0

Lemma 2.5 *Suppose that $\mathbb{G}$ is a $(\mathsf{V}), \mathbb{P}$-generic filter for a poset $\mathbb{P} \in \mathsf{V}$ and $j : \mathsf{V} \overset{\preceq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ such that, for cardinals κ, λ in V with $\kappa \leq \lambda$, $crit(j) = \kappa$ and $j''\lambda \in M$.*

(1) *For any set $A \in \mathsf{V}$ with $\mathsf{V} \models |\,A\,| \leq \lambda$, we have $j''A \in M$.*

(2) *$j \restriction \lambda$, $j \restriction \lambda^2 \in M$.*

(3) *For any $A \in \mathsf{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.*

(4) *$(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, Thus, if $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.*

(5) *$\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.*

(6) *$j \restriction A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.*

Proof. (1): In V, let $f : \lambda \to A$ be a surjection.

For each $a \in A$ with $a = f(\alpha)$, we have

$$(2.8) \qquad j(a) = j(f(\alpha)) = j(f)(j(\alpha))$$ pr-L-lt-conti-0

by elementarity. Thus $j''A = j(f)''(j''\lambda)$. Since $j(f)$, $j''\lambda \in M$, it follows that $j''A \in M$.

(2): Since $j''\lambda \in M$ and $(j \restriction \lambda)(\xi)$ for $\xi \in \lambda$ is the ξth element of $j''\lambda$, $j \restriction \lambda$ is definable subset of $\lambda \times j''\lambda$ in M and hence is an element of M. Similarly, $j \restriction \lambda^2 \in M$.

(3): Suppose that $A \in \mathsf{V}$ and $A \subseteq \lambda$ (the case of $A \subseteq \lambda^2$ can be treated similarly). Then $j''A \in M$ by (1). Thus, by (2), $A = (j \restriction \lambda)^{-1}{}''(j''A) \in M$.

(4): Suppose that $\mu < (\lambda^+)^{\mathsf{V}}$. Then there is $A \in \mathsf{V}$ with $A \subseteq \lambda^2$ such that A codes the order type of μ. $A \in M$ by (3). Thus $M \models$ "$|\,\mu\,| \leq \lambda$".

If $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, we have

$$(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]} \geq (\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}. \tag{2.9}$$

pr-L-lt-conti-1

(5): For $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$, let $U \in \mathsf{V}$ be such that $trcl(A) \subseteq U$ and $\mathsf{V} \models$ "$|\,U\,| = \lambda$". Let $c_A \subseteq \lambda^2$ and d_A, $e_A \subseteq \lambda$ be such that c_A, d_A, $e_A \in \mathsf{V}$ and

$$\langle \lambda, c_A, d_A, e_A \rangle \cong \langle U, \in\restriction U^2, trcl(A), A \rangle. \tag{2.10}$$

pr-L-lt-conti-2

By (3), c_A, d_A, $e_A \in M$ and hence $\langle \lambda, c_A, d_A, e_A \rangle \in M$. Since $trcl(A)$ and then A can be recovered from this quadruplet in M, it follows that $A \in M$.

(6): Suppose that $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$. It is enough to show that $j \restriction trcl(A) \in M$.

We have $trcl(A) \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$ and hence A, $trcl(A) \in M$ by (5). Thus $j''\,trcl(A)$, $j''(\in\restriction trcl(A)) \in M$ by (1). But then $(j \restriction trcl(A))^{-1}$ is the Mostowski collapse of $j''\,trcl(A)$. Thus $j \restriction trcl(A) \in M$. □ (Lemma 2.5)

The following Lemmas 2.6 and 2.7 should be well-known and easy to check. Lemma 2.7 can be proved similarly to Proposition 2.8, (2).

L-lt-conti-1-0

Lemma 2.6 *If κ is generically measurable for some poset $\mathbb{P}$, then κ is regular.*

Proof. Suppose that $\kappa = \sup\{\kappa_\xi : \xi < \mu\}$ for some $\mu < \kappa$ and $\kappa_\xi < \kappa$ for $\xi < \mu$. Let $S = \{\kappa_\xi : \xi < \mu\}$ and let $j : \mathsf{V} \overset{\preceq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$ be as in the definition of generic measurability. Then $j(S) = S$ by $crit(j) = \kappa$. By elementarity it follows that $M \models$ "$j(\kappa) = \sup S = \kappa$". This is a contradiction to $crit(j) = \kappa$. □ (Lemma 2.6)

L-lt-conti-1-1

Lemma 2.7 (1) *Suppose that κ is generically measurable for a poset $\mathbb{P}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$ $j : V \overset{\preceq}{\to} M$, $crit(j) = \kappa$. Then, in $V[\mathbb{G}]$,*

$$F = \{a \in (\mathcal{P}(\kappa))^{\mathsf{V}} \,:\, \kappa \in j(a)\} \tag{2.11}$$

lt-conti-2-2-0

is a V*-normal ultrafilter on (the Boolean algebra)* $(\mathcal{P}(\kappa))^V$.

(2) *If* $\mu < \kappa$ *and* κ *is generically measurable for a* μ*-cc poset* $\mathbb{P}$ *then there is a* μ*-saturated normal ideal over* κ *(in* V*). In particular,* κ *is* κ*-weakly Mahlo.*

Proof. (1): See the proof of Claim 2.8.1.

(2): Suppose that $\mathbb{P}$ is a μ-cc poset and $\mathbb{G}$ a $(\mathsf{V},\mathbb{P})$-generic filter with j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$, $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$ and $crit(j) = \kappa$. Let F be defined as in (2.11) and let $\underset{\sim}{F}$ be a $\mathbb{P}$-name of F such that all the properties of F we need below are forced for $\underset{\sim}{F}$ by $\mathbb{1}_{\mathbb{P}}$. In V, let

$$(\aleph 2.1) \quad I_0 = \{u \in \mathcal{P}(\kappa) : \Vdash_{\mathbb{P}} \text{“}\check{u} \not\varepsilon \underset{\sim}{F}\text{”}\}. \qquad \text{lt-conti-2-2-1}$$

It is easy to see that I_0 is an ideal on $\mathcal{P}(\kappa)$. We show that I_0 is as desired.

Let F_0 be the dual filter of I_0.

Cl-lt-conti-0

Claim 2.7.1 *I_0 is a normal ideal.*

$\vdash$ It is enough to show that F_0 is a normal filter. Suppose that $u_\alpha \in F_0$, for all $\alpha < \kappa$. This means that we have $\Vdash_{\mathbb{P}} \text{“}\check{u}_\alpha \varepsilon \underset{\sim}{F}\text{”}$ for all $\alpha < \kappa$. Since F is a V-normal ultrafilter, it follows that $\Vdash_{\mathbb{P}} \text{“}\surd_{\mathbb{P}}(\triangle_{\alpha<\kappa} u_\kappa) \varepsilon \underset{\sim}{F}\text{”}$. Thus, $\triangle_{\alpha<\kappa} u_\kappa \in F_0$. $\dashv$ (Claim 2.7.1)

Cl-lt-conti-1

Claim 2.7.2 *For any $a \in \mathcal{P}(\kappa)$, a is I_0-stationary if and only if there is $\mathbb{p} \in \mathbb{P}$ such that $\mathbb{p} \Vdash_{\mathbb{P}} \text{“}a \in \underset{\sim}{F}\text{”}$.*

$\vdash$ If there is $\mathbb{p} \in \mathbb{P}$ such that $\mathbb{p} \Vdash_{\mathbb{P}} \text{“}\check{a} \varepsilon \underset{\sim}{F}\text{”}$, then $\not\Vdash_{\mathbb{P}} \text{“}\check{a} \not\varepsilon \underset{\sim}{F}\text{”}$ and hence $a \notin I_0$. Thus a is I_0-stationary.

If there is no $\mathbb{p} \in \mathbb{P}$ such that $\mathbb{p} \Vdash_{\mathbb{P}} \text{“}\check{a} \varepsilon \underset{\sim}{F}\text{”}$, then we have $\Vdash_{\mathbb{P}} \text{“}\check{a} \not\varepsilon \underset{\sim}{F}\text{”}$. Thus $a_0 \in I_0$. $\dashv$ (Claim 2.7.2)

Cl-lt-conti-2

Claim 2.7.3 *I_0 is μ-saturated.*

$\vdash$ Suppose that a_ξ, $\xi < \mu$ are I_0-stationary. By Claim 2.7.2, there are $\mathbb{p}_\xi \in \mathbb{P}$, $\xi < \mu$ such that $\mathbb{p}_\xi \Vdash_{\mathbb{P}} \text{“}a_\xi \in \underset{\sim}{F}\text{”}$. By the μ-cc of $\mathbb{P}$, there are $\xi_0 < \xi_1 < \mu$ such that $\mathbb{p}_{\xi_0}$ and $\mathbb{p}_{\xi_1}$ are compatible in $\mathbb{P}$. Let $\mathbb{q} \in \mathbb{P}$ be such that $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_{\xi_0}$, $\mathbb{p}_{\xi_1}$. Then we have $\mathbb{q} \Vdash_{\mathbb{P}} \text{“}a_{\xi_0} \cap a_{\xi_1} \in \underset{\sim}{F}\text{”}$ and hence $a_{\xi_0} \cap a_{\xi_1}$ is I_0-stationary by Claim 2.7.2. $\dashv$ (Claim 2.7.3)

By Proposition 16.8 in Kanamori [9], it follows that κ is a κ-weakly Mahlo. ❑ (Lemma 2.7)

T-laver-1-1

Proposition 2.8 *Suppose that κ is generically supercompact for a class $\mathcal{P}$ of posets such that all $\mathbb{P} \in \mathcal{P}$ are μ-cc for some fixed $\mu \in Card$. Then*

(1) SCH *holds above* $\max\{2^{<\kappa}, \mu\}$.

(2) *For all regular $\lambda \geq \kappa$, there is a μ-saturated normal filter over $\mathcal{P}_\kappa(\lambda)$.*

Proof. (1): The following proof is a slight modification of the proof of Solovay's theorem on SCH above a strongly compact cardinal (see Theorem 20.8 and its proof in [8]).

Let $\lambda \geq \max\{2^{<\kappa}, \mu\}$ be a regular cardinal. It is enough to show that $\lambda^{<\kappa} = \lambda$.

Suppose that $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ be such that there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\preceq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Let

$$\mathcal{U} = \{A \in (\mathcal{P}(\mathcal{P}_\kappa(\lambda)))^{\mathsf{V}} : j''\lambda \in j(A)\}. \tag{2.12}$$ laver-1-6

Cl-laver-0

Claim 2.8.1 *$\mathcal{U}$ is a V-κ-complete fine ultrafilter on $(\mathcal{P}(\mathcal{P}_\kappa(\lambda)))^{\mathsf{V}}$. (Actually $\mathcal{U}$ is even a V-normal ultrafilter.)*

$\vdash$ For A, $B \in (\mathcal{P}(\mathcal{P}_\kappa(\lambda)))^{\mathsf{V}}$ with $A \cup B = (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}$, we have $j(A) \cup j(B) = j(A \cup B) = j((\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}) = (\mathcal{P}_{j(\kappa)}(j(\lambda)))^M \ni j''\lambda$. Hence $M \models j''\lambda \in j(A)$ or $M \models j''\lambda \in j(A)$. That is, $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

For A, $B \in (\mathcal{P}(\mathcal{P}_\kappa(\lambda)))^{\mathsf{V}}$, we can also show similarly that, if $A \in \mathcal{U}$ and $B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$ and that, if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.

To show the V-κ-completeness, suppose that $\vec{A} = \langle A_\xi : \xi < \eta \rangle \in \mathsf{V}$ be such that $\eta < \kappa$ and $A_\xi \in \mathcal{U}$ for all $\xi < \eta$. The last condition means that $j''\lambda \in j(A_\xi)$ for all $\xi < \eta$ by definition of $\mathcal{U}$. Note that $\langle j(A_\xi) : \xi < \eta \rangle = j(\vec{A}) \in M$. Thus $M \models j''\lambda \in \bigcap_{\xi<\eta} j(A_\eta) = j(\bigcap\{A_\xi : \xi < \mu\})$, i.e. $\bigcap\{A_\xi : \xi < \mu\} \in \mathcal{U}$.

For any $c \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}$, letting $A = \{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : c \subseteq a\}$, we have $M \models j''\lambda \in j(A)$ since $M \models j(c) = j''c \subseteq j''\lambda$. Thus $A \in \mathcal{U}$. This shows that $\mathcal{U}$ is fine.

To show the V-normality of $\mathcal{U}$, suppose that $\vec{U} = \langle U_\alpha : \alpha \in \lambda \rangle \in \mathsf{V}$ is a sequence of elements of $\mathcal{U}$. Let $U = \triangle \vec{U}$. Then we have

$$\mathsf{V} \models \text{“}(\forall x \in \mathcal{P}_\kappa(\lambda))(x \in U \leftrightarrow \forall \alpha \in x\,(x \in U_\alpha))\text{”}. \tag{ℵ2.2}$$

By elementarity

$$M \models \text{“}(\forall x \in \mathcal{P}_{j(\kappa)}(j(\lambda)))(x \in j(U) \leftrightarrow \forall \alpha \in x\,(x \in j(\vec{U})_\alpha))\text{”}. \tag{ℵ2.3}$$

Now, for $x = j''\lambda$, if $\alpha \in j''\lambda$, then there is $\beta \in \lambda$ such that $\alpha = j(\beta)$. Since $U_\beta \in \mathcal{U}$, we have $j''\lambda \in j(U_\beta) = j(\vec{U})_\alpha$.

Thus we have $M \models$ "$j''\lambda \in j(U)$" or equivalently $U \in \mathcal{U}$. $\dashv$ (Claim 2.8.1)

Let $f^* \in \mathsf{V}$ with $f^* : (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} \to \mathrm{On}$ be such that

$$(2.13) \quad [f^*]_{\mathcal{U}} = \sup j_{\mathcal{U}}{}''\lambda.$$ laver-1-6-0

Cl-laver-1

Claim 2.8.2 $\{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : f^*(a) < \lambda\} \in \mathcal{U}$.

