Fodor-type Reflection Principle and Balogh's reflection theorems [†]

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Abstract

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In this note, we show that the theorems in Z. Balogh [2] proved there under Axiom R are already provable under Fodor-type Reflection Principle (FRP) introduced in [9] or under a slight extension of FRP still much weaker than Axiom R.

1 Introduction

intro

The purpose of this note is to show that the theorems in [2] proved there under Axiom R are already provable under Fodor-type Reflection Principle (FRP) introduced in [9] or a slight extension of it still much weaker than Axiom R.

In Section 2, we begin with checking the proof of a slight extension of Dow's theorem mentioned in [2]. This is used in Section 3 to show that Balogh's theorem on reflection of metrizability (Theorem 2.2 in [2]) is a consequence of the reflection theorem on metrizability proved under FRP by Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba (Theorem 4.3 in [9]).

In Section 4, we prove that Balogh's reflection theorem on paracompactness (Theorem 1.6 in [2]) holds under FRP.

In Section 5, we consider another reflection theorem on paracompactness by Balogh (Theorem 1.4 in [2]) for which we need a slight strengthening of FRP which is provable from Axiom R. The status of the axiom we use here is still largely unknown (see Problems 2, 3) except that it is still much weaker than Axiom R.

In the following, we consider the topology of a space X as given either by an open base τ of X or by the family $\mathcal{O}$ of all open sets of X. We write $X = (X, \tau)$ or $X = (X, \mathcal{O})$. If $\mathcal{O}$ is generated from the open base τ we write $\mathcal{O} = \mathcal{O}_\tau$.

The approach $X = (X, \tau)$ with an open base τ is more convenient in connection with the method of elementary submodels. This is because, for an open base τ of a topological space X, $\tau \cap M$ is also an open base of $X \cap M$ for an elementary submodel M of $\mathcal{H}(\theta)$ for a sufficiently large cardinal θ with $(X, \tau) \in M$ while $\mathcal{O} \cap M$ for such M does not make up the set of all open sets of a topology on $X \cap M$ in general.

Here, we call a cardinal θ *sufficiently large* if it is regular and $2^{|X|}$, $2^{2^{|X|}}$, $\cdots < \theta$ for all (small) sets X relevant in the context following the declaration of θ being "sufficiently large".

A set M of cardinality $\aleph_1$ is *internally approachable* if M is the union of a continuously increasing chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$. If we consider M as an $\in$-structure, we assume also that each M_α is an elementary submodel of $M = \langle M, \in\rangle$. For an internally approachable M, the sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ as above is called *internally approachable filtration* of M.

A set M is *ω-bounding* if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. In modern terminology, the term ''internally cofinal'' is prefered to ''ω-bounding''. In the following we shall also use this expression.

For a regular uncountable θ any internally approachable $M \prec \mathcal{H}(\theta)$ is internally cofinal. It follows that there are cofinally may internally cofinal $M \prec \mathcal{H}(\theta)$ of cardinality $\aleph_1$.

A space is said to be *(countably) compact* here if it is Hausdorff and satisfies the usual (countably) compactness condition. So a compact space is normal. Note also that

(1.1) a first countable and countably compact space is regular. C-0

[Suppose that X is first countable and countably compact. For a closed $F \subseteq X$ and $p \in X \setminus F$, we have to show that there are O_0, $O_1 \in \mathcal{O}$ such that $F \subseteq O_0$, $p \in O_1$ and $O_0 \cap O_1 = \emptyset$.

Let $\mathcal{B}$ be a countable open neighborhood base of p. For each $x \in F$, let $O_x \in \mathcal{O}$ and $U_x \in \mathcal{B}$ be such that $x \in O_x$ and $O_x \cap U_x = \emptyset$. For each $U \in \mathcal{B}$, let $O_U = \bigcup\{O_x : x \in X, U_x = U\}$. Then O_U is an open set and $O_U \cap U = \emptyset$. Since F is countably compact and $\{O_U : U \in \mathcal{B}\}$ is a countable open cover of F, there is a finite $\mathcal{B}' \subseteq \mathcal{B}$ such that $\{O_U : U \in \mathcal{B}'\}$ already covers F. Then $O_0 = \bigcup\{O_U : U \in \mathcal{B}'\}$ and $O_1 = \cap\mathcal{B}'$ are as desired.]

Following the definition in Engelking [6], a Lindelöf space is a regular topological space X with the Lindelöf property:

every open cover of X has a countable subcover.

Similarly to the case of compact spaces, Lindelöf spaces are normal ([6, Theorem 3.8.2]).

For a property P of a topological space and a cardinal κ, we say that a given topological space X is $\leq\kappa$-P ($<\kappa$-P, respectively) if every subspace Y of X of cardinality $\leq \kappa$ ($< \kappa$, respectively) has the property P. In this notation, we shall always put '$\leq$' or '$<$' to the cardinal κ since very often "κ P" or "κ-P" is already used for some other notions (this is e.g. the case with "$\aleph_1$ meta-Lindelöf"). X is said to be *almost* P if X is $<|X|$- P, that is, if every subspace of X of cardinality $<|X|$ has the property P.

The following notation and the lemma have been introduced in [9].

For a family $\mathcal{F}$ of sets, let $\sim_{\mathcal{F}}$ be the intersection relation on $\mathcal{F}$, i.e. let $F \sim_{\mathcal{F}} G$ if and only if $F \cap G \neq \emptyset$ for F, $G \in \mathcal{F}$, and let $\approx_{\mathcal{F}}$ be the transitive closure of $\sim_{\mathcal{F}}$. An argument in elementary cardinal arithmetic shows the following:

ec

Lemma 1.1. *Let μ be an uncountable regular cardinal and $\mathcal{F}$ a family of sets such that, for all $F \in \mathcal{F}$, we have $|\{G \in \mathcal{F} : F \sim_{\mathcal{F}} G\}| < \mu$. Then every equivalence class of $\approx_{\mathcal{F}}$ has cardinality $< \mu$.* ❑

2 Dow's theorem

dow's thm

A. Dow [4] proved (in ZFC) that every countably compact $\leq\aleph_1$-metrizable space is metrizable. Z. Balogh [1] noted that practically the same proof of Dow's theorem as stated in [4] shows that every countably compact $\leq\aleph_1$-P space is metrizable where P here is the property: *there exists a point countable open base*. In this section we will check the details of the proof of this assertion (Theorem 2.8).

