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How to drive our families mad

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Abstract. Given a family $\mathcal{F}$ of pairwise almost disjoint (ad) sets on a countable set S, we study families $\tilde{\mathcal{F}}$ of maximal almost disjoint (mad) sets extending $\mathcal{F}$.

We define $\mathfrak{a}^+(\mathcal{F})$ to be the minimal possible cardinality of $\tilde{\mathcal{F}} \setminus \mathcal{F}$ for such $\tilde{\mathcal{F}}$ and $\mathfrak{a}^+(\kappa) = \max\{\mathfrak{a}^+(\mathcal{F}) : |\mathcal{F}| \leq \kappa\}$. We show that all infinite cardinals less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some ad $\mathcal{F}$ (Theorem 10) and that the inequalities $\aleph_1 = \mathfrak{a} < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ (Corollary 1) and $\mathfrak{a} = \mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ (Theorem 9) are both consistent.

We also give several constructions of mad families with some additional properties.

1. Introduction

Given a family $\mathcal{F}$ of pairwise almost disjoint countable sets, we can ask how the maximal almost disjoint (mad) families extending $\mathcal{F}$ look like. In this

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note and [5], we address some instances of this question and other related problems.

Let us begin with the definition of some notions and notation about almost disjointness we shall use here. Two countable sets A, B are said to be *almost disjoint* (*ad* for short) if $A \cap B$ is finite. A family $\mathcal{F}$ of countable sets is said to be *pairwise almost disjoint* (*ad* for short) if any two distinct A, $B \in \mathcal{F}$ are ad.

If $\mathcal{X} \subseteq [S]^{\aleph_0}$ and $S = \bigcup \mathcal{X}$, $\mathcal{F} \subseteq \mathcal{X}$ is said to be *mad in* $\mathcal{X}$ if $\mathcal{F}$ is ad and there is no ad $\mathcal{F}'$ such that $\mathcal{F} \subsetneqq \mathcal{F}' \subseteq \mathcal{X}$. Thus an ad family $\mathcal{F}$ is mad in $\mathcal{X}$ if and only if there is no $X \in \mathcal{X}$ which is ad from every $Y \in \mathcal{F}$. If $\mathcal{F}$ is mad in $[S]^{\aleph_0}$ for $S = \bigcup \mathcal{F}$, we say simply that $\mathcal{F}$ is a mad family (on S). S is called the *underlying set* of $\mathcal{F}$.

Let

$$(1.1) \quad \mathfrak{a}(\mathcal{X}) = \min\{|\,\mathcal{F}\,| \,:\, |\,\mathcal{F}\,| \geq \aleph_0 \text{ and } \mathcal{F} \text{ is mad in } \mathcal{X}\}.$$

Clearly, the cardinal invariant $\mathfrak{a}$ known as the almost disjoint number ([2]) can be characterized as:

Example 1. $\mathfrak{a} = \mathfrak{a}([S]^{\aleph_0})$ for any countable S.

In this paper we concentrate on the case where the underlying set $S = \bigcup \mathcal{X}$ (or $S = \bigcup \mathcal{F}$) is countable. In [5] and the forthcoming continuation of this paper, we will deal with the cases where S may be also uncountable.

As the countable $S = \bigcup \mathcal{X}$, we often use ω or $T = {}^{\omega>}2$ where T is considered as a tree growing downwards. That is, for b, $b' \in T$, we write $b' \leq_T b$ if $b \subseteq b'$. Each $f \in {}^{\omega}2$ induces the (maximal) branch

$$(1.2) \quad B(f) = \{f \restriction n \,:\, n \in \omega\} \subseteq T$$

in T.

In Section 2, we consider several cardinal invariants of the form $\mathfrak{a}(\mathcal{X})$ for some $\mathcal{X} \subseteq [T]^{\aleph_0}$.

For $\mathcal{X} \subseteq [S]^{\aleph_0}$ with $S = \bigcup \mathcal{X}$, let

$$(1.3) \quad \mathcal{X}^{\perp} = \{Y \in [S]^{\aleph_0} \,:\, \forall X \in \mathcal{X}\ |\,X \cap Y\,| < \aleph_0\}.$$

If $Y \in \mathcal{X}^{\perp}$ we shall say that Y is *almost disjoint* (ad) *to* $\mathcal{X}$.

For an ad family $\mathcal{F}$, let

$$(1.4) \quad \mathfrak{a}^{+}(\mathcal{F}) = \mathfrak{a}(\mathcal{F}^{\perp}).$$

For a cardinal κ, let

$$(1.5) \quad \mathfrak{a}^{+}(\kappa) = \sup\{\mathfrak{a}^{+}(\mathcal{F}) \,:\, \mathcal{F} \text{ is an ad family on } \omega \text{ of cardinality } \leq \kappa\}.$$

Clearly, $\mathfrak{a}^{+}(\omega) = \mathfrak{a}$ and $\mathfrak{a}^{+}(\kappa) \leq \mathfrak{a}^{+}(\lambda) \leq \mathfrak{c}$ for any $\kappa \leq \lambda \leq \mathfrak{c}$. In Section 3 we give several constructions of ad families $\mathcal{F}$ for which $\mathcal{F}^{\perp}$ has some particular property. Using these constructions, we show in Section 4 that $\mathfrak{a}^{+}(\mathfrak{c}) = \mathfrak{c}$ (actually we have $\mathfrak{a}^{+}(\bar{\mathfrak{o}}) = \mathfrak{c}$, see Theorem 7) and the consistency of the inequalities $\mathfrak{a} = \aleph_1 < \mathfrak{a}^{+}(\aleph_1) = \mathfrak{c}$ (see Corollary 1). We also show the consistency of $\mathfrak{a}^{+}(\aleph_1) < \mathfrak{c}$ (Theorem 9).

For undefined notions connected to the forcing, the reader may consult [7] or [8]. We mostly follow the notation and conventions set in [7] and/or [8]. In particular, the forcing is denoted in such a way that stronger conditions are smaller. We assume that $\mathbb{P}$-names are constructed just as in [8] for a poset $\mathbb{P}$ but different from [8] we use symbols with tilde below them like $\underset{\sim}{a}$, $\underset{\sim}{b}$ etc. to denote the $\mathbb{P}$-names corresponding to the sets a, b etc. in the generic extension. V denotes the ground model (in which we live). The canonical $\mathbb{P}$-names of elements a, b etc. of V are denoted by the same symbols with hat like $\hat{a}$, $\hat{b}$ etc. For a poset $\mathbb{P}$ (in V) we use $V^{\mathbb{P}}$ to denote a "generic" generic extension $V[G]$ of V by some $(V,\mathbb{P})$-generic filter G. Thus $V^{\mathbb{P}} \models \cdots$ is synonymous to $\Vdash_{\mathbb{P}}$ " $\cdots$ " or $V \models \Vdash_{\mathbb{P}}$ " $\cdots$ " and a phrase like: "Let $W = V^{\mathbb{P}}$" is to be interpreted as saying: "Let W be a generic extension of V by some/any $(V,\mathbb{P})$-generic filter".

For the notation connected to the set theory of reals see [1] and [2]. With $\mathfrak{c}$ we denote the size of the continuum $2^{\aleph_0}$. $\mathcal{M}$ and $\mathcal{N}$ are the ideals of meager sets and null sets (e.g. over the Cantor space ${}^{\omega}2$) respectively. For $I = \mathcal{M}$, $\mathcal{N}$ etc., $\mathsf{cov}(I)$ and $\mathsf{non}(I)$ are *covering number* and *uniformity* of I.