$\vdash$ Let $g \in \mathsf{V}$ with $g : (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} \to \lambda;\ a \mapsto \sup(a)$. Then, for $\gamma < \lambda$, $j_{\mathcal{U}}(\gamma) = [c_\gamma]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$ for all $\gamma < \lambda$ since $\mathcal{U}$ is fine. Hence $[f^*]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$. Thus, by Łoś's Theorem,

$$(2.14) \quad \mathcal{U} \ni \{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : f^*(a) \leq \underbrace{g(a)}_{= \sup a < \lambda}\} \subseteq \{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : f^*(a) < \lambda\}.$$

$\dashv$ (Claim 2.8.2)

By the Claim above, we may assume without loss of generality that $f^* : (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} \to \lambda$.

In $\mathsf{V}[\mathbb{G}]$, let $\mathcal{D} \subseteq \mathcal{P}(\lambda)^{\mathsf{V}}$ be defined by

$$(2.15) \quad \mathcal{D} = \{X \in \mathcal{P}(\lambda)^{\mathsf{V}} : f^{*-1}{}''X \in \mathcal{U}\}.$$ [2)] laver-1-7

Cl-laver-2

Claim 2.8.3 *$\mathcal{D}$ is a V-κ-complete uniform ultrafilter on $\mathcal{P}(\lambda)^{\mathsf{V}}$.*

$\vdash$ That $\mathcal{D}$ is an ultrafilter on $\mathcal{P}(\lambda)^{\mathsf{V}}$ is clear by definition.

Suppose that $\langle X_\alpha : \alpha < \mu\rangle \in \mathsf{V}$ is a sequence of elements of $\mathcal{D}$ for some $\mu < \kappa$. Then $\langle f^{*-1}(X_\alpha) : \alpha < \mu\rangle \in \mathsf{V}$ is a sequence of elements of $\mathcal{U}$. Hence $f^{*-1}{}''(\bigcap_{\alpha<\mu} X_\alpha) = \bigcap_{\alpha<\mu} f^{*-1}{}''X_\alpha \in \mathcal{U}$. Thus $\bigcap_{\alpha<\mu} X_\alpha \in \mathcal{D}$.

If $X \in ([\lambda]^{<\lambda})^{\mathsf{V}}$, then there is $\gamma < \lambda$ such that $X \subseteq \gamma$ by regularity of λ. Since $f^{*-1}{}''\gamma = \{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : f^*(a) < \gamma\}$ is disjoint with $\{a \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}} : f^*(a) \geq \gamma\}$ and since the latter set is in $\mathcal{U}$ by the definition (2.13), it follows that $X \notin \mathcal{D}$. $\dashv$ (Claim 2.8.3)

Cl-laver-3

Claim 2.8.4 $[id_\lambda]_{\mathcal{D}} = \sup(j_{\mathcal{D}}{}''\lambda)$.

$\vdash$ Suppose $\gamma < \lambda$. Then $\{\alpha < \lambda : c_\gamma(\alpha) < id_\lambda(\alpha)\} = \{\alpha < \lambda : \gamma < \alpha\} = \lambda\setminus(\gamma+1) \in \mathcal{D}$ since $\mathcal{D}$ is homogeneous by Claim 2.8.3. Thus $j_{\mathcal{D}}(\gamma) = [c_\gamma]_{\mathcal{D}} < [id_\lambda]_{\mathcal{D}}$.

Suppose now that $g : \lambda \to \mathrm{On}$ is such that $[g]_{\mathcal{D}} < [id_\lambda]_{\mathcal{D}}$. This means that

$$(2.16) \quad \{\alpha < \lambda : g(\alpha) < \underbrace{id_\lambda(\alpha)}_{= \alpha}\} \in \mathcal{D}$$ laver-1-8

[2)] Here, a more straightforward proof is possible e.g. by directly defining $\mathcal{D}$. We defined $\mathcal{D}$ via definition of $\mathcal{U}$ instead since we need $\mathcal{U}$ in the proof of (2) in any way.

$$\Leftrightarrow\ \mathcal{U} \ni f^{*-1}{}''\{\alpha<\lambda\,:\,g(\alpha)<\alpha\} = \{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,g(f^*(a))<f^*(a)\}.$$

By definition of f^*, there is $\gamma<\lambda$ such that $[gf^*]_{\mathcal{U}} \leq [c_\gamma]_{\mathcal{U}}$. That is, $\{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,g(f^*(a))\leq\gamma\}\in\mathcal{U}$. This is equivalent to $\{\alpha<\lambda\,:\,g(\alpha)\leq\gamma\}\in\mathcal{D}$ or $[g]_{\mathcal{D}}\leq j_{\mathcal{D}}(\gamma)$. $\dashv$ (Claim 2.8.4)

Cl-laver-4

Claim 2.8.5 (1) $\{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,f^*(a)=\sup a\cap f^*(a)\}\in\mathcal{U}$.

(2) $\{\alpha<\lambda\,:\,cf(\alpha)<\kappa\}\in\mathcal{D}$.

$\vdash$ (1): In V, let $g:\mathcal{P}_\kappa(\lambda)\to\lambda;\ a\mapsto\sup(a\cap f^*(a))$. Then we have $[g]_{\mathcal{U}}\leq[f^*]_{\mathcal{U}}$.

On the other hand, for each $\gamma<\lambda$, we have

$$(2.17)\quad \{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,\underbrace{c_\gamma(a)\leq g(a)}_{\Leftrightarrow\ \gamma\leq\sup(a\cap f^*(a))}\}$$
$$\supseteq \underbrace{\{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,\gamma\leq f^*(a)\}}_{\in\,\mathcal{U}\text{ (by the choice of }f^*)}\cap\underbrace{\{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,\gamma\in a\}}_{\in\,\mathcal{U}\text{ (since }\mathcal{U}\text{ is fine)}}\in\mathcal{U}.$$

Thus $j_{\mathcal{U}}(\gamma)\leq[g]_{\mathcal{U}}$. By the choice of f^* it follows that $[f^*]_{\mathcal{U}}\leq[g]_{\mathcal{U}}$.

(2): by the definition (2.15) of $\mathcal{D}$

$$(2.18)\quad \{\alpha<\lambda\,:\,cf(\alpha)<\kappa\}\in\mathcal{D}\ \Leftrightarrow\ \{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,cf(f^*(a))<\kappa\}\in\mathcal{U}.$$ laver-1-9

By (1),

$$(2.19)\quad \{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,cf(f^*(a)),\kappa\}$$
$$\supseteq\{a\in(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\,:\,f^*(a)=\sup(a\cap f^*(a))\}\in\mathcal{U}.$$ $\dashv$ (Claim 2.8.5)

In V, let

$$(2.20)\quad A_\alpha=\begin{cases}\text{a cofinal subset of }\alpha\text{ of order type }cf(\alpha), & \text{if } cf(\alpha)<\kappa;\\ \emptyset, & \text{otherwise}\end{cases}$$ laver-1-10

for $\alpha<\lambda$.

Let $\vec{A}=\langle A_\alpha\,:\,\alpha<\lambda\rangle$.

By Claim 2.8.4 and Claim 2.8.5, (2),

$$(2.21)\quad {}^{\lambda}\mathsf{V}/\mathcal{D}\models\text{“}[\vec{A}]_{\mathcal{D}}\text{ is a cofinal subset of }\sup(j_{\mathcal{D}}{}''\lambda)\text{”}.$$ laver-1-11

By the μ-cc of $\mathbb{P}$, there is a strictly and continuously increasing sequence $\langle\eta_\xi\,:\,\xi<\lambda\rangle$ of ordinals $<\lambda$ in V such that, letting $I_\xi=[\eta_\xi,\eta_{\xi+1})$,

$$(2.22)\quad \Vdash_{\mathbb{P}}\text{“}j_{\underset{\sim}{\mathcal{D}}}(I_\xi)\cap[\vec{A}]_{\underset{\sim}{\mathcal{D}}}\not\equiv\emptyset\text{”}$$ laver-1-12

where $\underset{\sim}{\mathcal{D}}$ is a $\mathbb{P}$-name for $\mathcal{D}$. Let

(2.23) $\quad M_\alpha = \{\xi < \lambda : I_\xi \cap A_\alpha \neq \emptyset\}$ laver-1-13

for $\alpha < \lambda$. Note that $\langle M_\alpha : \alpha < \lambda \rangle \in \mathsf{V}$.

Since $\lambda \geq 2^{<\kappa}$, (3) of the following Claim 2.8.6 implies that $\lambda^{<\kappa} = \lambda$ as desired.

Cl-laver-5

Claim 2.8.6 (1) *$|M_\alpha| < \kappa$ for all $\alpha < \lambda$.*

(2) *For any $\xi < \lambda$, $\{\alpha < \lambda : \xi \in M_\alpha\} \in \mathcal{D}$.*

(3) *For any $c \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}$, $\{\alpha < \lambda : c \subseteq M_\alpha\} \in \mathcal{D}$. In particular, there is some $\alpha < \lambda$ such that $c \subseteq M_\alpha$.*

$\vdash$ (1): Since $|A_\alpha| < \kappa$ (see (2.20)) and I_ξ's are pairwise disjoint, there can be only $< \kappa$ elements of M_α.

(2): Suppose $\xi < \lambda$. By (2.22), we have

(2.24) $\quad j_{\mathcal{D}}(I_\xi) \cap [\vec{A}]_{\mathcal{D}} \neq \emptyset.$ laver-1-14

By Łoś's Theorem, it follows that

(2.25) $\quad \{\alpha < \lambda : \underbrace{I_\xi \cap A_\alpha \neq \emptyset}_{\Leftrightarrow \quad \xi \in M_\alpha}\} \in \mathcal{D}.$ laver-1-15

(3): Suppose $c \in (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}$. For each $\xi \in c$, we have $\{\alpha < \lambda : \xi \in M_\alpha\} \in \mathcal{D}$ by (2). By κ-completeness of $\mathcal{D}$, it follows that

(2.26) $\quad \{\alpha < \lambda : c \subseteq M_\alpha\} = \bigcap_{\xi \in c} \{\alpha < \lambda : \xi \in M_\alpha\} \in \mathcal{D}.$ $\dashv$ (Claim 2.8.6)

(2) (of Proposition 2.8): Suppose that $\lambda \geq \kappa$ is a regular cardinal, $\mathbb{P} \in \mathcal{P}$, $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ are such that, M is transitive in $V[\mathbb{G}]$, $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$. Let $\mathcal{U}$ be defined by (2.12). By Claim 2.8.1, $\mathcal{U}$ is a V-normal ultrafilter.

In V, let $\underset{\sim}{\mathcal{U}}$ be a $\mathbb{P}$-name of $\mathcal{U}$ and let

(2.27) $\quad \mathcal{F} = \{U \in \mathcal{P}(\mathcal{P}_\kappa(\lambda)) : \Vdash_{\mathbb{P}} \text{“}\check{U} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}\text{”}\}.$ laver-1-15-0

Then $\mathcal{F}$ is a μ-saturated normal filter over $\mathcal{P}_\kappa(\lambda)$. ❏ (Proposition 2.8)

`Theorem 2.8, (2) follows from the next Lemma. Note that the normality of a filter over` $\mathcal{P}_\kappa(\lambda)$ `we use here is as defined in [9] p.301 and V-normality a natural modification of this definition.`

LA-laver-0

Lemma A 2.1 *Suppose $\kappa < \lambda$ and μ are uncountable regular cardinals and $\mathbb{P}$ a μ-cc poset such that $\mathbb{P}$ preserves the cardinal κ. If*

(ℵ2.4) $\quad \Vdash_{\mathbb{P}}$ “ *there is a V-normal ultrafilter $\mathcal{U}$ over $(\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}$* ”, laverA-0 *then there is a $<\mu$-saturated normal filter $\mathcal{F}$ on $\mathcal{P}_\kappa(\lambda)$.*

Proof. Let $\underset{\sim}{\mathcal{U}}$ be a $\mathbb{P}$ name such that

$$(\aleph 2.5) \quad \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{\mathcal{U}} \text{ is a V-normal ultrafilter over } (\mathcal{P}_\kappa(\lambda))^{\mathsf{V}}\text{"}. \qquad \text{laverA-1}$$

Let

$$(\aleph 2.6) \quad \mathcal{F} = \{U \in \mathcal{P}(\mathcal{P}_\kappa(\lambda)) : \Vdash_{\mathbb{P}} \text{"}\check{U} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}\text{"}\}. \qquad \text{laverA-2}$$

The following claim shows that $\mathcal{F}$ above is as desired.

ClA-laver-0

Claim A 2.1.1 (0) *$\mathcal{F}$ is a filter over $\mathcal{P}_\kappa(\lambda)$.*

(1) *$\mathcal{F}$ is $<\kappa$-complete.*

(2) *$\mathcal{F}$ is fine.*

(3) *$\mathcal{F}$ is normal.*

(4) *$\mathcal{F}$ is $<\mu$-saturated.*

$\vdash$ (0): By ($\aleph$2.5) and ($\aleph$2.6).

(1): Suppose that $U_\alpha \in \mathcal{F}$, $\alpha < \delta$ for some $\delta < \kappa$. Then $\underset{\sim}{S} = \{\langle \check{U}_\alpha, \mathbb{1}_{\mathbb{P}}\rangle : \alpha < \delta\}$ is a $\mathbb{P}$-name and

$$(\aleph 2.7) \quad \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{S} \subseteq \underset{\sim}{\mathcal{U}},\ |\underset{\sim}{S}| < \kappa,\ \underset{\sim}{S} \mathrel{\varepsilon} \mathsf{V}\text{"}. \qquad \text{laverA-3}$$

Thus $\Vdash_{\mathbb{P}}$ "$\bigcap \underset{\sim}{S} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$" by ($\aleph$2.5). Since $\Vdash_{\mathbb{P}}$ "$\bigcap \underset{\sim}{S} \equiv \surd_{\mathbb{P}}(\bigcap_{\alpha<\delta} U_\alpha)$", we have $\bigcap_{\alpha<\delta} U_\alpha \in \mathcal{F}$.