The following elegant proof of Proposition 2.1 is taken from Dow [4].

Recall that the weight $w(X)$ of a topological space $X = \langle X, \mathcal{O}\rangle$ is the minimal cardinality of an open base of X
($w(X) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{O}$ is an open base of $X\}$). Thus X has countable weight if and only if X is second countable. The density of X is the minimal cardinal of a dense subset of X ($d(X) = \min\{|D| : D \subseteq X$ is dense in $X\}$). We have $d(X) \leq w(X)$ [if $\mathcal{B}$ is an open base of X then for any choice function $f \in \prod \mathcal{B}$, $f''\mathcal{B}$ is a dense subset of X.]. For a metrizable space X we have $d(X) = w(X)$ [Suppose that d is a metric on X which induces the topology of X. If $D \subseteq X$ is dense then $\{B(d, \frac{1}{n}) : d \in D, n \in \omega \setminus 1\}$ is an open base of the topology of X.].

lemmaA-0

Lemma A 2.1. *Suppose that $X = \langle X, \mathcal{O}\rangle$ is a toplogical space with $w(X) \geq \aleph_0$. For an arbitrary open base $\mathcal{B}$ of X there is a subset $\mathcal{B}_1$ of $\mathcal{B}$ of cardinality $w(X)$ which is an open base of X.*

Proof. Suppose that $\mathcal{B}$ is an open base of X and $\mathcal{B}_0$ is an open base of X of cardinality $w(X)$.

(a2.1) For B_0 $B_1 \in \mathcal{B}_0$ with $B_0 \subseteq B_1$, if there is $B \in \mathcal{B}$ such that $B_0 \subseteq B \subseteq B_1$ then let $B_{B_0,B_1} \in \mathcal{B}$ be one of such B. Otherwise let $B_{B_0,B_1} = \emptyset$. xA-0

Let

(a2.2) $\mathcal{B}_1 = \{B_{B_0,B_1} : B_0, B_1 \in \mathcal{B}_0, B_0 \subseteq B_1\} \setminus \{\emptyset\}$.

Clearly $\mathcal{B}_1 \in [\mathcal{B}]^{\leq w(X)}$.

Claim A 2.1.1. *$\mathcal{B}_1$ is an open base of X.*

$\vdash$ Suppose that $x \in X$ and $O \in \mathcal{O}$ be such that $x \in O$. Since $\mathcal{B}_0$ is an open base of X, there is $B_1 \in \mathcal{B}_0$ such that $x \in B_1 \subseteq O$. Since $\mathcal{B}$ is also an open base of X, there is $B \in \mathcal{B}$ such that $x \in B \subseteq B_1$. Again since $\mathcal{B}_0$ is an open base of X, there is $B_0 \in \mathcal{B}_0$ such that $x \in B_0 \subseteq B$. By (a2.1), we have $x \in B_0 \subseteq B_{B_0,B_1} \subseteq B_1 \subseteq O$. $\dashv$ (Claim A 2.1.1)

$\square$ (Lemma A 2.1)

Note that by Arhangelskii's Theorem, all Hausdorff spaces with a countable base are of cardinality $\leq 2^{\aleph_0}$.

juhasz

Proposition 2.1 (Juhász [12]). *For any space X if every subspace of X of cardinality $\leq \aleph_1$ has countable weight then X has countable weight.*

Proof. Suppose that $X = (X, \mathcal{O})$ is as above. Let M be an internally cofinal elementary submodel of $\mathcal{H}(\theta)$ for a sufficiently large θ such that $|M| = \aleph_1$ and $\langle X, \mathcal{O}\rangle \in M$.

Claim 2.1.1. *$\mathcal{O} \cap M$ (or, more precisely, the family $\{O \cap (X \cap M) : O \in \mathcal{O} \cap M\}$) makes up an open base of the subspace topology of $X \cap M$.*

$\vdash$ Suppose that $x \in X \cap M$ and $x \in O \in \mathcal{O}$. It is enough to show that there is $O' \in \tau \cap M$ such that $x \in O' \cap M \subseteq O \cap M$.

Since $|(X \cap M) \setminus O| \leq \aleph_1$, $(X \cap M) \setminus O$ as a subspace of X has countable weight. Hence there is a countable $D \subseteq (X \cap M) \setminus O$ which is dense in $(X \cap M) \setminus O$. By the internal cofinality of M there is $D' \in [X \cap M]^{\aleph_0} \cap M$ such that $D \subseteq D'$.

Now, since $D' \cup \{x\}$ is a countable subspace of X, there is a countable $\mathcal{B} \subseteq \mathcal{O}$ such that $\{O \cap (D' \cup \{x\}) : O \in \mathcal{B}\}$ is an open base of the subspace topology of $D' \cup \{x\}$. By elementarity, we may assume that $\mathcal{B} \in M$. Since $\mathcal{B}$ is countable we have $\mathcal{B} \subseteq \mathcal{O} \cap M$. In particular, there is $O' \in \mathcal{B} \subseteq M$ such that $x \in O'$ and

$$\text{(a2.3)} \quad O' \cap D = (O' \cap (D' \cup \{x\})) \cap D \subseteq O \cap (D' \cup \{x\}) = \emptyset. \qquad \text{C-1}$$

We have $O' \cap M \subseteq O \cap M$: Otherwise $O' \cap ((X \cap M) \setminus O) \neq \emptyset$. Then there would be some $d \in D \cap O'$ since D is dense in $(X \cap M) \setminus O$. This is a contradiction to (a2.3). $\dashv$ (Claim 2.1.1)