For an infinite cardinal κ let $\mathcal{C}_\kappa = \mathrm{Fn}(\kappa, 2)$ or, more generally $\mathcal{C}_X = \mathrm{Fn}(X, 2)$ for any set X. $\mathcal{C}_\kappa$ is the Cohen forcing for adding κ many Cohen reals. $\mathcal{R}_\kappa$ denotes the random forcing for adding κ many random reals. $\mathcal{R}_\kappa$ is the poset consisting of Borel sets of positive measure in ${}^{\kappa}2$, which corresponds to the homogeneous measure algebra of Maharam type κ.

For a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$, $X \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, let

$$X \downarrow p = \{q \in X \,:\, q \leq_{\mathbb{P}} p\}.$$

2. Mad families and almost disjoint numbers

One of the advantages of using $T = {}^{\omega>}2$ as the countable underlying set is that we can define some natural subfamilies of $[T]^{\aleph_0}$ such as $\mathcal{O}_T$, $\mathcal{A}_T$, $\mathcal{B}_T$ etc. below.

For $X \subseteq T$, let

$$[X] = \{f \in {}^{\omega}2 \,:\, B(f) \subseteq X\}, \text{ and} \tag{2.1}$$

$$\lceil X \rceil = \{f \in {}^{\omega}2 \,:\, |\, B(f) \cap X \,| = \aleph_0\}. \tag{2.2}$$

Clearly, we have $[X] \subseteq \lceil X \rceil$. For $X \subseteq T$, let $X^{\uparrow}$ be the upward closure of X, that is:

$$X^{\uparrow} = \{t \restriction n \,:\, t \in X,\, n \leq \ell(t)\}. \tag{2.3}$$

Then we have $\lceil X \rceil \subseteq [X^{\uparrow}]$ for any $X \subseteq T$.

Definition 1 (Off-binary sets, [9]). *Let*

$$\mathcal{O}_T = \{X \in [T]^{\aleph_0} \,:\, \lceil X \rceil = \emptyset\}.$$

T. Leathrum [9] called elements of $\mathcal{O}_T$ off-binary sets. Note that $\lceil X \rceil = \emptyset$ if and only if there is no branch in T with infinite intersection with X.

Definition 2 (Antichains). *Let*

$$\mathcal{A}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ is an antichain in } T\}.$$

Clearly, we have $\mathcal{A}_T \subseteq \mathcal{O}_T$.

Using the notation above, the cardinal invariants $\mathfrak{o}$ and $\bar{\mathfrak{o}}$ introduced by Leathrum [9] can be characterized as:

(2.4) $\mathfrak{o} = \mathfrak{a}(\mathcal{O}_T)$,

(2.5) $\bar{\mathfrak{o}} = \mathfrak{a}(\mathcal{A}_T)$

(see [9]). Leathrum also showed $\mathfrak{a} \leq \mathfrak{o} \leq \bar{\mathfrak{o}}$. J. Brendle [3] proved $\mathsf{non}(\mathcal{M}) \leq \mathfrak{o}$.

Definition 3 (Sets without infinite antichains). *Let*

$$\mathcal{B}_T = \{X \in [T]^{\aleph_0} \,:\, X \textit{ does not contain any infinite antichain}\}.$$

Note that $\mathcal{B}_T = {\mathcal{A}_T}^{\perp}$. Elements of $\mathcal{B}_T$ are those infinite subsets of T which can be covered by finitely may branches:

Lemma 1 (K. Kunen). *Let $X \in [T]^{\aleph_0}$. Then $X \in \mathcal{B}_T$ if and only if X is covered by finitely may branches in T.*

Proof. If X is covered by finitely many branches in T then X clearly does not contain any infinite antichain since otherwise one of the finitely many branches would contain an infinite antichain.

Suppose now that X cannot be covered by finitely many branches. By induction on n, we choose $t_n \in 2^n$ such that $t_0 = \emptyset$, $t_{n+1} = t_n {}^\frown i$ for some $i \in 2$ and

(2.6) $X_{n+1} = X \downarrow t_{n+1}$ can not be covered by finitely many branches.

This is possible since $X_0 = X$ and $X_n \subseteq (X_n \downarrow (t_n {}^\frown 0)) \cup (X_n \downarrow (t_n {}^\frown 1)) \cup \{t_n\}$.

By (2.6), the branch $B = \{t_n \,:\, n < \omega\}$ does not cover X_n for each $n \in \omega$. So we can pick $s_n \in X_n \setminus B$. Let $S = \{s_n \,:\, n \in \omega\}$. S is an infinite subset of X since $\ell(s_n) \geq n$ for all $n \in \omega$. If C is a branch in T different from B then $t_n \notin C$ for some $n \in \omega$ and so $s_m \notin C$ for all $m \geq n$. Hence $S \cap C$ is finite. Moreover $S \cap B = \emptyset$. So we have $\lceil S \rceil = \emptyset$. Thus S should contain an infinite antichain by König's Lemma. □

Theorem 1 (K. Kunen). $\mathfrak{a}(\mathcal{B}_T) = \mathfrak{c}$.

Proof. Suppose that $\mathcal{F} \subseteq \mathcal{B}_T$ is an ad family of cardinality $< \mathfrak{c}$. We show that $\mathcal{F}$ is not mad. For each $X \in \mathcal{F}$ there is $b_X \in [{}^\omega 2]^{<\aleph_0}$ such that $X \subseteq \bigcup_{f \in b_X} B(f)$ by Lemma 1. Since $\mathcal{S} = \bigcup\{b_X \,:\, X \in \mathcal{F}\}$ has cardinality $\leq |\mathcal{F}| \cdot \aleph_0 < \mathfrak{c}$, there is $f^* \in {}^\omega 2 \setminus \mathcal{S}$. We have $B(f^*) \in \mathcal{B}_T$ and $B(f^*)$ is ad to $\mathcal{F}$. □

Let us say $X \subseteq T$ is *nowhere dense* if $\lceil X \rceil$ is nowhere dense in the Cantor space ${}^\omega 2$. It can be easily shown that X is nowhere dense if and only if

(2.7) $\forall t \in T\ \exists t' \leq_T t\ \forall t'' \leq_T t'\ (t'' \notin X)$.

Note that, if $X \subseteq T$ is not nowhere dense, then X is dense below some $t \in T$ (in terms of forcing). Also note that from (2.7) it follows that the property of being nowhere dense is absolute.

Definition 4 (Nowhere dense sets). *Let*

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : X \textit{ is nowhere dense}\}.$$

Note that, for $X \in [T]^{\aleph_0}$ with $X = \{t_n : n \in \omega\}$, we have

$$\lceil X \rceil = \bigcap_{n\in\omega} \bigcup_{m>n} [T \downarrow t_m].$$

In particular $\lceil X \rceil$ is a G_δ subset of ${}^\omega 2$. Hence by Baire Category Theorem we have

$$\mathcal{ND}_T = \{X \in [T]^{\aleph_0} : \lceil X \rceil \text{ is a meager subset of } {}^\omega 2\}.$$

Lemma 2. *If $X \in [T]^{\aleph_0}$ then there is $X' \in [X]^{\aleph_0}$ such that $X' \in \mathcal{ND}_T$.*

Proof. If $\lceil X \rceil = \emptyset$ then $X \in \mathcal{ND}_T$. Thus we can put $X' = X$. Otherwise let $f \in \lceil X \rceil$ and let $X' = X \cap B(f)$. □

Theorem 2. $\mathsf{cov}(\mathcal{M})$, $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$.

Proof. For the inequality $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathcal{ND}_T)$, suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathsf{cov}(\mathcal{M})$. Then $\bigcup\{\lceil X \rceil : X \in \mathcal{F}\} \neq {}^\omega 2$. Let $f \in {}^\omega 2 \setminus \bigcup\{\lceil X \rceil : X \in \mathcal{F}\}$. Then $B(f) \in \mathcal{ND}_T$ and $B(f)$ is ad from all $X \in \mathcal{F}$.