(2): By ($\aleph$2.5) and ($\aleph$2.6). (3): Similarly to (1).

For the proof of (4), we need the following:

Claim A 2.1.2 *For $X \subseteq \mathcal{P}_\kappa(\lambda)$, X is $\mathcal{F}$-stationary if and only if $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{X} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$" for some $\mathbb{p} \in \mathbb{P}$.*

$\vdash$ Suppose that $\mathbb{p} \not\Vdash_{\mathbb{P}}$ "$\check{X} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$" for all $\mathbb{p} \in \mathbb{P}$. Then we have $\Vdash_{\mathbb{P}}$ "$\check{X} \not\mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$". Since $\underset{\sim}{\mathcal{U}}$ is a $\mathbb{P}$-name of an ultrafilter, if follows that $\Vdash_{\mathbb{P}}$ "$(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \setminus \check{X}) \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$". Since $\Vdash_{\mathbb{P}}$ "$(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \setminus \check{X}) \equiv \surd_{\mathbb{P}}(\mathcal{P}_\kappa(\lambda) \setminus X)$", it follows that $(\mathcal{P}_\kappa(\lambda) \setminus X) \in \mathcal{F}$. Thus X is not $\mathcal{F}$-stationary.

Suppose now that $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{X} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$". Then for any $U \in \mathcal{F}$, we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{U} \cap \check{X} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$". Thus $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\check{U} \cap \check{X} \not\equiv \emptyset$" and hence $U \cap X \neq \emptyset$. This shows that X is $\mathcal{F}$-stationary. $\dashv$ (Claim A 2.1.2)

Continuation of the proof of Claim A 2.1.1:

(4): Suppose that $\langle X_\alpha : \alpha < \mu\rangle$ is a sequence of $\mathcal{F}$-stationary sets.

By Claim A 2.1.2, there are $\mathbb{p}_\alpha \in \mathbb{P}$, $\alpha < \mu$ such that $\mathbb{p}_\alpha \Vdash_{\mathbb{P}}$ "$\check{X}_\alpha \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$" for $\alpha < \mu$.

Since $\mathbb{P}$ is μ-cc, there are $\alpha < \alpha' < \mu$ such that $\mathbb{p}_\alpha$ and $\mathbb{p}_{\alpha'}$ are compatible in $\mathbb{P}$. Let $\mathbb{r} \in \mathbb{P}$ be such that $\mathbb{r} \leq_{\mathbb{P}} \mathbb{p}_\alpha, \mathbb{p}_{\alpha'}$. Then we have $\mathbb{r} \Vdash_{\mathbb{P}}$ "$\check{X}_\alpha \cap \check{X}_{\alpha'} \mathrel{\varepsilon} \underset{\sim}{\mathcal{U}}$". Again by Claim A 2.1.2, it follows that $X_\alpha \cap X_{\alpha'}$ is $\mathcal{F}$-stationary.

$\dashv$ (Claim A 2.1.1)

□ (Lemma A 2.1)

In case of $\mu \leq 2^{<\kappa}$, Proposition 2.8, (1) can be also proved as follows: By (the proof of) Corollary 3.5 and Theorem 5.1 in [11], we obtain the theorem asserting:

Theorem 2.9 (Matsubara, Sakai and Usuba [11]) *For a regular uncountable κ, if there are λ-saturated fine ideal over $\mathcal{P}_\kappa(\lambda)$ for all $\lambda \geq \kappa$, then* SCH *holds above $2^{<\kappa}$.* □

Proposition 2.8, (1) follows immediately from this theorem and Proposition 2.8, (2).

In [6], it is proved that, for a cardinal κ, if κ^+ is generically supercompact by $<\kappa$-closed posets then $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0}_{stat}, <\kappa^+)$ holds (this follows from Theorem 4.13, Theorem 4.7 and Lemma 4.1 in [6]).

In the next section, we show that, for a regular κ, the assertion "$(*)^{int+}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$" can be characterized in terms of SDLS for $\mathcal{L}^{\aleph_0}_{stat}$ in internal interpretation. This property "$(*)^{int+}_{<\kappa,\lambda}$ for all regular $\lambda \geq \kappa$" holds under a wider class of generic supercompactness (c.f. Theorem 5.9, (1) and (2)).

P-lt-conti-3

Theorem 2.10 *Suppose that κ is a generically supercompact cardinal by proper posets. Then $(*)^{int+}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$.*

In the proof of Theorem 2.10, we use the case $\kappa = \aleph_1$ of the following well-known fact:

P-lt-conti-4

Lemma 2.11 *Suppose that M is an inner model in* V. *For ordinals κ, λ with $\kappa \leq \lambda$ and* $\mathsf{V} \models$ "κ is a regular uncountable cardinal", *and for $A \in M$ with $M \models$ "$A \subseteq \mathcal{P}_\kappa(\lambda)$", if $\mathsf{V} \models$ "A is stationary in $\mathcal{P}_\kappa(\lambda)$", then $M \models$ "A is stationary in $\mathcal{P}_\kappa(\lambda)$".*

Proof. Suppose that $\mathsf{V} \models$ "A is stationary in $\mathcal{P}_\kappa(\lambda)$". Let $\mathcal{C} \in M$ be such that $M \models$ "$\mathcal{C}$ is a club in $\mathcal{P}_\kappa(\lambda)$". By Kueker's Theorem (see Exercise 38.10 in [8]) there is $F \in M$ such that

$$(\aleph 2.8) \quad M \models \text{“} F : [\lambda]^{<\aleph_0} \to \lambda \text{ and } \mathcal{C}_F = \{a \in \mathcal{P}_\kappa(\lambda) : a \cap \kappa \in \kappa \text{ and } a \text{ is closed under } F\} \subseteq \mathcal{C}\text{”}.$$ P-lt-conti-A-0

In V, there is $a \in A \cap \mathcal{C}_F{}^{\mathsf{V}}$. $a \in A \subseteq M$ and hence $M \models$ "$a \in A \cap \mathcal{C}_F$".

□ (Lemma 2.11)

Proof of Theorem 2.10: For a regular $\lambda \geq \kappa$, let $\lambda^* = |\,\mathcal{H}(\lambda)\,|$ and $\lambda^{**} = (2^{\lambda^*})^+$.

Suppose that $\mathfrak{A} = \langle \mathcal{H}(\lambda), \in, ...\rangle$ is a countable expansion of the structure $\langle \mathcal{H}(\lambda), \in\rangle$, $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ a sequence of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$ and $\mathcal{C} \subseteq [\mathcal{H}(\lambda)]^{<\kappa}$ a club.

Let $\iota : \mathcal{H}(\lambda) \to \lambda^*$ be a bijection and let $\mathfrak{A}^* = \langle \lambda^*, E^*, ...\rangle$ be a copy of $\mathfrak{A}$ translated by ι.

Let $\langle S^*_\alpha : \alpha \in \lambda^*\rangle$ be such that

(2.28) $\quad S^*_\alpha = \iota'' S_{\iota^{-1}(\alpha)}$ for $\alpha \in \lambda^*$ and lt-conti-2-3

(2.29) $\quad \mathcal{C}^* = \{\iota'' X : X \in \mathcal{C}\}$. lt-conti-2-4

Thus, we have

(2.30) $\quad EXT_{E^*}(S^*_\alpha)$ is a stationary subset of $[\lambda^*]^{\aleph_0}$ for all $\alpha \in \lambda^*$ and lt-conti-2-5

(2.31) $\quad \mathcal{C}^*$ is a club in $[\lambda^*]^{<\kappa}$ lt-conti-2-6

where, for $S \subseteq \lambda^*$, $EXT_{E^*}(S) = \{\{\beta \in \lambda^* : \beta\ E^*\ \alpha\} :$ for $\alpha \in S\}$ is the set of all extents of elements of S with respect to the element relation E^*.

It is enough to show that there is an $X \in \mathcal{C}^*$ such that

($2'$) $\mathfrak{A}^* \restriction X \prec \mathfrak{A}^*$, and

($3''$) $EXT_{E^*}(S^*_\alpha \cap X)$ is stationary in $[X]^{\aleph_0}$ for all $\alpha \in X$.

Let $\mathbb{P}$ be a proper poset and $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic filter such that there are transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \xrightarrow{\preceq} M$ such that

(2.32) $\quad \kappa = crit(j)$, lt-conti-3

(2.33) $\quad j(\kappa) > \lambda^{**}$ and lt-conti-4

(2.34) $\quad j''\lambda^{**} \in M$. lt-conti-5

Let $X = j''\lambda^*$. $X \in M$ by (2.34) and Lemma 2.5, (1). Thus, we also have $j \restriction \lambda^* \in M$ and

(2.35) $\quad M \models |\,X\,| = |\,\lambda^*\,| < j(\kappa)$. costat-2-3

Since $|\,\mathcal{C}^*\,| < \lambda^{**}$, we have $j''\mathcal{C}^* \in M$ by Lemma 2.5, (1). $M \models$ "$|\,j''\mathcal{C}^*\,| \leq |\,j(\mathcal{C}^*)\,| < j(\kappa)$". Also we have $M \models$ "$j''\mathcal{C}^*$ is directed and $\bigcup(j''\mathcal{C}^*) = X$". Since $M \models$ "$\mathcal{C}^*$ is a club in $[j(\lambda^*)]^{<j(\kappa)}$" by elementarity and (2.31), it follows that

(2.36) $X \in j(\mathcal{C}^*)$. costat-2-4

Let $\langle \tilde{S}^*_\alpha : \alpha < j(\lambda^*)\rangle = j(\langle S^*_\alpha : \alpha < \lambda^*\rangle)$.

By elementarity and applying Vaught's test, we can show

(2.37) $M \models j(\mathfrak{A}^*) \restriction X \prec j(\mathfrak{A}^*)$.

For $\alpha \in \lambda^*$, since $\tilde{S}^*_{j(\alpha)} = j(S^*_\alpha) \supseteq j''S^*_\alpha$, we have

(2.38) $M \models$ "$EXT_{j(E^*)}(\tilde{S}^*_{j(\alpha)} \cap X) \supseteq EXT_{j(E^*)}(j''S^*_\alpha)$". costat-2-5

By (2.30) and since $\mathbb{P}$ is proper, we have $\mathsf{V}[\mathbb{G}] \models$ "$EXT_{E^*}(S^*_\alpha)$ is stationary in $[\lambda^*]^{\aleph_0}$". Since $EXT_{j(E^*)}(j''S^*_\alpha)$ is a translation of $EXT_{E^*}(S^*_\alpha)$ induced by $j \restriction \lambda^*$, it follows that

(2.39) $\mathsf{V}[\mathbb{G}] \models$ "$EXT_{j(E^*)}(j''S^*_\alpha)$ is stationary in $[X]^{\aleph_0}$". costat-2-6

By Lemma 2.11, it follows that

(2.40) $M \models$ "$EXT_{j(E^*)}(j''S^*_\alpha)$ is stationary in $[X]^{\aleph_0}$". costat-2-7

Thus, we have

(2.41) $M \models$ "$\exists X \in j(\mathcal{C}^*) \big(j(\mathfrak{A}^*) \restriction X \prec j(\mathfrak{A}^*) \land EXT_{j(E^*)}(\tilde{S}^*_\xi \cap X)$ is a stationary subset of $[X]^{\aleph_0}$ for all $\xi \in X \big)$".

By elementarity, it follows that

(2.42) $\mathsf{V} \models$ "$\exists X \in \mathcal{C}^* \big(\mathfrak{A}^* \restriction X \prec \mathfrak{A}^* \ \land \ EXT_{E^*}(S^*_\xi \cap X)$ is a stationary subset of $[X]^{\aleph_0}$ for all $\xi \in X \big)$".

❑ (Theorem 2.10)

3 Internal interpretation of stationary logic

internal

For a structure $\mathfrak{A} = \langle A, ...\rangle$ of a countable signature, an $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi = \varphi(x_0, ..., X_0, ...)$[3] and $a_0, ... \in A$, $U_0, ... \in [A]^{\aleph_0} \cap A$, we define the internal interpretation of $\varphi(a_0, ..., U_0, ...)$ in $\mathfrak{A}$ (notation: $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$ for "$\varphi(a_0, ..., U_0, ...)$ holds internally in $\mathfrak{A}$") by induction on the construction of φ as follows:

If φ is "$x_i \varepsilon X_j$" then

(3.1) $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...) \ \Leftrightarrow \ a_i \in U_j$ internal-0

---

[3] As before, when we write $\varphi = \varphi(x_0, ..., X_0, ...)$, we always assume that the list $x_0, ...$ contains all the free first order variables of φ and $X_0, ...$ all the free weak second order variables of φ.

for a structure $\mathfrak{A} = \langle A, ...\rangle$, $a \in A$ and $U \in [A]^{\aleph_0} \cap A$.

For first-order connectives and quantifiers in $\mathcal{L}^{\aleph_0}_{stat}$, the semantics "$\models^{int}$" is defined exactly as for the first order "$\models$".

For an $\mathcal{L}^{\aleph_0}_{stat}$ formula φ with $\varphi = \varphi(x_0, ..., X_0, ..., X)$, assuming that the notion of $\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)$ has been defined for all $a_0, ... \in A$, $U_0, ..., U \in [A]^{\aleph_0} \cap A$, we stipulate

$$\text{(3.2)} \quad \mathfrak{A} \models^{int} \mathit{stat}\, X\, \varphi(a_0, ..., U_0, ..., X) \;\Leftrightarrow\; \{U \in [A]^{\aleph_0} \cap A \,:\, \mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ..., U)\} \text{ is stationary in } [A]^{\aleph_0}$$ internal-1

for a structure $\mathfrak{A} = \langle A, ...\rangle$ of a relevant signature, $a_0, ... \in A$ and $U_0, ... \in [A]^{\aleph_0} \cap A$.

For structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} = \langle B, ...\rangle$ and $\mathfrak{B} \subseteq \mathfrak{A}$, we define

(3.3) $\mathfrak{B} \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A} \;\Leftrightarrow$ internal-2

$\mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...)$ *if and only if* $\mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$ *for all* $\mathcal{L}^{\aleph_0}_{stat}$*-formulas* φ *in the signature of the structures with* $\varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ *and* $U_0, ... \in [B]^{\aleph_0} \cap B$.