By the assumption on X, by Claim 2.1.1 and by Lemma 2.1, there is a $\mathcal{B} \in [\mathcal{O} \cap M]^{\aleph_0}$ such that $\mathcal{B}$ makes up an open base of the subspace topology of $X \cap M$. Let $\mathcal{B}' \in [\mathcal{O}]^{\aleph_0} \cap M$ be such that $\mathcal{B} \subseteq \mathcal{B}'$. $\mathcal{B}'$ is also an open base of $X \cap M$. It follows that $M \models$ "$\mathcal{B}'$ is an open base of X". By elementarity, it follows that $\mathcal{B}'$ is really an open base of X. Since $\mathcal{B}'$ is countable this finishes the proof. ❏ (Proposition 2.1)

B-0

Lemma 2.2. *A countably compact space X is metrizable if and only if X has countable weight.*

Proof. Suppose that X is countably compact and metrizable. We show that X has countable weight. Let d be a metric on X which induces the topology of X. Let θ be a sufficiently large regular cardinal and $M \prec \mathcal{H}(\theta)$ be countable such that X, $d \in M$. It is enough to show that X is separative and for that it is enough to show that $X \cap M$ is dense in X.

Suppose that this is not the case. then there is $x^* \in O = X \setminus \overline{X \cap M}$. Let

(a2.4) $\mathcal{U} = \{O\} \cup \{U : U \in M, U$ is open in X such that $x^* \notin U\}$. B-0-0

Claim 2.2.1. *$\mathcal{U}$ is an open covering of X.*

$\vdash$ Suppose that $y \in \overline{X \cap M}$. Let $\varepsilon \in \mathbb{Q} \subseteq M$ be such that $0 < \varepsilon \leq d(y, x^*)$ and let $z \in B(y, \frac{1}{2}\varepsilon) \cap M$. Then $y \in U = B(z, \frac{1}{2}\varepsilon) \in M$ but $x^* \notin U$. Thus $y \in U \in \mathcal{U}$ and we have $y \in \bigcup \mathcal{U}$. $\dashv$ (Claim 2.2.1)

By countable compactness of X, there are $U_0, \ldots, U_{k-1} \in \mathcal{U} \setminus \{O\}$ such that $O \cup U_0 \cup \cdots \cup U_{k-1} = X$. Since $U_0 \cup \cdots \cup U_{k-1} \supseteq \overline{X \cap M} \supseteq X \cap M$, we have

(a2.5) $M \models$ "$\{U_0, \ldots, U_{k-1}\}$ is an open covering of X".

Hence, by elementarity, $\{U_0, \ldots, U_{k-1}\}$ should be really an open covering of X. But this is a contradiction to $x^* \notin U_0 \cup \cdots \cup U_{k-1}$.

Conversely, if X is countable compact and of countable weight then it is compact and regular (see (1.1)). Hence X is metrizable by Urysohn's Metrization Theorem. ❏ (Lemma 2.2)

lemmaA-1

Corollary A 2.2. *For a topological space X the following are equivalent:*

(a) *X is countably compact and $w(X) \leq \aleph_0$.*
(b) *X is countably compact and metrizable.*
(c) *X is compact and metrizable.*

Proof. The equivalence of (a) and (b) is Lemma 2.2. (c) $\Rightarrow$ (b) is trivial. Since countable compactness and countable weight implies (full) compactness, (b) ($\Leftrightarrow$ (a)) implies (c). ❏ (Corollary A 2.2)

Lemma 2.3. *For a topological space X and a subspace $Y \subseteq X$, we have $w(Y) \leq w(X)$. For any $y \in Y$, $\chi(y, Y) \leq \chi(y, X)$.* B-1

Proof. Suppose that $\{O_\alpha : \alpha < \kappa\}$ is an open base of X (neighborhood base of y in X resp.). Then $\{O_\alpha \cap Y : \alpha < \kappa\}$ is an open base of Y (neighborhood base of y in Y resp.). ❑ (Lemma 2.3)

Lemma 2.4. *Suppose that $X = (X, \tau)$, $Y \subseteq X$, $x \in Y$ and that X is regular at x. For $\mathcal{B} \subseteq \mathcal{O}_\tau$, if $\{U \cap Y : U \in \mathcal{B}\}$ is a neighborhood base of x in Y then $\{U \cap \overline{Y} : U \in \mathcal{B}\}$ is a neighborhood base of x in $\overline{Y}$. In particular, we have $\chi(x, Y) = \chi(x, \overline{Y})$.* B-2

Proof. Suppose that $O \in \mathcal{O}_\tau$ with $x \in O$. We have to show that there is $U \in \mathcal{B}$ such that $U \cap \overline{Y} \subseteq O \cap \overline{Y}$.

Now, since X is regular at x, there is $O' \in \mathcal{O}_\tau$ such that $x \in O'$ and $\overline{O'} \subseteq O$. Let $U \in \mathcal{B}$ be such that $U \cap Y \subseteq O' \cap Y$. Then we have

$$U \cap \overline{Y} \subseteq \overline{U} \cap \overline{Y} = \overline{U \cap Y} \subseteq \overline{O' \cap Y} = \overline{O'} \cap \overline{Y} \subseteq O \cap \overline{Y}.$$

This shows that $\mathcal{B}$ is also a neighborhood base of x in $\overline{Y}$. Thus $\chi(x, Y) \geq \chi(x, \overline{Y})$. We also have "$\leq$" by Lemma 2.3. ❑ (Lemma 2.4)

If X is metrizable then it has a point countable open base. [This can be seen from the well-known fact that if X is metrizable then X is paracompact. From this it is easy to see that a metrizable space has a σ-locally finite open base. But such a base is locally countable.]

Lemma 2.5 (Proposition 2.3 in [4])**.** *If a space $X = (X, \tau)$ has a point countable base then, for a sufficiently large θ and $M \prec \mathcal{H}(\theta)$ with $\langle X, \tau \rangle \in M$, $\tau \cap M$ is a base for (each point of) $\overline{X \cap M}$.* B-3

Proof. Suppose that $X = (X, \tau)$, θ and M are as above. By elementarity, there is a point countable base $\mathcal{B}$ of X with $\mathcal{B} \in M$.