To show $\mathfrak{a} \leq \mathfrak{a}(\mathcal{ND}_T)$ suppose that $\mathcal{F} \subseteq \mathcal{ND}_T$ is an ad family of cardinality $< \mathfrak{a}$. Then $\mathcal{F}$ is not a mad family in $[T]^{\aleph_0}$. Hence there is some $X \in [T]^{\aleph_0}$ ad to $\mathcal{F}$. By Lemma 2, there is $X' \subseteq X$ such that $X' \in \mathcal{ND}_T$. Since X' is also ad to $\mathcal{F}$, it follows that $\mathcal{F}$ is not mad in $\mathcal{ND}_T$. □

Let σ be the measure on Borel sets of the Cantor space ${}^\omega 2$ defined as the product measure of the probability measure on 2. For $X \subseteq T$, let $\mu(X) = \sigma(\lceil X \rceil)$.

Definition 5 (Null sets). *Let*

$$\mathcal{N}_T = \{X \in [T]^{\aleph_0} : \mu(X) = 0\}.$$

Theorem 3. $\mathsf{cov}(\mathcal{N})$, $\mathfrak{a} \leq \mathfrak{a}(\mathcal{N}_T)$.

Proof. Similarly to the proof of Theorem 2. □

Definition 6 (Nowhere dense null sets). *Let*

$$\mathcal{NDN}_T = \mathcal{ND}_T \cap \mathcal{N}_T.$$

Lemma 3. $\mathfrak{a}(\mathcal{ND}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$ *and* $\mathfrak{a}(\mathcal{N}_T) \leq \mathfrak{a}(\mathcal{NDN}_T)$.

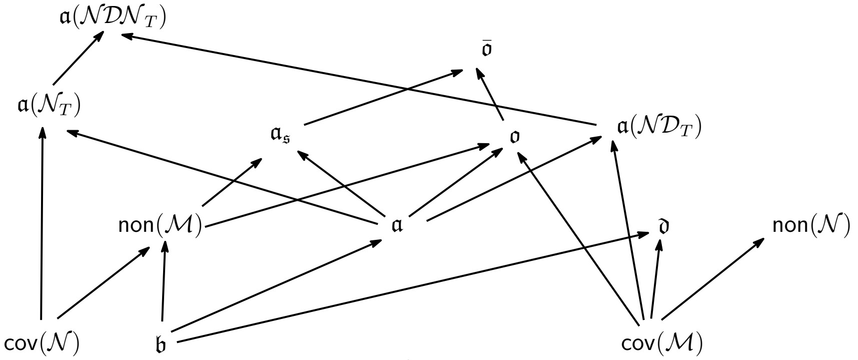

Fig. 1.

Proof. For the first inequality, suppose that $\mathcal{F}$ is a mad family in $\mathcal{NDN}_T$. Then $\mathcal{F}$ is an ad family in $\mathcal{ND}_T$. It is also mad in $\mathcal{ND}_T$. Suppose not. Then there is an $X \in \mathcal{ND}_T$ ad to $\mathcal{F}$. Let $X' \in [X]^{\aleph_0}$ be as in the measure analog of Lemma 2. Then $X' \in \mathcal{NDN}_T$. Hence $\mathcal{F}$ is not mad in $\mathcal{NDN}_T$. This is a contradiction. The second inequality can be also proved similarly. □

The diagram Fig. 1 summarizes the inequalities obtained in this section integrated into the cardinal diagram given in Brendle [4]. "$\kappa \to \lambda$" in the diagram means that "$\kappa \leq \lambda$ is provable in ZFC". There are still some open questions concerning the (in)completeness of this diagram. In particular:

Problem 1. (a) Are the inequalities between $\mathfrak{a}(\mathcal{N}_T)$, $\mathfrak{a}(\mathcal{ND}_T)$, $\mathfrak{a}(\mathcal{NDN}_T)$ consistently strict and complete?

(b) Are $\mathfrak{a}(\mathcal{ND}_T)$ etc. independent from $\mathfrak{o}$, $\bar{\mathfrak{o}}$, $\mathfrak{a}_{\mathfrak{s}}$?

3. Ad families $\mathcal{F}$ for which $\mathcal{F}^{\perp}$ is contained in a certain subfamily of $[T]^{\aleph_0}$

In this section we give several constructions of ad families with the property that the sets ad to them in a given generic extension are necessarily in a certain subfamily of $[T]^{\aleph_0}$. The constructions in this section are used in the proof of some results in the next sections.

Theorem 4. (CH) *There exists an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of size $\aleph_1$ such that for any cardinal κ we have*

(3.1) $V^{\mathcal{C}_\kappa} \models \mathcal{F}^{\perp} \subseteq \mathcal{ND}_T$.

Proof. Let

(3.2) $$\mathcal{S} = \{\langle p, \underset{\sim}{B}, t\rangle \,:\, p \in \mathcal{C}_\omega,\ \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of a subset of } T,\ t \in T \text{ and } p \Vdash_{\mathcal{C}_\omega} \text{“}\,\underset{\sim}{B} \text{ is dense below } t\,\text{”}.\}$$

Note that this set is of cardinality $\aleph_1$ by CH. Let $\langle\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha\rangle \,:\, \alpha < \omega_1 \setminus \omega\rangle$ be an enumeration of $\mathcal{S}$.

By induction on $\alpha < \omega_1$, we construct $A_\alpha \subseteq T$, $\alpha < \omega_1$ such that

(3.3) $A_\alpha \in \mathcal{A}_T$ for all $\alpha < \omega_1$,

(3.4) A_n, $n \in \omega$ is a partition of $T \setminus \{\emptyset\}$ (note that $\emptyset$ is the root of the tree T),

(3.5) $|\, A_\beta \cap A_\alpha \,| < \aleph_0$ for all $\beta < \alpha < \omega_1$, and

(3.6) if $\alpha \in \omega_1 \setminus \omega$, for each $q \leq_{\mathcal{C}_\omega} p_\alpha$ and $n \in \omega$, there are $r \leq_{\mathcal{C}_\omega} q$ and $t \in A_\alpha \downarrow t_\alpha$ such that $|\, t \,| \geq n$ and $r \Vdash_{\mathcal{C}_\omega}$ "$t \in \underset{\sim}{B}_\alpha$" (in particular, $p_\alpha \Vdash_{\mathcal{C}_\omega}$ "$|\, A_\alpha \cap \underset{\sim}{B}_\alpha \,| = \aleph_0$").

We show first that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ for A_α's as above satisfies (3.1). Since every subset of T in $V^{\mathcal{C}_\kappa}$ is contained in $V^{\mathcal{C}_X}$ for some countable $X \subseteq \kappa$, it is enough to show (3.1) for $\kappa = \omega$. Assume for contradiction that, for some $\mathcal{C}_\omega$-name $\underset{\sim}{B}^*$ of subset of T and $p \in \mathcal{C}_\omega$, we have $p \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{B}^* \in \mathcal{F}^\perp \setminus \mathcal{ND}_T$". Then there are $t^* \in T$ and $p^* \leq_{\mathcal{C}_\omega} p$ such that

(3.7) $p^* \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{B}^*$ is dense below t^* and $|\, \underset{\sim}{B}^* \cap A_\alpha \,| < \aleph_0$ for all $\alpha < \omega_1$".