Finally, for a regular $\kappa > \aleph_1$, the internal strong downward Löwenheim-Skolem Theorem $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ is defined by

$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$: *For any structure* $\mathfrak{A} = \langle A, ...\rangle$ *of countable signature with* $|\,A\,| \geq \kappa$, *there are stationarily many* $M \in [A]^{<\kappa}$ *such that* $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

'+' in "$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$" refers to the "stationarily many" existence of the reflection points M. Similarly to Lemma 2.1 in [6], this additional condition can be drooped if $\kappa = \aleph_2$. This is because the quantifier $Qx\,\varphi$ defined by $\mathit{stat}\, X \exists x\, (x \notin X \wedge \varphi$, $\mathfrak{A} \models^{int} Qx\, \varphi(x, ...))$ still implies that "there are uncountably many $a \in A$ with $\varphi(a, ...)$". Note that, if $\mathfrak{A} \models^{int} \neg \mathit{stat}\, X\, (x \equiv x)$, for a structure $\mathfrak{A} = \langle A, ...\rangle$, we can easily find even club many $X \in [A]^{<\kappa}$ for any regular $\aleph_1 \leq \kappa \leq |\,A\,|$ such that $\mathfrak{A} \restriction X \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

Proposition 3.1 *For a regular cardinal* $\kappa > \aleph_1$, *the following are equivalent:* P-internal-0

(a) $(*)^{int+}_{<\kappa,\lambda}$ *holds for all regular* $\lambda \geq \kappa$.

(b) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ *holds.*

Proof. This proof is a straightforward modification of the proof of Lemma 3.5,(1) in [6]. Nevertheless, we will give the complete proof since we have to modify it further to prove Proposition 4.1.

Suppose first that $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ holds. We show that $(*)^{int+}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\lambda \geq \kappa$. Let $\tilde{\mathfrak{A}}$ be a countable expansion of $\langle \mathcal{H}(\lambda), \in\rangle$, $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ a sequence of stationary subsets of $[\mathcal{H}(\lambda)]^{\aleph_0}$ and $\mathcal{D} \subseteq [\mathcal{H}(\lambda)]^{<\kappa}$ a club.

Let

$$\tilde{\mathfrak{A}}^* = \langle \underbrace{\mathcal{H}(\lambda), ..., \in}_{\tilde{\mathfrak{A}}}, \vec{\underline{S}}^{\tilde{\mathfrak{A}}^*}\rangle \tag{3.4}$$ internal-3

where $\vec{\underline{S}}$ is a binary relation symbol and

$$\vec{\underline{S}}^{\tilde{\mathfrak{A}}^*} = \{\langle a, s\rangle \in (\mathcal{H}(\lambda))^2 : s \in S_a\}. \tag{3.5}$$ internal-4

Let $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ be such that

$$M \in \mathcal{D} \text{ and} \tag{3.6}$$ internal-5

$$\tilde{\mathfrak{A}}^* \restriction M \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \tilde{\mathfrak{A}}^*. \tag{3.7}$$ internal-6

By the choice of $\tilde{\mathfrak{A}}^*$ and (3.7), $\tilde{\mathfrak{A}}^* \restriction M \models^{int} \forall x\, stat\, X \exists y\, (\vec{\underline{S}}(x, y) \land \forall z\, (z \,\varepsilon\, X \leftrightarrow z \in y))$ holds and hence, for all $a \in M$, $S_a \cap M$ is stationary in $[M]^{\aleph_0}$.

Suppose now that $(*)^{int+}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\mathfrak{A} = \langle A, ...\rangle$ be a structure in countable signature and of cardinality $\geq \kappa$, and $\mathcal{D} \subseteq [A]^{<\kappa}$ a club. Without loss of generality, we may assume that $\mathfrak{A}$ is a relational structure. Let λ be a regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\lambda)$. In particular, we have $A \subseteq \mathcal{H}(\lambda)$.

Let $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\lambda), \underbrace{\underline{A}^{\tilde{\mathfrak{A}}}, ...,}_{=\mathfrak{A}} \in\rangle$ where $\underline{A}$ is a unary relation symbol and $\underline{A}^{\tilde{\mathfrak{A}}} = A$.

For each $a \in \mathcal{H}(\lambda)$, let

$$S_a = \begin{cases} \{U \in [\mathcal{H}(\lambda)]^{\aleph_0} : |\, U \cap A\,| = \aleph_0,\ U \cap A \in A, \\ \qquad\qquad \mathfrak{A} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap A)\}, \\ \qquad \text{if } \psi = \psi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1}, X) \text{ is an } \mathcal{L}^{\aleph_0}_{stat}\text{-formula} \\ \qquad \text{in the signature of } \mathfrak{A},\ a_0, ... \in A,\ U_0, ... \in [A]^{\aleph_0} \cap A, \\ \qquad \mathfrak{A} \models^{int} stat\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X) \text{ and} \\ \qquad a = \langle \psi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1}\rangle; \\ [\mathcal{H}(\lambda)]^{\aleph_0}, \\ \qquad \text{otherwise.} \end{cases} \tag{3.8}$$ internal-7

Let

$$\tilde{\mathcal{D}} = \{U \in [\mathcal{H}(\lambda)]^{<\kappa} : U \cap A \in \mathcal{D}\}. \tag{3.9}$$ internal-8

$\tilde{\mathcal{D}}$ contains a club in $[\mathcal{H}(\lambda)]^{<\kappa}$.

By $(*)^{int+}_{<\kappa,\lambda}$, there is an $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

(3.10) $\quad M \in \tilde{\mathcal{D}}$, internal-9

(3.11) $\quad \tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}$ and internal-10

(3.12) $\quad S_a \cap \mathcal{P}_{\kappa\cap M}(M) \cap M$ is stationary in $\mathcal{P}_{\kappa\cap M}(M)$ for all $a \in M$. internal-11

We expand $\tilde{\mathfrak{A}}$ further by adding the relation $\{\langle u, a\rangle \in (\mathcal{H}(\lambda))^2 \,:\, u \in S_a\}$ to it as the interpretation of the binary relation symbol $\underset{\smile}{\vec{S}}$. For simplicity, we shall also call this expanded structure $\tilde{\mathfrak{A}}$[4].

Let $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \restriction M$ and let $B = \underset{\smile}{A}^{\tilde{\mathfrak{B}}} = A \cap M$ and $\mathfrak{B} = \mathfrak{A} \restriction B$. Denoting the underlying set of $\tilde{\mathfrak{B}}$ by $\tilde{B}$, we have $\tilde{B} = M$. $B \in \mathcal{D}$ by (3.10) and the definition (3.9) of $\tilde{\mathcal{D}}$.

By the elementarity $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ (3.11), the following Claim implies $\mathfrak{B} \prec^{int}_{\mathcal{L}^{\aleph_0}_{stat}} \mathfrak{A}$.

Claim 3.1.1 *For any $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1})$ in the signature of the structures $\mathfrak{A}$, $a_0, ..., a_{m-1} \in B$ and $U_0, ..., U_{n-1} \in [B]^{\aleph_0} \cap B$, we have*

$$(3.13) \quad \tilde{\mathfrak{B}} \models \text{“}\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)\text{”} \;\Leftrightarrow\; \mathfrak{B} \models^{int} \varphi(a_0, ..., U_0, ...).$$ internal-12

$\vdash$ By induction on φ. The crucial step in the induction is when φ is of the form $stat\, X\psi$ and (3.13) holds for ψ:

Suppose first that $\tilde{\mathfrak{B}} \models$ “$\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$” holds. Then, by elementarity and by the definition of $\tilde{\mathfrak{A}}$, we have $\mathfrak{A} \models^{int} stat\, X\psi(a_0, ..., U_0, ..., U_{n-1}, X)$. Thus, letting $a = \langle \varphi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1}\rangle$, we have $a \in \tilde{B}$ and

$$(3.14) \quad S_a = \{U \in [\mathcal{H}(\lambda)]^{\aleph_0} \,:\, |\,U \cap A\,| = \aleph_0,\, U \cap A \in A, \\ \mathfrak{A} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap A)\}$$ internal-14

by the definition (3.8) of S_a.

By (3.12), $S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}$ is stationary in $[\tilde{B}]^{\aleph_0}$. It follows that

$$\begin{aligned} (3.15) \quad & \{U \cap B \,:\, |\,U \cap B\,| = \aleph_0, U \in S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}\} \\ & = \{U \cap B \,:\, |\,U \cap B\,| = \aleph_0,\, U \cap B \in B, \\ & \qquad \tilde{\mathfrak{B}} \models \text{“}\mathfrak{A} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap B)\text{”}\} \\ & \qquad\qquad \text{(by elementarity and (3.14))} \\ & \subseteq \{V \in [B]^{\aleph_0} \cap B \,:\, \mathfrak{B} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)\} \\ & \qquad\qquad \text{(by induction hypothesis)} \end{aligned}$$ internal-15

is stationary. Thus $\mathfrak{B} \models^{int} stat\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X)$ holds, that is, $\mathfrak{B} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$.

Suppose now that $\tilde{\mathfrak{B}} \not\models$ “$\mathfrak{A} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$”. Then we have

---

[4] This expansion becomes necessary below when we would like to have $\{U \cap B \,:\, U \in S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}\} = \{V \in B \,:\, V = U \cap B$ for some $U \in S_a \cap [\tilde{B}]^{\aleph_0} \cap \tilde{B}\}$.

(3.16) $\tilde{\mathfrak{B}} \models$ "there is a club $\mathcal{C} \subseteq [\underline{A}]^{\aleph_0}$ such that $\mathfrak{A} \models^{int} \neg\psi(a_0, \ldots, U_0, \ldots, U_{n-1}, x)$ for all $x \in \mathcal{C}$". internal-16

By elementarity, there is a $\mathcal{C}_0 \in \tilde{B}$ such that $\mathcal{C}_0$ is a club $\subseteq [A]^{\aleph_0}$ and

(3.17) $\tilde{\mathfrak{B}} \models$ "$\mathfrak{A} \models^{int} \neg\psi(a_0, \ldots, a_{m-1}, U_0, \ldots, U_{n-1}, V)$" for all $V \in \mathcal{C}_0 \cap B$. internal-17

By induction hypothesis, it follows that

(3.18) $\{U \in [B]^{\aleph_0} \cap B \,:\, \mathcal{B} \models^{int} \psi(a_0, \ldots, U_0, \ldots, U)\} \cap \mathcal{C}_0 = \emptyset.$ internal-17-0

Thus $\mathfrak{B} \not\models^{int}$ *stat* $X\,\psi(a_0, \ldots, U_0, \ldots, U_{n-1}, X)$, i.e. $\mathfrak{B} \not\models^{int} \varphi(a_0, \ldots, U_0, \ldots)$.

⊣ (Claim 3.1.1)

□ (Proposition 3.1)

4 Stationarity quantifier in PKL logic

PKL

In this section we consider a $\mathcal{P}_\kappa(\lambda)$ version of weak second-order logic with stationarity quantifier $\mathcal{L}^{PKL}_{stat}$ and SDLS for this logic in internal interpretation.

One of the significant property of this SDLS is that it implies that the reflection cardinal is very large (see Corollary 4.4) and it is consistent (modulo supercompact cardinal) with the reflection cardinal being "$< 2^{\aleph_0}$".

For sets s and t we denote with $\mathcal{P}_s(t)$ the set $[t]^{<|s|} = \{a \in \mathcal{P}(t) \,:\, |a| < |s|\}$. We say $S \subseteq \mathcal{P}_s(t)$ is stationary if it is stationary in the sense of Jech [8].

The logic $\mathcal{L}^{PKL}_{stat}$ has a built-in unary predicate symbol $\underline{K}(\cdot)$. For a structure $\mathfrak{A} = \langle A, \underline{K}^{\mathfrak{A}}, \ldots\rangle$, the weak second-order variables X, Y,... run over elements of $\mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A)$.

We shall call a structure $\mathfrak{A}$ with $\underline{K}$ in its signature as a unary predicate symbol such that $|\,\underline{K}^{\mathfrak{A}}\,|$ is a regular uncountable cardinal, a *PKL-structure*.

$\mathcal{L}^{PKL}_{stat}$ has the unique second-order quantifier "*stat*" and the internal interpretation $\models^{int}$ of formulas in this logic is defined similarly to $\mathcal{L}^{\aleph_0}_{stat}$ with the crucial step in the inductive definition of $\models^{int}$ being

(4.1) $\mathfrak{A} \models^{int}$ *stat* $X\,\varphi(a_0, \ldots, U_0, \ldots, X) \;\Leftrightarrow$ PKL-0
$\{U \in \mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A) \cap A \,:\, \mathfrak{A} \models^{int} \varphi(a_0, \ldots, U_0, \ldots, U)\}$ is stationary in $\mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A)$

for an $\mathcal{L}^{PKL}_{stat}$-formula $\varphi = \varphi(x_0, \ldots, X_0, \ldots, X)$ (for which the relation $\models^{int}$ has been defined), a PKL-structure $\mathfrak{A} = \langle A, \underline{K}^{\mathfrak{A}}, \ldots\rangle$ of a relevant signature, $a_0, \ldots \in A$ and $U_0, \ldots \in \mathcal{P}_{\underline{K}^{\mathfrak{A}}}(A) \cap A$.

For PKL-structures $\mathfrak{A}$, $\mathfrak{B}$ of the same signature with $\mathfrak{B} = \langle B, \underline{K}^{\mathfrak{B}}, \ldots\rangle$ and $\mathfrak{B} \subseteq \mathfrak{A}$, we define:

(4.2) $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A} \;\Leftrightarrow$ PKL-1

$\mathfrak{B} \models^{int} \varphi(b_0, ..., U_0, ...)$ *if and only if* $\mathfrak{A} \models^{int} \varphi(b_0, ..., U_0, ...)$ *for all* $\mathcal{L}^{PKL}_{stat}$*-formulas* φ *in the signature of the structures with* $\varphi = \varphi(x_0, ..., X_0, ...)$, $b_0, ... \in B$ *and* $U_0, ... \in \mathcal{P}_{\underset{\sqcup}{K}^{\mathfrak{B}}}(B) \cap B$.