Suppose that

(2.1) $\quad x \in \overline{X \cap M}$ x-0

and $B_0 \in \mathcal{B}$ is a neighborhood of x. Let $O_0 \in \tau$ and $C_0 \in \mathcal{B}$ be such that $x \in C_0 \subseteq O_0 \subseteq B_0$. By (2.1), there is $y \in C_0 \cap (X \cap M) = C_0 \cap M$. Since there are only countably many $B \in \mathcal{B}$ with $y \in B$, all such B's are in M. In particular, we have C_0, $B_0 \in M$.

Again by elementarity, we have $M \models \exists O \in \tau (C_0 \subseteq O \subseteq B_0)$. Hence there is an $O_1 \in \tau \cap M$ such that $x \in C_0 \subseteq O_1 \subseteq B_0$. This shows that $\tau \cap M$ is a local base for x. ❑ (Lemma 2.5)

B-4

Lemma 2.6 (Proposition 2.4 in [4]). *Suppose that $X = (X, \tau)$ is a countably compact space. If $M \prec \mathcal{H}(\theta)$ is countable with $\langle X, \tau\rangle \in M$ and $\tau \cap M$ is not a base for (X, τ) then there is $z \in \overline{X \cap M}$ such that $\tau \cap M$ is not a base at z.*

Proof. If $\overline{X \cap M} = X$ then the assertion is just trivial. So assume that there is $x \in X \setminus \overline{X \cap M}$. Suppose, toward a contradiction, that $\tau \cap M$ is an open base at each $z \in \overline{X \cap M}$. Then we can choose $O_z \in \tau \cap M$ such that $z \in O_z$ and

$$(2.2) \quad x \notin O_z$$ x-1

for each $z \in \overline{X \cap M}$. Since $\overline{X \cap M}$ is countably compact and $\{O_z : z \in \overline{X \cap M}\} \subseteq \tau \cap M$ is a countable open covering of $\overline{X \cap M}$, there are $z_1, \ldots, z_n \in \overline{X \cap M}$ for some $n \in \omega$ such that $\overline{X \cap M} \subseteq O_{z_1} \cup \cdots \cup O_{z_n}$. It follows that $M \models$ "$O_{z_1}, \ldots, O_{z_n}$ covers X". By elementarity it follows that $O_{z_1}, \ldots, O_{z_n}$ really covers X. But this is a contradiction to (2.2). ❑ (Lemma 2.6)

Using the lemmas above, we can prove the following theorem of Miščenko:

B-5

Theorem 2.7 (Miščenko). *A countably compact* (Hausdorf) *space with a point countable open base has a countable open base .*

Proof. Suppose that θ is a sufficiently large regular cardinal and M is countable with $X \in M \prec \mathcal{H}(\theta)$. Then $\tau \cap M$ is an open base for $\overline{X \cap M}$ by Lemma 2.5. By Lemma 2.6 it follows that $\tau \cap M$ is an open base of X. But $\tau \cap M$ is countable. ❑ (Theorem 2.7)

Miščenko's Theorem improves Corollary A 2.2:

lemmaA-2

Corollary A 2.3. *For a toplogical space X the following are equivalent:*

(a) *X is countably compact (Hausdorff) with a point countable open base.*
(b) *X is compact metrizable.*

Proof. (a) $\Rightarrow$ (b): Suppose that X is countably compact (Hausdorff) with a point countable open base. Then $w(X) \leq \aleph_0$. By Lemma 2.2, it follows that X is metrizable. By Corollary A 2.2, X is compact metrizable.

(b) $\Rightarrow$ (a) is trivial (see the remark before Lemma 2.5.)

❑ (Corollary A 2.3)

We can even prove the following. Note that a countably compact space with a point countable open base is regular as noted in (1.1). Thus the following Theorem 2.8 indeed generalizes Miščenko's Theorem.

This theorem is also a (slight?) generalization of the original Dow's Theorem since every metrizable space has a σ-locally finite open base (this follows, e.g., from A.H. Stone's theorem which states every metrizable space is paracompact). A σ-locally finite open base is apparently point countable. The fact that every metrizable space has a σ-locally finite opne base is also a part of the Bin-Nagata-Smirnov Metrization Theorem:

Theorem A.2.4. (Bing-Nagata-Smirnow Metrization Theorem) *A Hausdorff space X is metrizable if and only if X is regular and it has a σ-locally finite open base.* ❑ lemmaA-3

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Theorem 2.8. (A variant of Theorem 3.1 in Dow [4]. See also [2]) *If X is a countably compact space such that every subspace of X of cardinality $\leq \aleph_1$ has a point countable open base, then X is metrizable.* dow's thm-x

Proof. Suppose, for contradiction, that there is a countably compact space $X = (X, \tau)$ such that

(2.3) all subspaces of X of cardinality $\leq \aleph_1$ have a point countable open base but B-6

(2.4) X is not metrizable. B-7

Let θ be sufficiently large and let M be an internally approachable elementary submodel of $\mathcal{H}(\theta)$ of cardinality $\aleph_1$ such that $\langle X, \tau \rangle \in M$.

Since $w(X) > \aleph_0$ (by (2.4) and Lemma 2.2), there is a $Z \in [X]^{\aleph_1}$ such that $w(Z) > \aleph_0$ by Juhász' Theorem (Proposition 2.1). By elementarity, there is such a $Z \in M$.

We have $w(\overline{Z}) > \aleph_0$ by Lemma 2.3. Since $\overline{Z}$ is countably compact, $\overline{Z}$ is non metrizable by Lemma 2.2. Thus we may assume without loss of generality $X = \overline{Z}$. For each $x \in X \cap M$, $Z \cup \{x\}$ has cardinality $\aleph_1$ and hence it has a point countable open base by (2.3). In particular, $\chi(x, Z \cup \{x\}) = \aleph_0$. Since $Z \cup \{x\} \in M$, it follows by Lemma 2.5 that $\tau \cap M$ is an open base of $(X \cap M, \tau)$. Thus

(2.5) $(X \cap M, \tau \cap M)$ has a point countable open base. x-1-0

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be an internally approachable filtration of M such that Z, $\langle X, \tau \rangle \in M_0$.