We may assume that $\underset{\sim}{B}^*$ is a nice $\mathcal{C}_\omega$-name. Let $\alpha < \omega_1 \setminus \omega$ be such that $\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha \rangle = \langle p^*, \underset{\sim}{B}^*, t^* \rangle$. Then $p^* \Vdash_{\mathcal{C}_\omega}$ "$|\, A_\alpha \cap \underset{\sim}{B}^* \,| = \aleph_0$" by (3.6). This is a contradiction.

To see that the construction of A_α, $\alpha < \omega_1$ is possible, assume that $\langle A_\beta : \beta < \alpha \rangle$ satisfying (3.3), (3.4), (3.5) and (3.6) has been constructed for $\alpha \in \omega_1 \setminus \omega$.

For $q \leq_{\mathcal{C}_\omega} p_\alpha$ let

$$I(\underset{\sim}{B}_\alpha, q) = \{t \in T : t \leq_T t_\alpha \wedge \exists r \leq_{\mathcal{C}_\omega} q \; \big(r \Vdash_{\mathcal{C}_\omega} \text{“} \hat{t} \in \underset{\sim}{B}_\alpha \text{”}\big)\}.$$

Note that $I(\underset{\sim}{B}_\alpha, q)$ is dense below t_α by the definition (3.2) of $\langle p_\alpha, \underset{\sim}{B}_\alpha, t_\alpha \rangle \in \mathcal{S}$.

Fix an enumeration $\{\langle q_i, n_i \rangle : i < \omega\}$ of $(\mathcal{C}_\omega \downarrow p_\alpha) \times \omega$ and an enumeration $\{\beta_i : i < \omega\}$ of α.

By induction on $m \in \omega$ we choose $u_m \in T$ and $r_m \in \mathcal{C}_\omega$ according to the following (3.8) – (3.12) and let

$$A_\alpha = \{u_m : m < \omega\}.$$

In the m'th step of the construction, let $u_m \in T$ and $r_m \in \mathcal{C}_\omega$ be such that

(3.8) $\{u_i : i \leq m\}$ is an antichain in $T \downarrow t_\alpha$ which is not maximal below t_α ;

(3.9) $u_m \in I(\underset{\sim}{B}_\alpha, q_m) \setminus \bigcup\{A_{\beta_i} : i < m\}$;

(3.10) $|u_m| \geq n_m$;

(3.11) $r_m \leq_{\mathcal{C}_\omega} q_m$; and

(3.12) $r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$".

This can be carried out. Indeed, at the m'th step if $\{u_i : i < m\}$ has been chosen so that it is a non-maximal antichain below t_α, then we can find

$u'_m \in T \downarrow t_\alpha$ distinct from all u_i, $i < m$ such that $\{u_i : i < m\} \cup \{u'_m\}$ is still a non-maximal antichain below t_α. We can also choose u'_m so that $|\, u'_m \,| \geq n_m$. Since $\{A_{\beta_i} : i < m\}$ are antichains we can find $u''_m \leq_T u'_m$ such that there is no $t \leq_T u''_m$ with $t \in \bigcup\{A_{\beta_i} : i < m\}$. Since $I(\underset{\sim}{B}_\alpha, q_m)$ is dense below t_α we can find $u_m \in I(\underset{\sim}{B}_\alpha, q_m)$ such that $u_m \leq_T u''_m$. By the definition of $I(\underset{\sim}{B}_\alpha, q_m)$ there is an $r_m \leq_{\mathcal{C}_\omega} q_m$ such that $r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$".

It is easy to see that A_α defined as above satisfies (3.3),(3.5) and (3.6): $A_\alpha \in \mathcal{A}_T$ by (3.8). $|\, A_\beta \cap A_\alpha \,| < \aleph_0$ for all $\beta < \alpha$ by (3.9). To show that A_α also satisfies (3.6), suppose that $q \leq_{\mathcal{C}_\omega} p_\alpha$ and $n \in \omega$. Let $m \in \omega$ be such that $\langle q, n \rangle = \langle q_m, n_m \rangle$. Then we have $r_m \leq_{\mathcal{C}_\omega} q$ by (3.11), $u_m \in A_\alpha$ by definition of A_α, $|\, u_m \,| \geq n$ by (3.10) and $r_m \Vdash_{\mathcal{C}_\omega}$ "$\hat{u}_m \in \underset{\sim}{B}_\alpha$" by (3.12). □

Problem 2. Is CH really necessary for the conclusion of Theorem 4?

In connection with the problem above, we can actually obtain a slightly stronger conclusion than that of Theorem 4 if our ground model is a generic extension of some inner model by adding uncountably may Cohen reals. Note that CH need not hold in such a model.

Theorem 5. *Suppose that $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ of cardinality $\aleph_1$ such that,*

(3.13) *for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^{\perp} \subseteq \mathcal{ND}_T$.*

Proof. Let $A \in [T]^{\aleph_0} \cap V$ be an antichain and let $\langle t^*_n : n \in \omega \rangle$ be a 1-1 enumeration of A.

Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. For $p \in \mathcal{C}_{\omega_1}$, $\alpha < \omega_1$ and $k \in \omega$, let

$$f^p_\alpha = \{\langle n, i \rangle \in \omega \times \omega : \langle \omega\alpha + 3n, i \rangle \in p\};$$

$$n^p_{\alpha,k} = \begin{cases} n, & \text{if } [\omega\alpha, \omega\alpha + 3n + 1] \subseteq \mathrm{dom}(p), \\ & \quad p(\omega\alpha + 3n + 1) = 1 \text{ and} \\ & \quad |\, \{m < n : p(\omega\alpha + 3m + 1) = 1\} \,| = k, \\ \text{undefined,} & \text{if there is no such } n \text{ as above;} \end{cases}$$

$$t^p_\alpha = \begin{cases} \{\langle n, i \rangle \in \omega \times \omega : n < n^p_{\alpha,0},\ \langle \omega\alpha + 3n + 2, i \rangle \in p\}, \\ \qquad\qquad\qquad \text{if } n^p_{\alpha,0} \text{ is defined,} \\ \text{undefined,} \qquad \text{otherwise} \end{cases}$$

and

$$t^p_{\alpha,k} = \begin{cases} \{\langle n, i \rangle \in \omega \times \omega : n < n^p_{\alpha,k+1},\ \langle \omega\alpha + 3n + 2, i \rangle \in p\}, \\ \qquad\qquad\qquad \text{if } n^p_{\alpha,k+1} \text{ is defined,} \\ \text{undefined,} \qquad \text{otherwise.} \end{cases}$$

Let

$f^G_\alpha = \bigcup_{p\in G} f^p_\alpha$,
$t^G_\alpha = t^p_\alpha$ for some $p \in G$ such that t^p_α is defined, and
$t^G_{\alpha,k} = t^p_{\alpha,k}$ for some $p \in G$ such that $t^p_{\alpha,k}$ is defined.

For $\alpha \in \omega_1$, let

(3.14) $A_\alpha = \{t^G_\alpha {}^\frown t^*_k {}^\frown t^G_{\alpha,k} \,:\, k \in \omega\}$.

Clearly each A_α is an antichain in T.

A_α, $\alpha < \omega_1$ are pairwise almost disjoint: Suppose that $\alpha < \beta < \omega_1$. Then there is $k_0 < \omega$ such that $t^G_{\alpha,k} \neq f^G_{\beta,k}$ for all $k \in \omega \setminus k_0$. It follows that $A_\alpha \cap A_\beta \subseteq \{t^G_\alpha {}^\frown t^*_k {}^\frown t^G_{\alpha,k} \,:\, k < k_0\}$.

We show that $\mathcal{F} = \{A_\alpha \,:\, \alpha < \omega_1\}$ satisfies (3.13).