Finally, we define the internal SDLS for this logic as follows:

Suppose that κ is a regular cardinal $> \aleph_1$.

$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$: *For any PKL-structure* $\mathfrak{A} = \langle A, \underset{\sqcup}{K}^{\mathfrak{A}}, ...\rangle$ *of countable signature with* $|\,A\,| \geq \kappa$ *and* $|\,\underset{\sqcup}{K}^{\mathfrak{A}}\,| = \kappa$*, there are stationarily many* $M \in [A]^{<\kappa}$ *such that* $\mathfrak{A} \restriction M$ *is a PKL-structure and* $\mathfrak{A} \restriction M \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

The following diagonal reflection characterizes $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$. For regular cardinals κ, λ with $\kappa \leq \lambda$, let

$(*)^{int+PKL}_{<\kappa,\lambda}$: *For any countable expansion* $\mathfrak{A}$ *of the structure* $\langle \mathcal{H}(\lambda), \kappa, \in\rangle$ *and any family* $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ *such that* S_a *is a stationary subset of* $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ *for all* $a \in \mathcal{H}(\lambda)$*, there are stationarily many* $M \in \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ *such that* $|\,\kappa \cap M\,|$ *is regular,* $\mathfrak{A} \restriction M \prec \mathfrak{A}$ *and* $S_a \cap \mathcal{P}_{\kappa\cap M}(M) \cap M$ *is stationary in* $\mathcal{P}_{\kappa\cap M}(M)$ *for all* $a \in M$.

The following Proposition 4.1 can be proved by a modification of the proof of Proposition 3.1.

P-PKL-0

Proposition 4.1 *For a regular cardinal* $\kappa > \aleph_1$*, the following are equivalent:*

(a) $(*)^{int+PKL}_{<\kappa,\lambda}$ *holds for all regular* $\lambda \geq \kappa$.

(b) $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ *holds.*

Proof. Suppose first that $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ holds. We show that $(*)^{int+PKL}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\lambda \geq \kappa$ and let $\tilde{\mathfrak{A}}$ be a countable expansion of $\langle \mathcal{H}(\lambda), \kappa, \in\rangle$ where κ is the interpretation of $\underset{\sqcup}{K}$ in this PKL-structure, $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ a sequence of stationary subsets of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ and $\mathcal{D} \subseteq [\mathcal{H}(\lambda)]^{<\kappa}$ a club.

Let

$$(\aleph 4.1)\qquad \tilde{\mathfrak{A}}^* = \langle \underbrace{\mathcal{H}(\lambda), \kappa, ..., \in}_{\tilde{\mathfrak{A}}}, \underset{\sqcup}{\vec{S}}^{\tilde{\mathfrak{A}}^*}\rangle$$ PKL-int-3

where $\underset{\sqcup}{\vec{S}}$ is a binary relation symbol and

$$(\aleph 4.2)\qquad \underset{\sqcup}{\vec{S}}^{\tilde{\mathfrak{A}}^*} = \{\langle a, s\rangle \in (\mathcal{H}(\lambda))^2 : s \in S_a\}.$$ PKL-int-4

Let $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ be such that

$$(\aleph 4.3) \quad M \in \mathcal{D} \text{ and} \qquad \text{PKL-int-5}$$

$$(\aleph 4.4) \quad \tilde{\mathfrak{A}}^* \restriction M \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \tilde{\mathfrak{A}}^*. \qquad \text{PKL-int-6}$$

By the choice of $\tilde{\mathfrak{A}}^*$ and ($\aleph$4.4), $\tilde{\mathfrak{A}}^* \restriction M \models^{int} \forall x\, stat\, X \exists y\, (\underset{\sim}{\vec{S}}(x,y) \wedge \forall z\, (z\, \varepsilon\, X \leftrightarrow z \in y))$ holds and hence, for all $a \in M$, $S_a \cap M$ is stationary in $\mathcal{P}_{\kappa \cap M}(M)$.

Suppose now that $(*)^{int+PKL}_{<\kappa,\lambda}$ holds for all $\lambda \geq \kappa$. Let $\mathfrak{A} = \langle A, \underset{\sim}{K}^{\mathfrak{A}}, ...\rangle$ be a structure in countable signature and of cardinality $\geq \kappa$ with $|\underset{\sim}{K}^{\mathfrak{A}}| = \kappa$ and $\mathcal{D} \subseteq [A]^{<\kappa}$ a club. Without loss of generality, we may assume that $\mathfrak{A}$ is a relational structure. We may also assume without loss of generality that $\kappa \subseteq A$ and $\underset{\sim}{K}^{\mathfrak{A}} = \kappa$. Let λ be a regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\lambda)$. Note that we have in particular $A \subseteq \mathcal{H}(\lambda)$.

Let $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\lambda), \underbrace{\underset{\sim}{A}^{\tilde{\mathfrak{A}}}, \kappa, ...}_{=\mathfrak{A}}, \in\rangle$ where $\underset{\sim}{A}$ is a unary relation symbol and $\underset{\sim}{A}^{\tilde{\mathfrak{A}}} = A$.

For each $a \in \mathcal{H}(\lambda)$, let

$$(\aleph 4.5) \quad S_a = \begin{cases} \{U \in \mathcal{P}_\kappa(\mathcal{H}(\lambda)) : U \cap A \in A, \\ \qquad \mathfrak{A} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap A)\}, \\ \qquad \text{if } \psi = \psi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1}, X) \text{ is an } \mathcal{L}^{PKL}_{stat}\text{-formula} \\ \qquad \text{in the signature of } \mathfrak{A},\ a_0, ..., a_{m-1} \in A,\ U_0, ..., U_{n-1} \in \mathcal{P}_\kappa(A) \cap A, \\ \qquad \mathfrak{A} \models^{int} stat\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X) \text{ and} \\ \qquad a = \langle \psi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1}\rangle; \\ \mathcal{P}_\kappa(\mathcal{H}(\lambda)), \\ \qquad \text{otherwise.} \end{cases} \qquad \text{PKL-int-7}$$

We expand $\tilde{\mathfrak{A}}$ further by adding the relation $\{\langle u, a\rangle \in (\mathcal{H}(\lambda))^2 : u \in S_a\}$ to it as the interpretation of the binary relation symbol $\underset{\sim}{\vec{S}}$. For simplicity, we shall also call this expanded structure $\tilde{\mathfrak{A}}$.

Let

$$(\aleph 4.6) \quad \tilde{\mathcal{D}} = \{U \in [\mathcal{H}(\lambda)]^{<\kappa} : U \cap A \in \mathcal{D}\}. \qquad \text{PKL-int-8}$$

$\tilde{\mathcal{D}}$ contains a club in $[\mathcal{H}(\lambda)]^{<\kappa}$.

By $(*)^{int+PKL}_{<\kappa,\lambda}$, there is an $M \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that

$$(\aleph 4.7) \quad M \in \tilde{\mathcal{D}}, \qquad \text{PKL-int-9}$$

$$(\aleph 4.8) \quad \tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}} \text{ and} \qquad \text{PKL-int-10}$$

(ℵ4.9) $S_a \cap [M]^{<\kappa} \cap M$ is stationary in $[M]^{<\kappa}$ for all $a \in M$. PKL-int-11

Let $\tilde{\mathfrak{B}} = \tilde{\mathfrak{A}} \restriction M$ and let $B = \underset{\sqcup}{A}^{\tilde{\mathfrak{B}}} = A \cap M$ and $\mathfrak{B} = \mathfrak{A} \restriction B$. By (ℵ4.8)

(ℵ4.10) $\kappa \cap M = \kappa \cap \tilde{B} = \kappa \cap B$. PKL-int-11-0

$B \in \mathcal{D}$ by (ℵ4.7) and the definition (ℵ4.6) of $\tilde{\mathcal{D}}$.

By the elementarity $\tilde{\mathfrak{B}} \prec \tilde{\mathfrak{A}}$ (ℵ4.8), the following Claim implies $\mathfrak{B} \prec^{int}_{\mathcal{L}^{PKL}_{stat}} \mathfrak{A}$.

Claim 4.1.1 *For any $\mathcal{L}^{PKL}_{stat}$-formula $\varphi(x_0, ..., x_{m-1}, Y_0, ..., Y_{n-1})$ in the signature of the structures $\mathfrak{A}$, $a_0, ..., a_{m-1} \in B$ and $U_0, ..., U_{n-1} \in \mathcal{P}_{\kappa \cap B}(B)$, we have*

(ℵ4.11) $\tilde{\mathfrak{B}} \models$ "$\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$" $\Leftrightarrow$ $\mathfrak{B} \models^{int} \varphi(a_0, ..., U_0, ...)$. PKL-int-12

$\vdash$ By induction on φ. The crucial step in the induction is when φ is of the form $stat\, X \psi$ and (ℵ4.11) holds for ψ:

Suppose first that $\tilde{\mathfrak{B}} \models$ "$\mathfrak{A} \models^{int} \varphi(a_0, ..., U_0, ...)$" holds. Then, by elementarity and by the definition of $\tilde{\mathfrak{A}}$, we have $\mathfrak{A} \models^{int} stat\, X \psi(a_0, ..., U_0, ..., U_{n-1}, X)$. Thus, letting $a = \langle \varphi, a_0, ..., a_{m-1}, U_0, ..., U_{n-1} \rangle$, we have $a \in \tilde{B}$ and

(ℵ4.12) $S_a = \{U \in \mathcal{P}_\kappa(\mathcal{H}(\lambda)) : U \cap A \in A,$
$\mathfrak{A} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap A)\}$ PKL-int-14

by the definition (ℵ4.5) of S_a.

By (ℵ4.9), $S_a \cap \mathcal{P}_{\kappa \cap \tilde{B}}(\tilde{B}) \cap \tilde{B}$ is stationary in $\mathcal{P}_{\kappa \cap \tilde{B}}(\tilde{B})$. It follows that

(ℵ4.13) $\{U \cap B : U \in S_a \cap \mathcal{P}_{\kappa \cap \tilde{B}}(\tilde{B}) \cap \tilde{B}\}$ PKL-int-15
$= \{U \cap B : U \cap B \in \mathcal{P}_{\kappa \cap B}(B) \cap B,$
$\tilde{\mathfrak{B}} \models$ "$\mathfrak{A} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, U \cap B)$"$\}$
(by elementarity, (ℵ4.10) and (ℵ4.12))
$\subseteq \{V \in \mathcal{P}_{\kappa \cap B}(B) \cap B : \mathfrak{B} \models^{int} \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)\}$
(by induction hypothesis)

is stationary. Thus $\mathfrak{B} \models^{int} stat\, X\, \psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, X)$ holds, that is, $\mathfrak{B} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$.

Suppose now that $\tilde{\mathfrak{B}} \not\models$ "$\mathfrak{A} \models^{int} \varphi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1})$". Then we have

(ℵ4.14) $\tilde{\mathfrak{B}} \models$ "there is a club $\mathcal{C} \subseteq [\underset{\sqcup}{A}]^{<\kappa}$ such that
$\mathfrak{A} \models^{int} \neg\psi(a_0, ..., U_0, ..., U_{n-1}, x)$ for all $x \in \mathcal{C}$". PKL-int-16

By elementarity, there is a $\mathcal{C}_0 \in \tilde{B}$ such that $\mathcal{C}_0$ is a club $\subseteq [A]^{<\kappa}$ and

$(\aleph 4.15)$ $\tilde{\mathfrak{B}} \models$ "$\mathfrak{A} \models^{int} \neg\psi(a_0, ..., a_{m-1}, U_0, ..., U_{n-1}, V)$" for all $V \in \mathcal{C}_0 \cap B$. PKL-int-17

By induction hypothesis, it follows that

$(\aleph 4.16)$ $\{U \in \mathcal{P}_{\kappa\cap B}(B) \cap B \,:\, \mathcal{B} \models^{int} \psi(a_0, ..., U_0, ..., U)\} \cap \mathcal{C}_0 = \emptyset$. PKL-int-17-0

Thus $\mathfrak{B} \not\models^{int} stat\, X\, \psi(a_0, ..., U_0, ..., U_{n-1}, X)$, i.e. $\mathfrak{B} \not\models^{int} \varphi(a_0, ..., U_0, ...)$.

$\dashv$ (Claim 4.1.1)

□ (Proposition 4.1)

For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is said to be 2-*stationary* if, for any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there is an $a \in \mathcal{S}$ such that $|\,\kappa \cap a\,|$ is a regular uncountable cardinal and $\mathcal{T} \cap \mathcal{P}_{\kappa\cap a}(a)$ is stationary in $\mathcal{P}_{\kappa\cap a}(a)$. A regular cardinal κ has the 2-*stationarity property* if $\mathcal{P}_\kappa(\lambda)$ is 2-stationary (as a subset of itself) for all $\lambda \geq \kappa$. More generally, we can define α-stationarity for $\alpha \leq \kappa$ and show that these generalized stationarities are compatible with κ being continuum ([2]).

P-PKL-1

Lemma 4.2 *For a regular uncountable κ, $\mathsf{SDLS}_+^{int}(\mathcal{L}_{stat}^{PKL}, <\kappa)$ implies that κ is 2-stationary.*

Proof. The property (a) in Proposition 4.1 is a strengthening of the 2-stationarity of κ. □ (Lemma 4.2)

P-PKL-2

Lemma 4.3 *Suppose that κ is a regular uncountable cardinal.*

(1) *If κ is 2-stationary then κ is a limit cardinal.*

(2) *For any $\lambda \geq \kappa$, 2-stationary $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$, and any stationary $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$, there are stationarily many $r \in \mathcal{S}$ such that $\mathcal{T} \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*

(3) *If κ is 2-stationary then κ is a weakly Mahlo cardinal.*

Proof. (1): Suppose that $\kappa = \mu^+$. Then $\mathcal{C} = \{a \in \mathcal{P}_\kappa(\lambda) \,:\, |\,a\,| = \mu\}$ is a club and hence stationary. But, for any $r \in \mathcal{P}_\kappa(\lambda)$, $|\,\kappa \cap r\,| \leq \mu$ and hence $\mathcal{C} \cap \mathcal{P}_{\kappa\cap r}(r) = \emptyset$. Thus κ is not 2-stationary.