Since $w(X) > \aleph_0$ and M_α is countable, $\tau \cap M_\alpha$ is not an open base of (X, τ) for any $\alpha < \omega_1$. Thus, by Lemma 2.6, there is $z \in \overline{X \cap M_\alpha}$ such that $\tau \cap M_\alpha$ is not an open base at z. Since $M_\alpha \in M_{\alpha+1}$, there is such z in $M_{\alpha+1}$ by elementarity.

Let N be a countable elementary submodel of $\mathcal{H}(\theta)$ such that

(2.6) $X, Z, M, \langle M_\alpha : \alpha < \omega_1\rangle \in N$. x-2

Let $\alpha^* = \omega_1 \cap N$. By the remark above there is $z^* \in M_{\alpha^*+1}$ such that

(2.7) $z^* \in \overline{X \cap M_{\alpha^*}}$ and $\tau \cap M_{\alpha^*}$ does not contain any neighborhood base at z^*. x-3

On the other hand, by (2.6), we have

$$(\tau \cap M) \cap N = \bigcup\{\tau \cap M_\beta : \beta < \alpha^*\} = \tau \cap M_{\alpha^*}.$$

Hence by (2.5) and Lemma 2.5, $\tau \cap M_{\alpha^*}$ is a neighborhood base for any $z \in \overline{X \cap M_{\alpha^*}}$. This is a contradiction. ❑ (Theorem 2.8)

Corollary A 2.5. (Dow's Metrization Theorem) *A counably compact (Hausdorff) space X is metrizable if and only if every subspace Y of X of cardinality $< \aleph_2$ is metrizable.* lemmaA-4

Proof. If X is metrizable then all subspaces Y of X are metrizable. In paticular all subspaces of X of cardinality $< \aleph_2$ are metrizable.

Suppose now that X is countably compact (Hausdorff) and all subspace Y of X of cardinality $< \aleph_2$ are metrizable. Then by Theorem A 2.4 (see also the remark above Theorem A 2.4), all of such Y have a point countable base. By Theorem 2.8, it follows that X is metrizable. ❑ (Theorem A 2.5)

The condition ''of cardinality $< \aleph_2$'' is optimal. First let me cite the following theorem for rererence:

Theorem A.2.6. (Mazurkiewicz-Sierpiński 1920, see [milliet]) *Any countable and compact (Hausdorff) space X is isomorphic to a countable ordinal with the order topology.* ❑

By Theorem 2.8, it follows that all countable and compact Hausdorff spaces are metrizable since all countable ordinals are embeddable in $\mathbb{R}$. Actually more general statement holds:

lemmaA-5

Lemma A 2.7. *Any countable linear ordering $X = (X, \leq_X)$ can be embedded order-preservingly and continuously in $[0, 1]$. In particular, every countable topological space with linear order topology is metrizable.*

Proof. Let $\langle r_n : n \in \omega\rangle$ be a sequence of positive real numbers such that $\sum_{n\in\omega} r_n = 1$.

For a countable lenear ordering $X = (X, \leq_X)$, let $f : X \to \omega$ be a 1-1 mapping. Let $g : X \to \mathbb{R}$ be defined by

(2.8) $\quad g(x) = \sum_{n \in I_x} r_n$ B-8

for $x \in X$ where $I_x = \{n \in \omega : n = f(y)$ for some $y <_X x\}$. Then g is a order-preserving continuous embedding of X in $\mathbb{R}$. ❑ (Lemma A 2.7)

Example A 2.9. (in ZFC) *There is a compact, first countable non-metrizable space X such that all countable subspace of X are metrizable.* lemmaA-6

Proof. Let $X = [0,1] \times \{0,1\}$ be with the order topology for the lexicographical ordering on $[0,1] \times \{0,1\}$.

claimA-0

Claim A 2.7.1. *X is compact.*

$\vdash$ X as a linear order topological space is Hausdorff. Suppose that $\mathcal{U} \subseteq \mathcal{O}_X$ is an open covering of X. Without loss of generality we may assume that each $I \in \mathcal{U}$ is an open interval with the form

(a2.6) $\; I = (\langle a_I, i_I \rangle, \langle b_I, j_I \rangle)_X$ B-9

where $(x, y)_X$ for x, $y \in X$ denotes the interval $\{z \in X : x <_X z <_X y\}$ in X.

Let $\mathcal{V}_0 = \{(a_I, b_I) : I \in \mathcal{U}\}$. If $x \in [0,1] \setminus \bigcup \mathcal{V}_0$, then there are I, $I' \in \mathcal{U}$ such that $b_I = b_{I'} = x$. Thus

(a2.7) $\; \mathcal{V}_1 = \mathcal{V}_0 \cup \{(a_I, b_{I'}) : I, I' \in \mathcal{U},\, b_I = a_{I'}\}$ B-10

is an open covering of $[0,1]$.

By compactness of $[0,1]$ there is a finite $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\{(a_I, b_I) : I \in \mathcal{U}_0\} \cup \{(a_I, b_{I'}) : I, I' \in \mathcal{U}_0, b_I = a_{I'}\}$ is a covering of $[0,1]$. $\mathcal{U}_0$ is a finite covering of X. $\dashv$ (Claim A 2.7.1)

Claim A 2.7.2. *X is first countable.*

$\vdash$ $\langle 0,0 \rangle$ and $\langle 1,1 \rangle$ are isolated in X and hence $\{\{\langle 0,0 \rangle\}\}$ and $\{\{\langle 1,1 \rangle\}\}$ are neighborhood bases of $\langle 0,0 \rangle$ and $\langle 1,1 \rangle$ respectively.