Suppose that $\mathbb{P}$ is a c.c.c. poset (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0}$ is not nowhere dense then X is not almost ad to $\mathcal{F}$.

By the c.c.c. of $\mathcal{C}_{\omega_1} * \hat{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1$ such that $X \in V[G \restriction \mathcal{C}_{\omega\alpha^*}][H]$. Let $t \in T$ be such that X is dense below t. Then

$$D = \{p \in \mathcal{C}_{\omega_1\setminus\omega\alpha^*} \,:\, t^p_\alpha \supseteq t \text{ for some } \alpha \in \omega_1 \setminus \omega\alpha^*\}$$

is dense in $\mathcal{C}_{\omega_1\setminus\omega\alpha^*}$.

For $p \in D$ and $\alpha \in \omega_1 \setminus \omega\alpha^*$ such that $t^p_\alpha \supseteq t$, letting $\underset{\sim}{A}_\alpha$ a $\mathcal{C}_{\omega_1\setminus\alpha^*}$-name of A_α, we have $p \Vdash_{\mathcal{C}_{\omega_1\setminus\omega\alpha^*}}$ "$|\, \underset{\sim}{A}_\alpha \cap X \downarrow t \,| = \aleph_0$" by (3.14) and since X is dense below t.

By genericity, it follows that, in $W[G]$, there is $\alpha < \omega_1$ such that $|\, A_\alpha \cap X \,| = \aleph_0$. □

A measure version of Theorem 5 also holds:

Theorem 6. *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F}$ in $\mathcal{N}_T$ of cardinality $\aleph_1$ such that for any c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, we have $W^{\mathbb{P}} \models \mathcal{F}^\perp \subseteq \mathcal{O}_T$.*

For the proof of Theorem 6 we note first the following:

Lemma 4. *Suppose that $X \subseteq T$ is such that $X = \{t_k \,:\, k \in \omega\}$ for some enumeration t_k, $k \in \omega$ of X with $\ell(t_k) \geq k$ for all $k \in \omega$. Then $X \in \mathcal{N}_T$.*

Proof. For all $n \in \omega$, we have $\lceil X \rceil \subseteq \bigcup_{k\in\omega\setminus n} \lceil T \downarrow t_k \rceil$. Hence

$$\mu(X) = \sigma(\lceil X \rceil) \leq \textstyle\sum_{k\in\omega\setminus n} \sigma(\lceil T \downarrow t_k \rceil) \leq \sum_{k\in\omega\setminus n} 2^k = 2^{-n}.$$

It follows that $\mu(X) = 0$. □

Proof (of Theorem 6). Let G be a $(V, \mathcal{C}_{\omega_1})$-generic filter and $W = V[G]$. In W, let

$f^G_\alpha = \{\langle n, i\rangle \,:\, \langle \omega\alpha + n, i\rangle \in p \text{ for some } p \in G\}$

for $\alpha < \omega_1$ and let $g^G_\alpha \in {}^\omega\omega$ be the increasing enumeration of $\left(f^G_\alpha\right)^{-1}[\{1\}]$.

Further in W, we construct inductively $A_\alpha \in \mathcal{N}_T$, $\alpha < \omega_1$ as follows.

For $n \in \omega$, let $A_n \in \mathcal{N}_T$ be such that $\langle A_n : n \in \omega\rangle$ is a partition of T in V. We can be easily find such A_n's by Lemma 4.

For $\omega \leq \alpha < \omega_1$, suppose that pairwise almost disjoint A_β, $\beta < \alpha$ have been constructed. Let $\langle B_\ell : \ell \in \omega\rangle$ be an enumeration of $\{A_\beta : \beta < \alpha\}$ and, for each $n \in \omega$, let $\langle b_{n,m} : m \in \omega\rangle$ be an enumeration of

$$C_n = T \setminus ({}^{n>}2 \cup \{B_\ell : \ell < n\}). \tag{3.15}$$

Let

$$A_\alpha = \{b_{n,g^G_\alpha(n)} : n \in \omega\}. \tag{3.16}$$

$A_\alpha \in \mathcal{N}_T$ by (3.15) and Lemma 4. A_α is ad to $\{A_\beta : \beta < \alpha\}$ by (3.15) and (3.16).

We show that $\mathcal{F} = \{A_\alpha : \alpha < \omega_1\}$ is as desired. Suppose that $\mathbb{P}$ is c.c.c. (in W) and $\mathbb{P} \in V$. Let H be a $(W, \mathbb{P})$-generic filter. It is enough to show that, in $W[H]$, if $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ then X is not ad to $\mathcal{F}$. So suppose that (in $W[H]$) $X \in [T]^{\aleph_0} \setminus \mathcal{O}_T$ and $f \in \lceil X \rceil$. Let $B = X \cap B(f)$. By the c.c.c. of $\mathcal{C}_{\omega_1} * \hat{\mathbb{P}} \sim \mathcal{C}_{\omega_1} \times \mathbb{P}$, there is an $\alpha^* \in \omega_1 \setminus \omega$ such that $B \in V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$. If $B \cap A_\alpha$ is infinite for some $\alpha < \alpha^*$ then we are done. So assume that B is ad to all A_α, $\alpha < \alpha^*$. Then $B \cap C_n$ is infinite for all $n \in \omega$. Since $f^G_{\alpha^*}$ is a Cohen real generic over $V[(G \restriction \mathcal{C}_{\omega\alpha^*})][H]$, it follows that $B \cap A_{\alpha^*}$ is infinite. □

4. Almost disjoint numbers over ad families

In this section we turn to questions on the possible values of $\mathfrak{a}^+(\cdot)$.

Theorem 7. (K. Kunen) $\mathfrak{a}^+(\bar{\mathfrak{o}}) = \mathfrak{c}$.

Proof. Let $\mathcal{F}$ be any mad family in $\mathcal{A}_T$ of cardinality $\bar{\mathfrak{o}}$. By maximality of $\mathcal{F}$ we have $\mathcal{F}^\perp = \mathcal{B}_T$. If $\mathcal{G} \subseteq [T]^{\aleph_0}$ is disjoint from $\mathcal{F}$ and $\mathcal{F} \cup \mathcal{G}$ is mad then $\mathcal{G}$ is mad in $\mathcal{B}_T$ and hence $|\,\mathcal{G}\,| = \mathfrak{c}$ by Theorem 1. □

Theorem 8. *$V^{\mathcal{C}_\kappa} \models \mathfrak{a}^+(\aleph_1) \geq \kappa$ for all regular κ.*

Proof. If $\kappa = \omega_1$ this is trivial. So suppose that $\kappa > \omega_1$. Let $W = V^{\mathcal{C}_{\omega_1}}$. Then $V^{\mathcal{C}_\kappa} = W^{\mathcal{C}_{\kappa \setminus \omega_1}}$. Let $\mathcal{F}$ be as in the proof of Theorem 5. Suppose that $\tilde{\mathcal{F}} \supseteq \mathcal{F}$ is mad on T in $V^{\mathcal{C}_\kappa}$. Then $\tilde{\mathcal{F}} \subseteq (\mathcal{ND}_T)^{V^{\mathcal{C}_\kappa}}$. Since $V^{\mathcal{C}_\kappa} \models \mathsf{cov}(\mathcal{M}) \geq \kappa$, it follows that $|\,\tilde{\mathcal{F}}\,| \geq \kappa$ by Theorem 2. □

Corollary 1. *The inequality $\mathfrak{a} = \aleph_1 < \mathfrak{a}^+(\aleph_1) = \mathfrak{c}$ is consistent.*

Proof. Start from a model V of CH. Since there is a $\mathcal{C}_\kappa$-indestructible mad family in V it follows that $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a} = \aleph_1$ (see e.g. [8], Theorem 2.3). On the other hand we have $V^{\mathcal{C}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) = \aleph_2 = \mathfrak{c}$ by Theorem 8. □