(2): Suppose that $\mathcal{S} \subseteq \mathcal{P}_\kappa(\lambda)$ is 2-stationary and $\mathcal{T} \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Let $\mathcal{C} \subseteq \mathcal{P}_\kappa(\lambda)$ be a club. We have to show that there is $r \in \mathcal{S}\cap\mathcal{C}$ such that $\mathcal{T}\cap\mathcal{P}_{\kappa\cap r}(r)$ is stationary.

Let $f : {}^{\omega>}\lambda \to \lambda$ be such that

(4.3) $\mathcal{C}_f = \{a \in \mathcal{P}_\kappa(\lambda) \,:\, \kappa \cap a \in \kappa$ and a is closed under $f\} \subseteq \mathcal{C}$.

Since $\mathcal{C}_f$ is a club, $\mathcal{T} \cap \mathcal{C}_f$ is stationary. Let $r \in \mathcal{S}$ be such that $|\,\kappa \cap r\,|$ is regular and $(\mathcal{T} \cap \mathcal{C}_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary in $\mathcal{P}_{\kappa\cap r}(r)$. We have

Cl-a-stat-0

Claim 4.3.1 $\kappa \cap r \in \kappa$.

$\vdash$ Otherwise, there is an $\alpha \in \sup(\kappa \cap r) \setminus r$. Let $\mathcal{Y} = \{b \in \mathcal{P}_{\kappa\cap r}(r) : \sup(\kappa \cap b) > \alpha\}$. $\mathcal{Y}$ is club in $\mathcal{P}_{\kappa\cap r}(r)$ but $\mathcal{Y} \cap \mathcal{C}_f = \emptyset$. Thus $(\mathcal{T} \cap \mathcal{C}_f) \cap \mathcal{P}_{\kappa\cap r}(r) \cap \mathcal{Y} = \emptyset$. This is a contradiction to the stationarity of $(\mathcal{T} \cap \mathcal{C}_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ in $\mathcal{P}_{\kappa\cap r}(r)$. $\dashv$ (Claim 4.3.1)

Cl-a-stat-1

Claim 4.3.2 *r is closed under f.*

$\vdash$ This is clear since $\mathcal{T} \cap \mathcal{C}_f$ is cofinal in $\mathcal{P}_{\kappa\cap r}(r)$ with respect to $\subseteq$) and elements of $\mathcal{T} \cap \mathcal{C}_f$ are closed under f. $\dashv$ (Claim 4.3.2)

From the Claims above, it follows that $r \in \mathcal{C}_f \subseteq \mathcal{C}$ is as desired.

(3): Let $\mathcal{T} = \{a \in \mathcal{P}_\kappa(\lambda) : \kappa \cap a \in \kappa\}$. $\mathcal{T}$ is a club and hence stationary. Let $r \in \mathcal{P}_\kappa(\lambda)$ be such that $|\,\kappa \cap r\,|$ is regular and

(4.4) $\mathcal{T} \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary in $\mathcal{P}_{\kappa\cap r}(r)$. a-stat-2

Similarly to Claim 4.3.1, we have $\kappa \cap r \in \kappa$.

Cl-a-stat-2

Claim 4.3.3 *$\kappa \cap r$ is a cardinal.*

$\vdash$ Otherwise there is $\mu < \kappa \cap r$ such that $|\,\kappa \cap r\,| = \mu$. But then the set $\{a \in \mathcal{P}_{\kappa\cap r}(r) : \sup(a \cap \kappa) \geq \mu\}$ is a club in $\mathcal{P}_{\kappa\cap r}(r)$ disjoint from $\mathcal{T}$. $\dashv$ (Claim 4.3.3)

Cl-a-stat-3

Claim 4.3.4 *$\kappa \cap r$ is a regular cardinal.*

$\vdash$ Otherwise there is an $s \subseteq \kappa \cap r$ cofinal in $\kappa \cap r$ with $|\,s\,| < \kappa \cap r$. But then the set $\{a \in \mathcal{P}_{\kappa\cap r}(r) : \sup(a \cap \kappa) \supseteq s\}$ is a club in $\mathcal{P}_{\kappa\cap r}(r)$ disjoint from $\mathcal{T}$.

$\dashv$ (Claim 4.3.4)

Since there are stationarily many r with (4.4) by (2), it follows from Claim 4.3.4 that κ is weakly Mahlo. ❑ (Lemma 4.3)

Note that we can continue in the proof of (3) above to show that κ is weakly hyper Mahlo, weakly hyper hyper Mahlo. etc.

Cl-a-stat-4

Corollary 4.4 $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ *implies that κ is weakly Mahlo, weakly hyper Mahlo, etc.*

Proof. By Lemma 4.2, Lemma 4.3, (3) and the remark below it. ❑ (Corollary 4.4)

P-lt-conti-3:

Theorem 4.5 *Suppose that κ is a generically supercompact cardinal by μ-cc posets for some $\mu < \kappa$. Then $(*)^{int+PKL}_{<\kappa,\lambda}$ holds for all regular $\lambda \geq \kappa$.*

The proof of Theorem 4.5 can be done analogously to the proof of Theorem 2.10 noting Lemma 2.11 and

P-lt-conti-4:

Lemma 4.6 *Suppose that M is an inner model in V, λ an ordinal and $\mu \leq \lambda$ a regular cardinal in V.*

If $\mathbb{P}$ is a μ-cc poset and $\mathcal{S} \subseteq \mathcal{P}_\mu(\lambda)$ is stationary in $\mathcal{P}_\mu(\lambda)$, then $\Vdash_{\mathbb{P}}$ "$\check{\mathcal{S}}$ is stationary in $\mathcal{P}_\mu(\lambda)$".

Proof. Suppose that $\underset{\sim}{\mathcal{C}}$ is a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathcal{C}}$ is a club in $\mathcal{P}_\mu(\lambda)$".

In V, let $\mathcal{C} = \{C \in \mathcal{P}_\mu(\lambda) : \Vdash_{\mathbb{P}} \text{"}\check{C} \varepsilon \underset{\sim}{\mathcal{C}}\text{"}\}$. Then $\mathcal{C}$ is club by the μ-cc of $\mathbb{P}$. Hence $\mathcal{S} \cap \mathcal{C} \neq \emptyset$. Since $\Vdash_{\mathbb{P}}$ "$\check{\mathcal{C}} \subseteq \underset{\sim}{\mathcal{C}}$", it follows that $\Vdash_{\mathbb{P}}$ "$\check{\mathcal{S}} \cap \underset{\sim}{\mathcal{C}} \neq \emptyset$". □ (Lemma 4.6)

Proof of Theorem 4.5: For a regular $\lambda \geq \kappa$, let $\lambda^* = |\mathcal{H}(\lambda)|$ and $\lambda^{**} = (2^{\lambda^*})^+$.

Suppose that $\mathfrak{A} = \langle \mathcal{H}(\lambda), \in, \kappa, ...\rangle$ is a countable expansion of the structure $\langle \mathcal{H}(\lambda), \in, \kappa\rangle$, $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ a sequence of stationary subsets of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ and $\mathcal{C} \subseteq \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ a club.

Without loss of generality, we may assume that $\mathfrak{A}$ contains the relation $\{\langle b, a\rangle \in (\mathcal{H}(\lambda))^2 : b \in S_a\}$ coding the sequence $\langle S_a : a \in \mathcal{H}(\lambda)\rangle$ in its structure.

Let $\iota : \mathcal{H}(\lambda) \to \lambda^*$ be a bijection such that $\iota \restriction \kappa = id_\kappa$ and let $\mathfrak{A}^* = \langle \lambda^*, E^*, \kappa, ...\rangle$ be a copy of $\mathfrak{A}$ translated by ι.

Let $\langle S^*_\alpha : \alpha \in \lambda^*\rangle$ be such that

($\aleph$4.17) $S^*_\alpha = \iota'' S_{\iota^{-1}(\alpha)}$ for $\alpha \in \lambda^*$ and lt-conti-2-3:

($\aleph$4.18) $\mathcal{C}^* = \{\iota'' X : X \in \mathcal{C}\}$. lt-conti-2-4:

Thus, we have

($\aleph$4.19) $EXT_{E^*}(S^*_\alpha)$ is a stationary subset of $\mathcal{P}_\kappa(\lambda^*)$ for all $\alpha \in \lambda^*$ and lt-conti-2-5:

($\aleph$4.20) $\mathcal{C}^*$ is a club in $\mathcal{P}_\kappa(\lambda^*)$ lt-conti-2-6:

where, for $S \subseteq \lambda^*$, $EXT_{E^*}(S) = \{\{\beta \in \lambda^* : \beta \; E^* \; \alpha\} : \text{ for } \alpha \in S\}$ is the set of all extents of elements of S with respect to the element relation E^*.

It is enough to show that there is an $X \in \mathcal{C}^*$ such that

($2'$) $\mathfrak{A}^* \restriction X \prec \mathfrak{A}^*$, $\kappa \cap X < \kappa$ and

($3''$) $EXT_{E^*}(S^*_\alpha \cap X)$ is stationary in $\mathcal{P}_{\kappa \cap X}(X)$ for all $\alpha \in X$.

Let $\mathbb{P}$ be a μ-cc poset and $\mathbb{G}$ a $(\mathsf{V},\mathbb{P})$-generic filter such that there are transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preccurlyeq}{\to} M$ with

$$(\aleph 4.21)\quad \kappa = crit(j), \qquad \text{lt-conti-3:}$$

$$(\aleph 4.22)\quad j(\kappa) > \lambda^{**} \text{ and} \qquad \text{lt-conti-4:}$$

$$(\aleph 4.23)\quad j''\lambda^{**} \in M. \qquad \text{lt-conti-5:}$$

Let $X = j''\lambda^*$. $X \in M$ by ($\aleph$4.23) and Lemma 2.5, (1). Thus, we also have $j \restriction \lambda^* \in M$ and hence

$$(\aleph 4.24)\quad M \models |X| = |\lambda^*| < j(\kappa). \qquad \text{costat-2-3:}$$

Since $|\mathcal{C}^*| < \lambda^{**}$, we have $j''\mathcal{C}^* \in M$ by Lemma 2.5, (1). $M \models$ "$|j''\mathcal{C}^*| \leq |j(\mathcal{C}^*)| < j(\kappa)$". Also we have $M \models$ "$j''\mathcal{C}^*$ is directed and $\bigcup(j''\mathcal{C}^*) = X$". Since $M \models$ "$\mathcal{C}^*$ is a club in $\mathcal{P}_{j(\kappa)}(j(\lambda^*))$" by elementarity and ($\aleph$4.20), it follows that

$$(\aleph 4.25)\quad X \in j(\mathcal{C}^*). \qquad \text{costat-2-4:}$$

Let $\langle \tilde{S}^*_\alpha : \alpha < j(\lambda^*)\rangle = j(\langle S^*_\alpha : \alpha < \lambda^*\rangle)$.

By elementarity and applying Vaught's test, we can show

$$(\aleph 4.26)\quad M \models j(\mathfrak{A}^*) \restriction X \prec j(\mathfrak{A}^*).$$

For $\alpha \in \lambda^*$, since $\tilde{S}^*_{j(\alpha)} = j(S^*_\alpha) \supseteq j''S^*_\alpha$, we have

$$(\aleph 4.27)\quad M \models \text{"}EXT_{j(E^*)}(\tilde{S}^*_{j(\alpha)} \cap X) \supseteq EXT_{j(E^*)}(j''S^*_\alpha)\text{"}. \qquad \text{costat-2-5:}$$

By ($\aleph$4.19) and since $\mathbb{P}$ is μ-cc, we have
$\mathsf{V}[\mathbb{G}] \models$ "$EXT_{E^*}(S^*_\alpha)$ is stationary in $\mathcal{P}_\kappa(\lambda^*)$" by Lemma 4.6.
Since $EXT_{j(E^*)}(j''S^*_\alpha)$ is a translation of $EXT_{E^*}(S^*_\alpha)$ induced by $j \restriction \lambda^*$, it follows that

$$(\aleph 4.28)\quad \mathsf{V}[\mathbb{G}] \models \text{"}EXT_{j(E^*)}(j''S^*_\alpha) \text{ is stationary in } \mathcal{P}_\kappa(X)\text{"}. \qquad \text{costat-2-6:}$$

By Lemma 2.11, it follows that

$$(\aleph 4.29)\quad M \models \text{"}EXT_{j(E^*)}(j''S^*_\alpha) \text{ is stationary in } \mathcal{P}_\kappa(X)\text{"}. \qquad \text{costat-2-7:}$$

Noting that $j(\kappa) \cap X = \kappa$,

$$(\aleph 4.30)\quad M \models \text{"}\exists X \in j(\mathcal{C}^*) \big(j(\mathfrak{A}^*) \restriction X \prec j(\mathfrak{A}^*) \land EXT_{j(E^*)}(\tilde{S}^*_\xi \cap X) \text{ is a stationary subset of } \mathcal{P}_{j(\kappa)\cap X}(X) \text{ for all } \xi \in X\big)\text{"}.$$

By elementarity, it follows that

$$(\aleph 4.31)\quad \mathsf{V} \models \text{"}\exists X \in \mathcal{C}^* \big(\mathfrak{A}^* \restriction X \prec \mathfrak{A}^* \ \land \ EXT_{E^*}(S^*_\xi \cap X) \text{ is a stationary subset of } \mathcal{P}_{\kappa\cap X}(X) \text{ for all } \xi \in X\big)\text{"}.$$

□ (Theorem 4.5)

5 Laver-generic large cardinals

laver

In the following, we assume that the classes $\mathcal{P}$ of posets we consider are always closed under isomorphism and forcing equivalence.