$\{(\langle r - \frac{1}{n}, 1 \rangle, \langle r, 1 \rangle)_X : n \in \omega \setminus 1\}$ is a countable neighborhood base of $\langle r, 0 \rangle$ for all $r \in (0,1]$. $\{(\langle r, 0 \rangle, \langle r + \frac{1}{n}, 0 \rangle)_X : n \in \omega \setminus 1\}$ is a countable neighborhood base of $\langle r, 1 \rangle$ for all $r \in [0,1)$. $\dashv$ (Claim A 2.7.2)

Claim A 2.7.3. *$w(X) = 2^{\aleph_0}$. In particular, X is not metrizable.*

$\vdash$ Suppose that $\mathcal{B}$ is an open base of X. For each $r \in [0,1]$ there is $B_r \in \mathcal{B}$ such that $\langle r, 0 \rangle \in B \subseteq (\langle 0,0 \rangle, \langle r, 1 \rangle)_X$. Since B_r's $r \in [0,1]$ should be distinct to each other we have $|\mathcal{B}| \geq 2^{\aleph_0}$. It is also easy to see that there is an open base $\mathcal{B}$ of cardinality $\leq 2^{\aleph_0}$.

Since compact metrizable space must be of countable weight by Corollary A 2.2, it follows from Claim A 2.7.1 that X is not metrizable. $\dashv$ (Claim A 2.7.3)

By Lemma A 2.7, all countable subspace of X are metrizable. Thus this X is as desired. ❑ (Example A 2.7)

3 Balogh's metrization theorem under FRP

metrization

The following two theorems were proved in S. Fuchino, I. Juhasz, L. Soukup, Z. Szentmiklóssy and T. Usuba [9].

`Recall that a topological space` X `is meta-Lindelöf if every open cover` $\mathcal{B}$ `of` X `has a point countable open refinement.`

meta-L in ZFC

Theorem 3.1 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba, [9, Theorem 4.2]). *Suppose that X is a locally countably compact and meta-Lindelöf space. If X is $\leq \aleph_1$-metrizable then it is actually metrizable.* ❑

ssmL

Theorem 3.2 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9, Theorem 4.3]). (1) *Assume that $\mathsf{FRP}(\kappa)$ holds for every regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ and X is a locally separable, countably tight space with $L(X) \leq \lambda$. If X is $\leq \aleph_1$-meta-Lindelöf then X is actually meta-Lindelöf.*

(2) *Under FRP every locally separable, countably tight and $\leq \aleph_1$-meta-Lindelöf space is meta-Lindelöf.* ❑

Here, for a regular cardinal $\kappa \geq \omega_1$, $\mathsf{FRP}(\kappa)$ (*The Fodor-type Reflection Principle for κ*) is the following statement:

$\mathsf{FRP}(\kappa)$: For any stationary $S \subseteq E^\kappa_\omega = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

(3.1) $\mathrm{cf}(I) = \omega_1$; c-0

(3.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$; c-1

(3.3) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$. c-2

FRP is the axiom which asserts that $\mathsf{FRP}(\kappa)$ holds for all regular cardinal $\kappa \geq \aleph_2$. Note that we can only demand $\mathsf{FRP}(\kappa)$ for a regular κ since $\mathsf{FRP}(\kappa)$ for a singular κ is easily shown to be inconsistent (see Lemma 2.2 in [9]).

In [9], it is shown that $\mathsf{FRP}(\kappa)$ for a regular cardinal κ follows from $\mathsf{RP}(\kappa)$ which is a weakening of of Axiom R for κ. Thus FRP is a consequence of Axiom R. On the other hand, it is also proved in [9] that $\mathsf{FRP}(\kappa)$ is preserved by c.c.c.-extension of the universe. Thus FRP is strictly weaker than Axiom R.

Here, the Reflection Principle $\mathsf{RP}(\kappa)$ and Axiom R for κ (Notation: $\mathsf{AR}(\kappa)$) are defined as follows:

$\mathsf{RP}(\kappa)$: For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

(3.4) $\omega_1 \subseteq I$; RP-0

(3.5) $\mathrm{cf}(I) = \omega_1$; RP-1

(3.6) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$. RP-2

$\mathsf{AR}(\kappa)$: For any stationary $S \subseteq [\kappa]^{\aleph_0}$ and ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, there is $I \in \mathcal{T}$ such that $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$

where $\mathcal{T} \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or *tight and unbounded* in Fleissner's terminology in [7]) if

(3.7) $\mathcal{T}$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and tight-0

(3.8) for any increasing chain $\langle I_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{T}$ of length ω_1, we have $\bigcup_{\alpha<\omega_1} I_\alpha \in \mathcal{T}$. tight-1

Axiom R is the assertion that $\mathsf{AR}(\kappa)$ holds for all cardinals $\kappa \geq \aleph_2$ and RP is the assertion that $\mathsf{RP}(\kappa)$ holds for all cardinals κ with $\kappa \geq \aleph_2$.

It is easy to see that $\mathsf{AR}(\kappa)$ implies $\mathsf{RP}(\kappa)$. R.E. Beaudoin [3] proved that Axiom R follows from $\mathsf{MA}^+(\sigma\text{-closed})$.

By the theorems above and by Theorem 2.8, we can prove the following improvement of Theorem 2.2 in Z. Balogh [2] where the assertion (2) of the following theorem was proved under Axiom R.

balogh-thm2.2

Theorem 3.3. (1) *Let λ be a cardinal such that for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ we have $\mathsf{FRP}(\kappa)$. If X is a regular locally countably compact space with $L(X) \leq \lambda$ and*

(3.9) *every subspace of X of cardinality $\leq \aleph_1$ has a point countable open base,* x-4

then X is metrizable.

(2) *Assume FRP. If X is a regular locally countably compact space satisfying* (3.9)*, then X is metrizable.*

Proof. We prove only (1) since (2) clearly follows from (1).