Theorem 9. *The inequality $\mathfrak{a}^+(\aleph_1) < \mathfrak{c}$ is consistent.*

For the proof of the theorem we use the following forcing notions: for a family $\mathcal{I} \subseteq \{A \in [\omega]^{\aleph_0} : |\,\omega \setminus A\,| = \aleph_0\}$ closed under union, let $\mathbb{Q}_{\mathcal{I}} = \langle \mathbb{Q}_{\mathcal{I}}, \leq_{\mathbb{Q}_{\mathcal{I}}} \rangle$ be the poset defined by

$$\mathbb{Q}_{\mathcal{I}} = \mathcal{C}_\omega \times \mathcal{I}\,;$$

For all $\langle s, A \rangle$, $\langle s', A' \rangle \in \mathbb{Q}_{\mathcal{I}}$

$$\begin{aligned} \text{(4.1)} \quad \langle s', A' \rangle \leq_{\mathbb{Q}_{\mathcal{I}}} \langle s, A \rangle \quad \Leftrightarrow \quad & s \subseteq s',\ A \subseteq A' \text{ and} \\ & \forall n \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)\ (n \in A \ \rightarrow\ s'(n) = 0). \end{aligned}$$

Clearly $\mathbb{Q}_{\mathcal{I}}$ is σ-centered.

For a $(V, \mathbb{Q}_{\mathcal{I}})$-generic G, let

$$\begin{aligned} & f_G = \bigcup \{s : \langle s, A \rangle \in G \text{ for some } A \in \mathcal{I}\} \text{ and} \\ & A_G = f_G^{-1}{}''\{1\}. \end{aligned}$$

Let $\tilde{\mathcal{I}}$ be the ideal in $[\omega]^{\aleph_0}$ generated from $\mathcal{I}$ (i.e. the downward closure of $\mathcal{I}$ with respect to $\subseteq$). By the genericity of G and the definition of $\leq_{\mathbb{Q}_{\mathcal{I}}}$ it is easy to see that A_G is infinite and

$$\text{(4.2)} \quad \text{for every } B \in ([\omega]^{\aleph_0})^V,\ A_G \text{ is almost disjoint from } B \ \Leftrightarrow\ B \in \tilde{\mathcal{I}}.$$

Proof (of Theorem 9). Working in a ground model V of $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$, let

$$\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \omega_2,\ \beta < \omega_2 \rangle$$

be the finite support iteration of c.c.c. posets defined as follows: for $\beta < \omega_2$, let $\underset{\sim}{\mathbb{Q}}_\beta$ be the $\mathbb{P}_\beta$-name of the finite support (side-by-side) product of

$$\text{(4.3)} \quad \mathbb{Q}_{\tilde{\mathcal{F}}},\ \tilde{\mathcal{F}} \in \varPhi$$

where

$$\begin{aligned} \varPhi = \{\tilde{\mathcal{F}} : \tilde{\mathcal{F}} & \text{ is an ideal in } [\omega]^{\aleph_0} \\ & \text{generated from an ad family in } [\omega]^{\aleph_0} \text{ of cardinality } \aleph_1\} \end{aligned}$$

in $V^{\mathbb{P}_\beta}$. We have

$$V^{\mathbb{P}_\beta} \models \underset{\sim}{\mathbb{Q}}_\beta \text{ satisfies the c.c.c.}$$

since $V^{\mathbb{P}_\beta} \models \mathbb{Q}_{\tilde{\mathcal{F}}}$ is σ-centered for all $\tilde{\mathcal{F}} \in \varPhi$. By induction on $\alpha \leq \omega_2$, we can show that $\mathbb{P}_\alpha$ satisfies the c.c.c. and $|\,\mathbb{P}_\alpha\,| \leq 2^{\aleph_1} = \aleph_3$ for all $\alpha \leq \omega_2$. It follows that

$$\text{(4.4)} \quad V^{\mathbb{P}_{\omega_2}} \models 2^{\aleph_0} = 2^{\aleph_1} = \aleph_3.$$

Thus the following claim finishes the proof:

Claim. $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} = \mathfrak{a}^+(\aleph_1) = \aleph_2$.

$\vdash$ Working in $V^{\mathbb{P}_{\omega_2}}$, suppose that $\mathcal{F}$ is an ad family in $[\omega]^{\aleph_0}$ of cardinality $\aleph_1$. By the c.c.c. of $\mathbb{P}_{\omega_2}$, there is some $\alpha^* < \omega_2$ such that $\mathcal{F} \in V^{\mathbb{P}_{\alpha^*}}$. By (4.3) and (4.2), there are A_α, $\alpha \in \omega_2 \setminus \alpha^*$ such that

(4.5) for every $B \in ([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, A_α is ad from $B \Leftrightarrow B \in$ the ideal generated from $\mathcal{F} \cup \{A_\beta : \beta \in \alpha \setminus \alpha^*\}$.

Since $([\omega]^{\aleph_0})^{V^{\mathbb{P}_{\omega_2}}} = \bigcup_{\alpha<\omega_2}([\omega]^{\aleph_0})^{V^{\mathbb{P}_\alpha}}$, it follows that $\mathcal{F} \cup \{A_\alpha : \alpha \in \omega_2 \setminus \alpha^*\}$ is a mad family in $V^{\mathbb{P}_{\omega_2}}$. This shows that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a}^+(\aleph_1) \leq \aleph_2$.

We also have $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{a} \geq \aleph_2$: for any ad family $\mathcal{G} \subseteq ([\omega]^{\aleph_0})^{V^{\mathbb{P}_{\omega_2}}}$ of cardinality $\leq \aleph_1$, there is some $\alpha^* < \omega_2$ such that $\mathcal{G} \in V^{\mathbb{P}_{\alpha^*}}$. But $\underset{\sim}{\mathbb{Q}}_{\alpha^*}$ adds an infinite subset of ω almost disjoint to every element of $\mathcal{G}$. Hence $\mathcal{G}$ is not mad. $\dashv$ □

Clearly, the method of the proof of Theorem 9 cannot produce a model of $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$.

Problem 3. Is $\mathfrak{a}^+(\aleph_1) = \aleph_1 < \mathfrak{c}$ consistent?

All infinite cardinals less than or equal to the continuum $\mathfrak{c}$ can be represented as $\mathfrak{a}^+(\mathcal{F})$ for some $\mathcal{F}$.

Theorem 10. *For any infinite $\kappa \leq \mathfrak{c}$, there is an ad family $\mathcal{F} \subseteq [T]^{\aleph_0}$ of cardinality $\mathfrak{c}$ such that $\mathfrak{a}^+(\mathcal{F}) = \kappa$.*

Proof. Let $\mathcal{F}'$ be a mad family in $\mathcal{A}_T$. Then by Lemma 1, we have

(4.6) $\mathcal{F}'^{\perp} = \mathcal{B}_T$.

Let X and X' be disjoint with ${}^{\omega}2 = X \cup X'$, $|X| = \mathfrak{c}$ and $|X'| = \kappa$. Let

$$\mathcal{F} = \mathcal{F}' \cup \{B(f) : f \in X\}.$$

Clearly $\mathcal{F}$ is an ad family. By (4.6) we have $\mathcal{F}^{\perp} \subseteq \mathcal{B}_T$.