For a cardinal κ and a class $\mathcal{P}$ of posets, we call κ a *Laver-generically supercompact for* $\mathcal{P}$ if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there are a poset $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$ and $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ such that there are an inner model $M \subseteq \mathsf{V}[\mathbb{H}]$ and a class $j \subseteq \mathsf{V}[\mathbb{H}]$ with

(5.1) $j : \mathsf{V} \stackrel{\preceq}{\rightarrow} M$, laver-a

(5.2) $crit(j) = \kappa$, $j(\kappa) > \lambda$, laver-0

(5.3) $\mathbb{P}, \mathbb{H} \in M$ and laver-0-0

(5.4) $j''\lambda \in M$. laver-1

κ is *Laver-generically superhuge* (*Laver-generically super almost-huge* resp.) *for* $\mathcal{P}$ if κ satisfies the definition of Laver-generic supercompactness for $\mathcal{P}$ with (5.4) replaced by

(5.4)$'$ $j''j(\kappa) \in M$ ($j''\mu \in M$ for all $\mu < j(\kappa)$ resp.).

κ is *tightly Laver-generically supercompact* (*tightly Laver-generically superhuge, tightly Laver-generically super almost-huge*, resp.) if the definition of Laver-generically supercompact (Laver-generically superhuge, Laver-generically super almost-huge, resp.) holds with (5.2) replaced by

(5.2)$'$ $crit(j) = \kappa$, $j(\kappa) = |\,\mathbb{Q}\,| > \lambda$.

All consistency proofs of the existence of Laver-generic very large cardinals we know actually show the existence of tightly Laver-generic very large cardinals (see the proof of Theorem 5.2).

The following is clear by definition.

L-laver-0

Lemma 5.1 *Suppose that $\mathcal{P}$ is a class of posets.* (1) *If κ is Laver-generically superhuge for $\mathcal{P}$ then κ is Laver-generically super almost-huge for $\mathcal{P}$. If κ is Laver-generically super almost-huge for $\mathbb{P}$ then κ is Laver-generically supercompact for $\mathcal{P}$. If κ is Laver-generically supercompact for $\mathcal{P}$ then κ is generically supercompact by $\mathcal{P}$.*

(2) *If κ is generically supercompact by $\mathcal{P}$ then κ is generically measurable by some $\mathbb{P} \in \mathcal{P}$.*

(3) *If κ is tightly Laver-generically supercompact (super almost-huge, superhuge, resp.) for $\mathcal{P}$ then κ is Laver-generically supercompact (super almost-huge, huge, resp.) for $\mathcal{P}$.*

(4) *If κ is Laver-generically supercompact for $\mathcal{P}$ then, for any $\mathbb{P} \in \mathcal{P}$, there is $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \leqslant \mathbb{Q}$ such that κ is generically measurable by $\mathbb{Q}$.* ❑

The following Theorem 5.2 can be still improved and extended: among other things the assumption of the consistency of the existence of a superhuge cardinal in (2) and (3) can be reduced to that of the existence of a super almost-huge cardinal. We shall discuss more about these improvements in [7].

T-laver-a

Theorem 5.2 (1) *Suppose that* ZFC + *"there exists a supercompact cardinal (super almost-huge cardinal, superhuge cardinal, resp.)" is consistent. Then* ZFC + *"there exists a tightly Laver-generically supercompact cardinal (super almost-huge cardinal, superhuge cardinal, resp.) for σ-closed posets" is consistent as well.*

(2) *Suppose that* ZFC + *"there exists a superhuge cardinal" is consistent. Then* ZFC + *"there exists a tightly Laver-generically super almost-huge cardinal for proper posets" is consistent as well.*

(3) *Suppose that* ZFC + *"there exists a supercompact cardinal (superhuge cardinal, resp.)" is consistent. Then* ZFC + *"there exists a tightly Laver-generically supercompact cardinal (super almost-huge cardinal, resp.) for ccc posets" is consistent as well.*

Proof. (1): Suppose that κ is a supercompact cardinal (the case with a super almost-huge cardinal or a superhuge cardinal can be treated similarly).

We show that $\mathbb{C} = \mathrm{Col}(\omega_1, \kappa)$ forces that κ is Laver-generically supercompact cardinal for σ-closed posets.

Let $\mathbb{G}_0$ be a $(\mathsf{V}, \mathbb{C})$-generic filter and $\mathbb{P}$ be a σ-closed poset in $\mathsf{V}[\mathbb{G}_0]$ and $\lambda \geq \kappa$. We may assume that $\mathsf{V}[\mathbb{G}_0] \models$ "$\lambda \geq |\,\mathbb{P}\,|$".

Let $j : \mathsf{V} \overset{\preceq}{\to} M \subseteq \mathsf{V}$ be such that $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $[M]^\lambda \subseteq M$. Note that, by the σ-closedness of $\mathbb{C}$ and closedness property of M, we have $\mathrm{Col}(\omega_1, j(\kappa))^{\mathsf{V}} = \mathrm{Col}(\omega_1, j(\kappa))^{\mathsf{V}[\mathbb{G}_0]} = \mathrm{Col}(\omega_1, j(\kappa))^M$. Thus we denote this poset simply by $\mathrm{Col}(\omega_1, j(\kappa))$.

We have

$$M \models \text{“} j(\mathbb{C}) = \mathrm{Col}(\omega_1, j(\kappa)) \sim \mathbb{C} \times \mathrm{Col}(\omega_1, j(\kappa) \setminus \kappa) \sim \mathbb{C} \times \mathrm{Col}(\omega_1, j(\kappa))\text{”}. \tag{5.5}$$ laver-1-16

By Corollary 1.6 we also have

$$\mathsf{V}[\mathbb{G}_0] \models \text{“}\mathbb{P} \leqslant \mathrm{Col}(\omega_1, j(\kappa))\text{”}. \tag{5.6}$$ laver-1-16-0

Note that $\mathsf{V}[\mathbb{G}_0] \models$ "$|\operatorname{Col}(\omega_1, j(\kappa))| = j(\kappa)$".

Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}_0], \operatorname{Col}(\omega_1, j(\kappa)))$-generic filter and let $\mathbb{H}^*$ be the $(\mathsf{V}, j(\mathbb{C}))$-generic filter extending $\mathbb{G}_0$ which corresponds to $\mathbb{G}_0 * \mathbb{H}$ via (5.5). We have $\mathsf{V}[\mathbb{G}_0][\mathbb{H}] = \mathsf{V}[\mathbb{H}^*]$. Let

$$(5.7) \qquad j^* : \mathsf{V}[\mathbb{G}_0] \xrightarrow{\preceq} M[\mathbb{G}^*] \subseteq \mathsf{V}[\mathbb{G}^*];\ \underset{\sim}{a}[\mathbb{G}_0] \mapsto j(\underset{\sim}{a})[\mathbb{H}^*].$$ laver-1-17

Then we have $\mathit{crit}(j^*) = \mathit{crit}(j) = \kappa$, $j^*(\kappa) > \lambda$, $j^{*\prime\prime}\lambda = j''\lambda \in M \subseteq M[\mathbb{H}^*]$ and $M[\mathbb{H}^*]$ is an inner model of $\mathsf{V}[\mathbb{G}_0][\mathbb{H}]$. We also have $\mathbb{H} \in M[\mathbb{H}^*]$ since $\underset{\sim}{\mathbb{H}}^* \in M$. Since λ can be taken arbitrarily large, this shows that κ in $\mathsf{V}[\mathbb{G}_0]$ is tightly Laver-generically supercompact for σ-closed forcing.

(2): Suppose that κ is a superhuge cardinal. Then, by [3], there is a super almost-huge Laver-function $f : \kappa \to \mathsf{V}_\kappa$. Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a CS-iteration of proper posets such that

$$(5.8) \qquad \underset{\sim}{\mathbb{Q}}_\beta = \begin{cases} f(\beta), & \text{if } f(\alpha) \text{ is a } \mathbb{P}_\beta\text{-name and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\alpha) \text{ is a proper poset”;} \\ \mathbb{P}_\alpha\text{-name of the trivial poset,} & \text{otherwise.} \end{cases}$$ laver-1-18

Let $\mathbb{G}_0$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. We show that, in $\mathsf{V}[\mathbb{G}_0]$, κ is a tightly Laver-generically super almost-huge cardinal for proper posets.

Working in $\mathsf{V}[\mathbb{G}_0]$, suppose that $\mathbb{P}$ is a proper poset and $\lambda \geq \kappa$. Let $\underset{\sim}{\mathbb{P}}$ be be $\mathbb{P}_\kappa$-name of $\mathbb{P}$.

Back in V, let $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}$ be such that

$$(5.9) \qquad \mathit{crit}(j) = \kappa,$$ laver-1-19

$$(5.10) \qquad j(\kappa) > \lambda,$$ laver-1-20

$$(5.11) \qquad [M]^{<j(\kappa)} \subseteq M, \text{ and}$$ laver-1-21

$$(5.12) \qquad f(\kappa) = \underset{\sim}{\mathbb{P}}.$$ laver-1-22

By elementarity, we have

$$(5.13) \qquad M \models \text{“} j(\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle) \text{ and } j(\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta < \kappa \rangle) \text{ make up a CS-iteration of proper posets of length } j(\kappa) \text{, and each name in the sequence } j(\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta < \kappa \rangle) \text{ is of size } < j(\kappa)\text{”}.$$ laver-1-23

By (5.11), it follows that the statement in (5.13) also holds in V. That is,

$$(5.14) \qquad V \models \text{“} j(\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle) \text{ and } j(\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta < \kappa \rangle) \text{ make up a CS-iteration of proper posets of length } j(\kappa) \text{, and each name in the sequence } j(\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta < \kappa \rangle) \text{ is of size } < j(\kappa)\text{”}.$$ laver-1-23-0

In V, let

(5.15) $\langle \mathbb{P}^*_\alpha : \alpha \leq j(\kappa) \rangle = j(\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle)$ and laver-1-24

(5.16) $\langle \underset{\sim}{\mathbb{Q}}^*_\beta : \beta < j(\kappa) \rangle = j(\langle \underset{\sim}{\mathbb{Q}}_\beta : \beta < \kappa \rangle)$. laver-1-25

By elementarity of j and since $crit(j) = \kappa$, we have $\mathbb{P}_\alpha = \mathbb{P}^*_\alpha$ for all $\alpha \leq \kappa$ and $\underset{\sim}{\mathbb{Q}}^*_\beta = \underset{\sim}{\mathbb{Q}}_\beta$ for all $\beta < \kappa$. In $\mathsf{V}[\mathbb{G}_0]$, let $\mathbb{Q} = j(\mathbb{P}_\kappa)/\mathbb{G}_0$.

By (5.14), (5.12), (5.15), (5.16) and Factor Lemma, $\mathbb{Q}$ is a proper poset of cardinality $j(\kappa)$ and $\mathbb{P} \lessdot \mathbb{Q}$.

Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}_0], \mathbb{Q})$-generic filter and

(5.17) $\tilde{\jmath} : \mathsf{V}[\mathbb{G}_0] \overset{\preceq}{\to} M[\mathbb{G}_0 * \mathbb{H}] \subseteq \mathsf{V}[\mathbb{G}_0][\mathbb{H}];\ \underset{\sim}{a}[\mathbb{G}_0] \mapsto j(\underset{\sim}{a})[\mathbb{G}_0 * \mathbb{H}]$. laver-1-26

Then $\mathbb{Q} \in M[\mathbb{G}_0 * \mathbb{H}]$ and $\tilde{\jmath}''\mu = j''\mu \in M \subseteq M[\mathbb{G}_0 * \mathbb{H}]$ for all $\mu < j(\kappa)$.

(3): The proof is quite similar to that of (2). Starting from a supercompact (super almost-huge, resp.) Laver function, we define a FS-iteration of ccc posets with the definition of $\underset{\sim}{\mathbb{Q}}_\beta$ similarly to (5.8). The step corresponding to the one from (5.13) to (5.15) is now easily done by $[M]^{\aleph_1} \subseteq M$ instead of (5.11) since we only need to check subsets of a poset of size $\aleph_1$ to conclude that the poset has the ccc.

❑ (Theorem 5.2)

We show in the following that generic large cardinal property of κ have very strong influence on the cardinal arithmetic around κ.

A poset $\mathbb{P}$ is *ω_1-preserving* if it satisfies $\Vdash_{\mathbb{P}}$ "$(\omega_1)^{\mathsf{V}} \equiv \omega_1$".

L-laver-1

Lemma 5.3 *Suppose that κ is generically measurable by a ω_1-preserving $\mathbb{P}$. Then $\kappa > \omega_1$.*

Proof. Suppose otherwise. Since κ cannot be ω, we have then $\kappa = \omega_1$. Thus there is an ω_1-preserving poset $\mathbb{P}$ and a $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ such that there are transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preceq}{\to} M$ with $crit(j) = \omega_1$. Since $M \models$ "$(\omega_1)^{\mathsf{V}} < j(\kappa) = \omega_1$", we have $M \models$ "$(\omega_1)^{\mathsf{V}}$ is countable" and hence $\mathsf{V}[\mathbb{G}] \models$ "$(\omega_1)^{\mathsf{V}}$ is countable". This is a contradiction to the ω_1-preserving of $\mathbb{P}$. ❑ (Lemma 5.3)

L-laver-2

Lemma 5.4 *Suppose that κ is Laver-generically supercompact for ω_1-preserving $\mathcal{P}$ with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$. *Then we have $\kappa = \omega_2$.*

Proof. $\kappa > \omega_1$ by Lemma 5.3. Suppose, toward a contradiction, that $\kappa > \omega_2$. Let $\mathbb{P} = \mathrm{Col}(\omega_1, \{\omega_2\})$ and let $\mathbb{Q} \in \mathcal{P}$ be such that $\mathbb{P} \lessdot \mathbb{Q}$ and, for a $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$, there are transitive $M \subseteq \mathsf{V}[\mathbb{H}]$ and $j : \mathsf{V} \overset{\preceq}{\to} M$ with $crit(j) = \kappa$. Since $(\omega_2)^{\mathsf{V}} < \kappa$, $j((\omega_2)^{\mathsf{V}}) = (\omega_2)^{\mathsf{V}}$. Thus, by elementarity, $M \models$ "$(\omega_2)^{\mathsf{V}}$ is the second uncountable cardinal". But since the $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{H} \cap \mathbb{P}$ is in M, $M \models$ "$|(\omega_2)^{\mathsf{V}}| = \aleph_1$". This is a contradiction. ❑ (Lemma 5.4)

T-laver-1

Lemma 5.5 *Suppose that $\mathcal{P}$ is a class of posets containing a poset $\mathbb{P}$ such that any $(\mathsf{V},\mathbb{P})$-generic filter $\mathbb{G}$ codes a new real. If κ is a Laver-generically supercompact for $\mathcal{P}$, then $\kappa \leq 2^{\aleph_0}$.*

Proof. Let $\mathbb{P} \in \mathcal{P}$ be such that any generic filter over $\mathbb{P}$ codes a new real.