Let X be as in (1). Then every point of X has a countably compact neighborhood, and this neighborhood is compact metrizable by Theorem 2.8. By Lemma 2.2, it follows that X is both locally separable and countably tight[1]. Also X is $\leq \aleph_1$-meta-Lindelöf by (3.9). Hence X is meta-Lindelöf by Theorem 3.2 (1). By Theorem 3.1, it follows that X is metrizable. ❑ (Theorem 3.3)

Theorem 3.3 implies the following theorem which can be also derived directly form Theorem 3.2:

[1]Recall that a topological space X is countably tight if, for every $x \in X$ and $Y \subseteq X$, $x \in \overline{Y}$ always implies that there is a $Y' \in [Y]^{\leq\aleph_0}$ such that $x \in \overline{Y'}$.

Theorem 3.4 (Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9]). (1) *Let λ be a cardinal such that for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$ we have* $\mathsf{FRP}(\kappa)$*. If X is a locally countably compact and $\aleph_1$-metrizable space with $L(X) \leq \lambda$ then X is metrizable.* balogh revisited

(2) *Assume* FRP*. Then every locally countably compact and $\aleph_1$-metrizable space is metrizable.* ❑

In S. Fuchino, H. Sakai, L. Soukup and T. Usuba [11], it is proved that the assertion of Theorem 3.2, (1) as well as Theorem 3.4, (1) are equivalent to:

$\mathsf{FRP}(\leq\lambda)$: $\mathsf{FRP}(\kappa)$ holds for each regular cardinal κ with $\omega_1 < \kappa \leq \lambda$

over ZFC. Thus also we obtain the following:

Theorem 3.5. *The assertion of* Theorem 3.3, (1) *is equivalent to* $\mathsf{FRP}(\leq\lambda)$ *over* ZFC. ❑

4 Reflection of paracompactness in countably tight locally Lindelöf spaces

paracompact

In this section we prove that Theorem 1.6 in Balogh [2] is already provable under FRP (Theorem 4.6).

Recall that a space X is *locally Lindelöf* if every point x of X has an open neighborhood O such that $\overline{O}$ is a Lindelöf subspace of X.

L-a-0

Lemma 4.1. *For a topological space $X = (X, \mathcal{O})$, if $\mathcal{F} \subset \mathcal{P}(X)$ is locally finite, then we have $\bigcup\{\overline{Y} : Y \in \mathcal{F}\} = \overline{\bigcup\mathcal{F}}$.*

Proof. The inclusion "$\subseteq$" is clear. To show the other inclusion "$\supseteq$", suppose $x \in \overline{\cup\mathcal{F}}$. Let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. Then we have $x \in \overline{\bigcup\mathcal{F}_0} = \bigcup\{\overline{Y} : Y \in \mathcal{F}_0\}$. Thus $x \in \bigcup\{\overline{Y} : Y \in \mathcal{F}\}$. ❑ (Lemma 4.1)

L-a-1

Lemma 4.2. *For a topological space $X = (X, \mathcal{O})$, if $\mathcal{F} \subseteq \mathcal{P}(X)$ is locally finite, then $\overline{\mathcal{F}} = \{\overline{Y} : Y \in \mathcal{F}\}$ is also locally finite.*

Proof. For $x \in X$, let $O \in \mathcal{O}$ be such that $x \in O$ and $\mathcal{F}_0 = \{Y \in \mathcal{F} : O \cap Y \neq \emptyset\}$ is finite. For any $y \in O$ if $y \in \overline{Y}$ for some $Y \in \mathcal{F}$ then $O \cap Y \neq \emptyset$, i.e. $Y \in \mathcal{F}_0$. So we have $\{Y \in \mathcal{F} : O \cap \overline{Y} \neq \emptyset\} = \mathcal{F}_0$. ❑ (Lemma 4.2)

Recall that a topological space X is paracompact if X is Hausdorff and every open cover of X has a locally finite open refinement. Morita's theorem states that every Lindelöf space is paracompact.

The following characterization of paracompactness of locally Lindelöf spaces was already mentioned in [2]. In the proof of Theorem 4.6 we actually only use the trivial direction "(a) $\Rightarrow$ (b)" of this characterization. Nevertheless the characterization explains the need to look at open partitions of a given locally Lindelöf space to prove the paracompactness of the space.

A topological space X is para-Lindelöf if any open covering of X has a locally countable open refinement. We have the following implication[2] for any topological space:

$$\begin{array}{ccccc} \text{metrizable} & \rightarrow & \text{paracompact} & \rightarrow & \text{metacompact} \\ & & \downarrow & & \downarrow \\ & & \text{para-Lindelöf} & \rightarrow & \text{meta-Lindelöf} \end{array}$$

Lemma 4.3. *Suppose that X is a locally Lindelöf space*[3]*. Then the following are equivalent:* L-0

(a) *X can be partitioned into open Lindelöf subspaces.*

(b) *X is paracompact.*

(c) *X is para-Lindelöf.*

Proof. "(a) $\Rightarrow$ (b)": Suppose that X is partitioned into open Lindelöf subspaces. By Morita's theorem each subspace in the partition is paracompact. Hence it follows that the whole space is paracompact as well.

"(b) $\Rightarrow$ (c)" is trivial.

"(c) $\Rightarrow$ (a)": Suppose now that X is a locally Lindelöf para-Lindelöf space. We show that there is a partition of X into clopen Lindelöf subspaces. Let $\mathcal{A} \subseteq \mathcal{O}$ be an open covering of X such that $\overline{Y}$ is Lindelöf for all $Y \in \mathcal{A}$. Let $\mathcal{B}$ be a locally countable open refinement of $\mathcal{A}$. Then elements of $\mathcal{B}' = \{\overline{Y} : Y \in \mathcal{B}\}$ are Lindelöf and $\mathcal{B}'$ is still locally countable by Lemma 4.2.

Claim 4.3.1. *For any $Y \in \mathcal{B}'$, $\{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$ is countable.*

$\vdash$ Suppose $Y \in \mathcal{B}'$. Let $S = \{Z \in \mathcal{B}' : Y \cap Z \neq \emptyset\}$. For each $y \in Y$, let $O_y \in \mathcal{O}$ be such that $y \in O_y$ and $\{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset\}$ is countable. Note

---

[2] A topological space X is meta-Lindelöf if any open covering of X has a point countable open refinement. The implication "metrizable $\rightarrow$ paracompact" is Stone's Theorem (see e.g. [6], p.280).