We claim $\mathfrak{a}^+(\mathcal{F}) = \kappa$: Since $\mathcal{F} \cup \{B(f) : f \in X'\}$ is a mad family by Lemma 1, we have $\mathfrak{a}^+(\mathcal{F}) \leq \kappa$. Again by Lemma 1, if $\mathcal{G} \subseteq \mathcal{F}^{\perp}$ is an ad family of cardinality $< \kappa$, then there is $f \in X'$ such that $B(f)$ is ad from every $B \in \mathcal{G}$. Thus $\mathfrak{a}^+(\mathcal{F}) \geq \kappa$. □

5. Destructibility of mad families

For a poset $\mathbb{P}$, a mad family $\mathcal{F}$ in $[T]^{\aleph_0}$ is said to be $\mathbb{P}$-*destructible* if

$$V^{\mathbb{P}} \models \mathcal{F} \text{ is not mad in } [T]^{\aleph_0}.$$

Otherwise it is $\mathbb{P}$-*indestructible.*

The results in Section 3 can be also formulated in terms of destructibility of mad families.

Theorem 11. (1) (CH) *There is an ad family $\mathcal{F} \subseteq \mathcal{A}_T$ which cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family in any generic extension of the ground model of the form $V^{\mathcal{C}_\kappa}$.*

(2) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{ND}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{C}_\omega$-indestructible mad family.*

(3) *Let $W = V^{\mathcal{C}_{\omega_1}}$. Then, in W, there is an ad family $\mathcal{F} \subseteq \mathcal{N}_T$ of cardinality $\aleph_1$ such that, in any generic extension of W by a c.c.c. poset $\mathbb{P}$ with $\mathbb{P} \in V$, $\mathcal{F}$ cannot be extended to a $\mathcal{R}_\omega$-indestructible mad family.*

Proof. (1): The family $\mathcal{F}$ as in Theorem 4 will do. Since we have $\mathcal{F}' \subseteq \mathcal{ND}_T$ for any mad $\mathcal{F}'$ extending $\mathcal{F}$ in $V^{\mathcal{C}_\kappa}$, a further Cohen real over $V^{\mathcal{C}_\kappa}$ introduces a branch almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no longer mad in $V^{\mathcal{C}_\kappa * \mathcal{C}_\omega}$.

(2): By Theorem 5 and by an argument similar to the proof of (1).

(3): In W, let $\mathcal{F}$ be as in the proof of Theorem 6. Then any mad $\mathcal{F}' \supseteq \mathcal{F}$ on T in any $W^{\mathbb{P}}$ for $\mathbb{P}$ as above is included in $\mathcal{N}_T$ by $\mathcal{O}_T \subseteq \mathcal{N}_T$. Hence, in $W^{\mathbb{P}*\mathcal{R}_\omega}$, the random real f over $W^{\mathbb{P}}$ introduces the branch $B(f)$ almost avoiding all elements of $\mathcal{F}'$. Thus $\mathcal{F}'$ is no longer mad in $W^{\mathbb{P}*\mathcal{R}_\omega}$. □

6. κ-almost decided and λ-minimal mad families

In this final section we collect several other constructions of mad families with some additional properties.

Given an ad family $\mathcal{F}$ on T let $\mathcal{I}(\mathcal{F})$ be the ideal on T generated by $\mathcal{F} \cup [T]^{<\omega}$, i.e. for $S \subset T$ we have $S \in \mathcal{I}(\mathcal{F})$ if $S \subset^* \cup\mathcal{F}'$ for some finite subfamily $\mathcal{F}'$ of $\mathcal{F}$.

Let $\mathcal{F}$ be mad family on T and $\mathcal{B} \subseteq \mathcal{F}$. Clearly $\mathcal{B}^\perp \supseteq \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. We say that $\mathcal{B}$ *almost decides* $\mathcal{F}$ if $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. A mad family $\mathcal{F}$ is said to be *κ-almost decided* if every $\mathcal{B} \in [\mathcal{F}]^\kappa$ almost decides $\mathcal{F}$.

Theorem 12. *Assume that* MA(σ*-centered*) *holds. Then there is a* $\mathfrak{c}$*-almost decided mad family* $\mathcal{F}$ *on* T*.*

Proof. Let $\langle B_\beta : \beta < \mathfrak{c}\rangle$ be an enumeration of $[T]^{\aleph_0}$. We define A_α, $\alpha < \mathfrak{c}$ inductively such that

(6.1) $\{A_n : n \in \omega\}$ is a partition of T into infinite subsets;

For all $\alpha \in \mathfrak{c} \setminus \omega$

(6.2) A_α is ad from A_β for all $\beta < \alpha$;

(6.3) For $\beta < \alpha$, if $B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$ then $|\, A_\alpha \cap B_\beta \,| = \aleph_0$;

Claim. The construction of A_α, $\alpha < \mathfrak{c}$ as above is possible.

$\vdash$ Suppose that $\alpha \in \mathfrak{c} \setminus \omega$ and A_β, $\beta < \alpha$ have been constructed according to (6.1), (6.2) and (6.3). Let

$$S_\alpha = \{\beta < \alpha : B_\beta \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})\}.$$

Let $\mathbb{P}_\alpha = \{\langle \varphi, s\rangle : \varphi \in \mathrm{Fn}(T, 2),\ s \in [\alpha]^{<\aleph_0}\}$ be the poset with the ordering defined by

$$\begin{aligned} &\langle \varphi', s'\rangle \leq_{\mathbb{P}_\alpha} \langle \varphi, s\rangle \ \Leftrightarrow \\ &\qquad \varphi \subseteq \varphi',\ s \subseteq s' \text{ and} \\ &\qquad \forall t \in \mathrm{dom}(\varphi') \setminus \mathrm{dom}(\varphi)\ (\varphi'(t) = 1 \ \rightarrow\ t \notin A_\delta \text{ for all } \delta \in s) \end{aligned}$$

for $\langle \varphi, s\rangle$, $\langle \varphi', s'\rangle \in \mathbb{P}_\alpha$.

$\mathbb{P}_\alpha$ is σ-centered since $\langle \varphi, s\rangle$, $\langle \varphi', s'\rangle \in \mathbb{P}_\alpha$ are compatible if $\varphi = \varphi'$.

For $\beta < \alpha$, let

$$C_\beta = \{\langle \varphi, s\rangle \in \mathbb{P}_\alpha \,:\, \beta \in s\}$$

and, for $\beta \in S_\alpha$ and $n \in \omega$, let

$$D_{\beta,n} = \{\langle \varphi, s\rangle \in \mathbb{P}_\alpha \,:\, \exists t \in \mathrm{dom}(\varphi)\ (\ell(t) \geq n \wedge \varphi(t) = 1 \wedge t \in B_\beta)\}.$$

It is easy to see that C_β, $\beta < \alpha$ and $D_{\beta,n}$, $\beta \in S_\alpha$, $n \in \omega$ are dense in $\mathbb{P}_\alpha$. Let

$$\mathcal{D} = \{C_\beta \,:\, \beta < \alpha\} \cup \{D_{\beta,n} \,:\, \beta \in S_\alpha,\, n \in \omega\}.$$

Since $|\,\mathcal{D}\,| < \mathfrak{c}$, we can apply MA($\sigma$-centered) to obtain a $(\mathcal{D}, \mathbb{P}_\alpha)$-generic filter G. Let

$$A_\alpha = \{t \in T \,:\, \varphi(t) = 1 \text{ for some } \langle \varphi, s\rangle \in G\}.$$

Then this A_α is as desired. $\dashv$

Let $\mathcal{F} = \{A_\alpha \,:\, \alpha < \mathfrak{c}\}$. $\mathcal{F}$ is infinite by (6.2) and mad by (6.3).