Suppose that $\mu < \kappa$. We show that $2^{\aleph_0} > \mu$. Let $\vec{a} = \langle a_\xi : \xi < \mu\rangle$ be a sequence of subsets of ω. It is enough to show that $\vec{a}$ does not enumerate $\mathcal{P}(\omega)$. By Laver-generic supercompactness of κ for $\mathcal{P}$, there are $\mathbb{Q} \in \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$, $(\mathsf{V},\mathbb{Q})$-generic $\mathbb{H}$, transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$ with $crit(j) = \kappa$ and $\mathbb{P}, \mathbb{H} \in M$. Since $\mu < \kappa$, we have $j(\vec{a}) = \vec{a}$. Since $\mathbb{G} \in M$ where $\mathbb{G} = \mathbb{H} \cap \mathbb{P}$ and $\mathbb{G}$ codes a new real not in V, we have

(5.18) $\quad M \models$ "$j(\vec{a})$ does not enumerate $2^{\aleph_0}$".

By elementarity, it follows that

(5.19) $\quad \mathsf{V} \models$ "$\vec{a}$ does not enumerate $2^{\aleph_0}$". ❑ (Lemma 5.5)

T-laver-1-a

Lemma 5.6 *Suppose that $\mathcal{P}$ is a class of posets such that elements of $\mathcal{P}$ do not add any reals. If κ is generically supercompact by $\mathcal{P}$, then we have $2^{\aleph_0} < \kappa$.*

Let $\lambda > 2^{\aleph_0}$ and let $\mathbb{P} \in \mathcal{P}$ be such that there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ for a $(\mathsf{V},\mathbb{P})$-generic filter $\mathbb{G}$ with $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Since $\mathbb{P}$ does not add any new reals $\mathsf{V}[\mathbb{G}] \models$ "$2^{\aleph_0} < j(\kappa)$". By Lemma 2.5, (3), it follows that $M \models$ "$2^{\aleph_0} < j(\kappa)$". Thus, by elementarity, $\mathsf{V} \models$ "$2^{\aleph_0} < \kappa$". ❑ (Lemma 5.6)

For a class $\mathcal{P}$ of posets and cardinals μ, κ, we consider the following strengthening of the forcing axiom for $\mathcal{P}$:

$\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\kappa)$: *For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$* is a stationary subset of ω_1" *for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.*

For a poset $\mathbb{P}$, $\mathbb{P}$-name $\underset{\sim}{S}$ of a set of subsets of On and a filter $\mathbb{G}$ on $\mathbb{P}$, let

(5.20) $\quad \underset{\sim}{S}(\mathbb{G}) = \{b : b = \{\alpha \in \mathrm{On} : \mathbb{p} \Vdash_{\mathbb{P}} \text{"}\check{\alpha}\ \varepsilon\ \underset{\sim}{s}\text{"} \text{ for a } \mathbb{p} \in \mathbb{G}\} \text{ for a } \mathbb{P}\text{-name } \underset{\sim}{s}$ laver-1-27
$\text{such that } \Vdash_{\mathbb{P}} \text{"}\underset{\sim}{s}\ \varepsilon\ \underset{\sim}{S} \text{ and } \sup(\underset{\sim}{s}) \equiv \sup(b)\text{"}\}.$

Note that if $\mathbb{G}$ is a $(\mathsf{V},\mathbb{P})$-generic filter, then $\underset{\sim}{S}(\mathbb{G}) = \underset{\sim}{S}[\mathbb{G}]$. [...]

For uncountable cardinals μ and $\kappa > \aleph_1$, let $\mathsf{MA}^{++\mu}(\mathcal{P}, {<}\kappa)$ be the strengthening of $\mathsf{MA}^{+\mu}(\mathcal{P}, {<}\kappa)$ defined by:

$\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$: *For any* $\mathbb{P}\in\mathcal{P}$*, any family* $\mathcal{D}$ *of dense subsets of* $\mathbb{P}$ *with* $|\mathcal{D}|<\kappa$ *and any family* $\mathcal{S}$ *of* $\mathbb{P}$*-names such that* $|\mathcal{S}|\leq\mu$ *and* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" *for some* $\omega<\eta_{\underset{\sim}{S}}\leq\theta_{\underset{\sim}{S}}\leq\mu$ *with* $\eta_{\underset{\sim}{S}}$ *regular, for all* $\underset{\sim}{S}\in\mathcal{S}$*, there is a* $\mathcal{D}$*-generic filter* $\mathbb{G}$ *over* $\mathbb{P}$ *such that* $\underset{\sim}{S}(\mathbb{G})$ *is stationary in* $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ *for all* $\underset{\sim}{S}\in\mathcal{S}$*.*

Clearly $\mathsf{MA}^{++\omega_1}(\mathcal{P},<\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P},<\kappa)$.

T-laver-0

Theorem 5.7 *For an arbitrary class* $\mathcal{P}$ *of posets, if* $\kappa>\aleph_1$ *is a Laver-generically supercompact for* $\mathcal{P}$*, then* $\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$ *holds for all* $\mu<\kappa$*.*

Proof. Let $\mathbb{P}\in\mathcal{P}$ and $\mu<\kappa$. Let $\mathcal{D}$ and $\mathcal{S}$ be as in the definition of $\mathsf{MA}^{++\mu}(\mathcal{P},<\kappa)$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}$ is some cardinal λ_0 and elements of $\mathcal{S}$ are nice $\mathbb{P}$-names.

Let $\lambda>\lambda_0$ be sufficiently large and let $\mathbb{Q}\in\mathcal{P}$ be such that $\mathbb{P}\lessdot\mathbb{Q}$ and, for a $(\mathsf{V},\mathbb{Q})$-generic filter $\mathbb{H}$, there are transitive $M\subseteq\mathsf{V}[\mathbb{H}]$ and $j:\mathsf{V}\xrightarrow{\preceq}M$ with

(5.2) $\quad crit(j)=\kappa$, $j(\kappa)>\lambda$,

(5.3) $\quad \mathbb{P}$, $\mathbb{H}\in M$ and

(5.4) $\quad j''\lambda\in M$.

Without loss of generality, we may assume that $\mathbb{P}$ is a complete subordering of $\mathbb{Q}$.

By the choice of λ, (5.4) and Lemma 2.5, (5), we have $\mathbb{P}$, $\mathcal{D}$, $\mathcal{S}\in M$. Let $\mathbb{G}=\mathbb{H}\cap\mathbb{P}$. Then $\mathbb{G}\in M$ by (5.3). Thus $\mathbb{G}$ witnesses

(5.21) $\quad M\models$ "there is a $\mathcal{D}$-generic filter G over $\mathbb{P}$ laver-1-3
such that $\underset{\sim}{S}(G)$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S}\in\mathcal{S}$".

Since $j(\mathcal{D})=\{j(D):D\in\mathcal{D}\}$ and $j(\mathcal{S})=\{j(S):S\in\mathcal{S}\}$ by $|\mathcal{D}|$, $|\mathcal{S}|<\kappa$, $j(D)\supseteq j''D$ for all $D\in\mathcal{D}$, $j(S)\supseteq j''S$ for all $S\in\mathcal{S}$ and $j''\mathbb{G}\in M$ by Lemma 2.5, (6), it follows that

(5.22) $\quad M\models$ "there is a $j(\mathcal{D})$-generic filter G over $j(\mathbb{P})$ laver-1-4
such that $\underset{\sim}{S}(G)$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S}\in j(\mathcal{S})$".

By elementarity, it follows that

(5.23) $\quad \mathsf{V}\models$ "there is a $\mathcal{D}$-generic filter G over $\mathbb{P}$ laver-1-5
such that $\underset{\sim}{S}(G)$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S}\in\mathcal{S}$".

□ (Theorem 5.7)

At the moment we do not know if Laver-generic supercompactness of κ for ccc posets implies $\kappa = 2^{\aleph_0}$ (note that we have $\kappa \leq 2^{\aleph_0}$ by Lemma 5.5). However

T-laver-1-2

Theorem 5.8 *If κ is tightly Laver-generically superhuge for ccc posets, then $\kappa = 2^{\aleph_0}$.*

Proof. Suppose that κ is tightly Laver-generically superhuge for ccc posets. By Lemma 5.5, we have $2^{\aleph_0} \geq \kappa$.

To prove $2^{\aleph_0} \leq \kappa$, let $\lambda \geq \kappa$, $2^{\aleph_0}$ be large enough and let $\mathbb{Q}$ be a ccc poset such that there are $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{H}]$ with $crit(j) = \kappa$, $|\,\mathbb{Q}\,| \leq j(\kappa) > \lambda$, $\mathbb{H} \in M$ and $j''j(\kappa) \in M$.

Since $M \models$ "$j(\kappa)$ is regular" by elementarity, $j(\kappa)$ is regular in V (e.g. by Lemma 2.5, (3)). Thus, we have $\mathsf{V} \models$ "$j(\kappa)^{\aleph_0} = j(\kappa)$" by Proposition 2.8, (1). Since $\mathbb{Q}$ has the ccc and $|\,\mathbb{Q}\,| \leq j(\kappa)$, it follows that $\mathsf{V}[\mathbb{G}] \models$ "$2^{\aleph_0} \leq j(\kappa)$". Now by Lemma 2.5, (4), $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}[\mathbb{G}]}$. Thus $M \models$ "$2^{\aleph_0} \leq j(\kappa)$".

By elementarity, it follows that $\mathsf{V} \models$ "$2^{\aleph_0} \leq \kappa$". □ (Theorem 5.8)

T-laver-2

Theorem 5.9 (1) *Suppose that κ is Laver-generically supercompact for σ-closed posets. Then $2^{\aleph_0} = \aleph_1$, $\kappa = \aleph_2$, $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ and hence also $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ holds.*

(2) *Suppose that κ is Laver-generically supercompact for proper posets. Then $2^{\aleph_0} = \kappa = \aleph_2$, $\mathsf{PFA}^{+\omega_1}$ and hence also $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <2^{\aleph_0})$ holds.*

(3) *Suppose that κ is Laver-generically supercompact for ccc posets. Then $2^{\aleph_0} \geq \kappa$ and $\mathcal{P}_\kappa(\lambda)$ for any regular $\lambda \geq \kappa$ carries an $\aleph_1$-saturated normal ideal. In particular, κ is κ-weakly Mahlo. $\mathsf{MA}^{++\mu}(ccc, <\kappa)$ for all $\mu < \kappa$, $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ and $\mathsf{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, <\kappa)$ also hold.*

Proof. (1): Assume that κ is Laver-generically supercompact for σ-closed posets. Then $\kappa = \aleph_2$ by Lemma 5.4. Hence $2^{\aleph_0} = \aleph_1$ by Lemma 5.6. $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ holds by Theorem 5.7. $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ (see [4]). Hence, by Corollary 1.3, (4), $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ holds.

(2): Assume that κ is Laver-generically supercompact for proper posets. Then $\kappa = \aleph_2$ by Lemma 5.4. $\kappa \leq 2^{\aleph_0}$ by Lemma 5.5 and $\mathsf{PFA}^{+\omega_1}$ by Theorem 5.7. Since PFA implies $2^{\aleph_0} = \aleph_2$, we obtain $\kappa = 2^{\aleph_0}$.

$\mathsf{PFA}^{+\omega_1}$ implies $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ and $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ follows from $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$ ([4]). Thus, by Corollary 1.3, (3), $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2)$ holds.

(3): Assume that κ is Laver-generically supercompact for ccc posets. Then $\kappa \leq 2^{\aleph_0}$ by Lemma 5.5.

$\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa)$ holds by Theorem 2.10 and Proposition 3.1. $\mathsf{SDLS}^{int}(\mathcal{L}^{PKL}_{stat}, < \kappa)$ holds by Theorem 4.5 and Proposition 4.1. For any regular $\lambda \geq \kappa$ $\mathcal{P}_\kappa(\lambda)$ carries an $\aleph_1$-saturated normal ideal by Lemma 2.7, (2). $\mathsf{MA}^{++\mu}(ccc, < \kappa)$ for all $\mu < \kappa$ by Theorem 5.7. ❏ (Theorem 5.9)

References

ref

[1] James E. Baumgartner and Alan D. Taylor, Saturation properties of Ideals in generic extensions I, Transactions of the American Mathematical Society, Vol.270, No.2, (1982), 557–574.

[2] Hazel Brickhill, Sakaé Fuchino and Hiroshi Sakai, On α-stationary subsetes of $\mathcal{P}_\kappa(\lambda)$, in preparation.

[3] Paul Corazza, Laver Sequences for Extendible and Super-Almost-Huge Cardinals, The Journal of Symbolic Logic, Vol.64, No.3 (1999), 963–983.

[4] Sean Cox, The diagonal reflection principle, Proceedings of the American Mathematical Society, Vol.140, No.8 (2012), 2893-2902.

[5] ________, Forcing axioms, approachability, and stationary set reflection, preprint.

[6] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, preprint. http://fuchino.ddo.jp/papers/SDLS-x.pdf

[7] ________________, Strong downward Löwenheim-Skolem theorems for stationary logics, III — more on Laver-generically large cardinals and mixed support iterations, in preparation.

[8] T. Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[9] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994/2003).

[10] Bernhard König, Generic compactness reformulated, Archive of Mathematical Logic 43, (2004), 311–326.

[11] Yo Matsubara, Hiroshi Sakai and Toshimichi Usuba, On the existence of skinny stationary subsets, Annals of Pure and Applied Logic 170, (2019) 539–557.