We show that $\mathcal{F}$ is $\mathfrak{c}$-almost decided. First, note that we have $\mathfrak{a} = \mathfrak{c}$ by the assumptions of the theorem. By (6.3), we have:

(6.4) For any $B \in [T]^{\aleph_0}$, if $B \notin \mathcal{I}(\{A_\alpha \,:\, \alpha < \mathfrak{c}\})$ then $|\,\{\alpha < \mathfrak{c} \,:\, |\,A_\alpha \cap B\,| < \aleph_0\}\,| < \mathfrak{c}$.

Suppose that $\mathcal{H} \in [\mathcal{F}]^{\mathfrak{c}}$ and $B \in \mathcal{H}^\perp$. Then $|\,\{\alpha < \mathfrak{c} \,:\, |\,A_\alpha \cap B\,| < \aleph_0\}\,| = \mathfrak{c}$ and so $B \in \mathcal{I}(\mathcal{F})$ by (6.4). Thus there is a finite $\mathcal{F}' \subset \mathcal{F}$ such that $B \subset^* \cup\mathcal{F}'$ and $F \cap B$ is infinite for each $F \in \mathcal{F}'$. But $B \in \mathcal{H}^\perp$ so $\mathcal{F}' \cap \mathcal{H} = \emptyset$. Thus $\mathcal{F}'$ witnesses that $B \in \mathcal{I}(\mathcal{F} \setminus \mathcal{H})$ which was to be proved. □

For a mad family $\mathcal{F}$ on T, $\mathcal{C} \subseteq \mathcal{F}$ is said to be *minimal in* $\mathcal{F}$ if $\mathfrak{a}^+(\mathcal{F}\setminus\mathcal{C}) = |\,\mathcal{C}\,|$. A mad family $\mathcal{F}$ is said to be λ-*minimal* if every $\mathcal{C} \in [\mathcal{F}]^\lambda$ is minimal in $\mathcal{F}$.

Lemma 5. *Suppose that $\mathcal{F}$ is a mad family on T.*
(1) *If $\mathcal{F}$ is $|\,\mathcal{F}\,|$-minimal then $|\,\mathcal{F}\,| = \mathfrak{a}$.*
(2) *If $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$ and $\mathcal{F} \setminus \mathcal{B}$ is infinite then $\mathcal{F} \setminus \mathcal{B}$ is minimal in $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is κ-almost decided for $\kappa = |\,\mathcal{F}\,|$ then $\mathcal{F}$ is λ-minimal for all $\omega \leq \lambda < \kappa$.*
(4) *If $|\,\mathcal{F}\,| = \mathfrak{a}$ and $\mathcal{F}$ is $\mathfrak{a}$-almost decided then $\mathcal{F}$ is $\mathfrak{a}$-minimal.*

Proof. (1): If $\mathcal{F}$ is $|\,\mathcal{F}\,|$-minimal then $\mathcal{F}$ itself is minimal in $\mathcal{F}$. Thus $\mathfrak{a} = \mathfrak{a}^+(\emptyset) = \mathfrak{a}^+(\mathcal{F} \setminus \mathcal{F}) = |\,\mathcal{F}\,|$.

(2): First, note that, for any infinite ad $\mathcal{F}$, we have $\mathfrak{a}(\mathcal{I}(\mathcal{F})) = |\,\mathcal{F}\,|$.

Suppose that $\mathcal{F}$ is a mad family on T and $\mathcal{B} \subseteq \mathcal{F}$ almost decides $\mathcal{F}$, i.e. $\mathcal{B}^\perp = \mathcal{I}(\mathcal{F} \setminus \mathcal{B})$. Hence

$$\mathfrak{a}^+(\mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{B})) = \mathfrak{a}^+(\mathcal{B}) = \mathfrak{a}(\mathcal{B}^\perp) = \mathfrak{a}(\mathcal{I}(\mathcal{F} \setminus \mathcal{B})) = |\,\mathcal{F} \setminus \mathcal{B}\,|.$$

(3): Suppose that $\kappa = |\,\mathcal{F}\,|$ and $\mathcal{F}$ is κ-almost decided. If $\mathcal{C} \in [\mathcal{F}]^{\lambda}$ for some $\omega \leq \lambda < \kappa$ then $|\,\mathcal{F} \setminus \mathcal{C}\,| = \kappa$ and hence $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. By (2) it follows that $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is minimal in $\mathcal{F}$.

(4): Suppose that $|\,\mathcal{F}\,| = \mathfrak{a}$ and $\mathcal{F}$ is $\mathfrak{a}$-almost decided. Suppose that $\mathcal{C} \in [\mathcal{F}]^{\mathfrak{a}}$. If $|\,\mathcal{F} \setminus \mathcal{C}\,| < \mathfrak{a}$, then clearly $\mathfrak{a}^{+}(\mathcal{F} \setminus \mathcal{C}) = \mathfrak{a} = |\,\mathcal{C}\,|$. Hence $\mathcal{C}$ is minimal in $\mathcal{F}$. If $|\,\mathcal{F} \setminus \mathcal{C}\,| = \mathfrak{a}$ then $\mathcal{F} \setminus \mathcal{C}$ almost decides $\mathcal{F}$. Thus, by (2), $\mathcal{C} = \mathcal{F} \setminus (\mathcal{F} \setminus \mathcal{C})$ is again minimal in $\mathcal{F}$. □

Corollary 2. *Assume that* MA(σ*-centered*) *holds. Then there is a mad family* $\mathcal{F}$ *on* T *which is* λ*-minimal for all* $\omega \leq \lambda \leq \mathfrak{c}$.

Proof. By Theorem 12 and Lemma 5, (3), (4). □

Theorem 12 can be further improved to the following theorem:

Theorem 13. *Assume that* MA(σ*-centered*) *holds. Let* $\kappa = \mathfrak{c}$. *Then there is a* $\mathcal{C}_\omega$*-indestructible mad family* $\mathcal{F}$ *(of size* κ*) such that*

(6.5) $V^{\mathcal{C}_\omega} \models \mathcal{F}$ *is* κ*-almost decided on* T.

Proof. Let $\langle\langle t_\beta, \underset{\sim}{B}_\beta\rangle : \beta < \kappa\rangle$ be an enumeration of

$$T \times \{\,\underset{\sim}{B} \;:\; \underset{\sim}{B} \text{ is a nice } \mathcal{C}_\omega\text{-name of an element of } [T]^{\aleph_0} \text{ in } V^{\mathcal{C}_\omega}\}.$$

Let A_α, $\alpha < \kappa$ be then defined inductively just as in the proof of Theorem 12 with

(6.3)$'$ For $\beta < \alpha$, if $t \Vdash_{\mathcal{C}_\omega}$ "$\underset{\sim}{B}_\alpha \notin \mathcal{I}(\{A_\delta : \delta < \alpha\})$" then $t \Vdash_{\mathcal{C}_\omega}$ "$|\,A_\alpha \cap \underset{\sim}{B}_\beta\,| = \aleph_0$"

in place of (6.3). □

Corollary 3. *For any cardinal* $\kappa \geq \mathfrak{c}$ *in the ground model* V *there is a cardinal preserving generic extension* W *of* V *such that, in* W, $\kappa < \mathfrak{c}$ *and there is a* κ*-almost decided mad family* $\mathcal{F}$ *of size* κ *(furthermore* $\mathcal{F}$ *is* λ*-minimal for all* $\omega \leq \lambda \leq \kappa$*).*

Proof. First extend V to a model V' of $\kappa = \mathfrak{c}$ and MA(σ-centered). In V', let $\mathcal{F}$ be as in Theorem 13. Then $\mathcal{F}$ is as desired in $V^{\mathcal{C}_\mu}$ for any $\mu > \kappa$. The claim in the parentheses follows from Lemma 5, (3) and (6.3)$'$. □

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