[3] We assume that a Lindelöf space is a regular space with Lindelöf property. A topological space X is locally Lindelöf if for every $x \in X$ there is an open set $x \in O \subseteq X$ such that $\overline{O}$ is a Lindelöf space in the supspace topology. In particular, a locally Lindel"of space is locally regular.

that we can find such O_y since $\mathcal{B}'$ is locally finite. Since Y is Lindelöf, there is a countable $Y_0 \subseteq Y$ such that $\{O_y : y \in Y_0\}$ is a cover of Y. Then we have $S \subseteq \{Z \in \mathcal{B}' : O_y \cap Z \neq \emptyset \text{ for some } y \in Y_0\}$ and the right side of the inclusion is easily seen to be countable. $\dashv$ (Claim 4.3.1)

Let $\sim_{\mathcal{B}'}$ be the intersection relation[4] on $\mathcal{B}'$ and $\approx_{\mathcal{B}'}$ be its transitive closure. Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_{\mathcal{B}'}$. By the claim above, it follows that each $e \in \mathbb{E}$ is countable. Thus $\bigcup e$ is Lindelöf and $\bigcup e$ is closed by Lemma 4.1. Since $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X, each $\bigcup e$ for $e \in \mathbb{E}$ is also open.

Thus $\{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen Lindelöf subspaces of X.

❑ (Lemma 4.3)

A similar proof shows the following:

Lemma 4.4. *For a locally (separable & Lindelöf) space X, the following are equivalent:*

(a) *X has an open partition into Lindelöf spaces;*

(b) *X is paracompact;*

(c) *X is meta-Lindelöf.*

Proof. "(a) $\Rightarrow$ (b)": If $\mathcal{A}$ is an open partition of X into Lindelöf spaces then each $Y \in \mathcal{A}$ is paracompact by Morita's theorem. Hence X is also paracompact.

"(b) $\Rightarrow$ (c)" is trivial.

"(c) $\Rightarrow$ (a)": Suppose that X is meta-Lindelöf. Let $\mathcal{A}$ be an open covering of X consisting of separable Lindelöf subspaces and $\mathcal{A}'$ be its point countable open refinement. Note that elements of $\mathcal{A}'$ are still separable as open subspaces of separable spaces.

Claim 4.4.1. *For each $Y \in \mathcal{A}'$, the set $\{Z \in \mathcal{A}' : Y \cap Z \neq \emptyset\}$ is coutable.*

$\vdash$ Let $D \in [Y]^{\aleph_0}$ be a dense subset of Y. Let $\mathcal{B} = \{Z \in \mathcal{A}' : Z \cap D \neq \emptyset\}$. $\mathcal{B}$ is countable, since $\mathcal{A}'$ is point countable. We show that $\mathcal{B} = \{Z \in \mathcal{A}' : Y \cap Z \neq \emptyset\}$. "$\subseteq$" is clear. To show "$\supseteq$", suppose that $Z \in \mathcal{A}'$ is such that $Y \cap Z \neq \emptyset$. Then as a nonempty open subset of Y, $Y \cap Z$ contains some element of D which means that $Z \in \mathcal{B}$. $\dashv$ (Claim 4.4.1)

Let $\approx_{\mathcal{A}'}$ be the transitive closure of the intersection relation on $\mathcal{A}'$. Then each equivalence class $e \subseteq \mathcal{A}'$ with respect to $\approx_{\mathcal{A}'}$ is countable by Claim 4.4.1. Since $\bigcup e$ is also closed. $\bigcup e = \bigcup \{\overline{Z} : Z \in e\}$. Since each $\overline{Z}$, $Z \in e$ is Lindelöf as a closed subspace of a Lindelöf space, it follows that $\bigcup e$ is also Lindelöf. Thus $\{\bigcup e : e \in \mathcal{A}'/\approx_{\mathcal{A}'}\}$ is a partition of X as in (a). ❑ (Lemma 4.4)

[4] That is, for $Y, Y' \in \mathcal{B}'$, $Y \sim_{\mathcal{B}'} Y' \iff Y \cap Y' \neq \emptyset$.

Lemma 4.5 (Proposition 1.1 in Balogh [2])**.** *If a topological space $X = (X, \mathcal{O})$ is locally Lindelöf, then $\mathcal{B} = \{V \subseteq X : V$ is an open Lindelöf subspace of $X\}$ forms an open base of X.* L-1

Proof. Note that a closed subspace of a Lindelöf space is also Lindelöf. Hence, for $x \in X$ and $x \in O \in \mathcal{O}$, there is a $U \in \mathcal{O}$ such that $x \in U \subseteq O$ and $\overline{U}$ is Lindelöf. Since $\overline{U}$ is a Lindelöf space and thus normal, we can construct a sequence $\langle O_i : i \in \omega \rangle$ of open sets such that

$$(4.1) \quad x \in O_0 \subseteq \overline{O_0} \subseteq O_1 \subseteq \overline{O_1} \subseteq \cdots \subseteq U.$$

Let $O^* = \bigcup_{i\in\omega} O_i$. Then O^* is an open neighborhood of x and $O^* \subseteq O$. O^* is Lindelöf since we can also represent O^* as the countable union of Lindelöf spaces, namely as $O^* = \bigcup_{i\in\omega} \overline{O_i}$. ❑ (Lemma 4.5)

Z. Balogh [2] proved the following theorem under Axiom R.

Theorem 4.6 (FRP)**.** *Suppose that X is locally Lindelöf and countably tight. If every open subspace Y of X with $L(Y) \leq \aleph_1$ is paracompact then X itself is paracompact.* L-2

Proof. A variation of the proof of Theorem 4.3 in S. Fuchino, I. Juhasz, L. Soukup, Z. Szentmiklóssy and T. Usuba [9] will do.

It is enough to prove that the following $(4.2)_\kappa$ holds for all cardinal κ by induction on κ:

$(4.2)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) \leq \kappa$, if every open subspace of X of Lindelöf degree $\leq \aleph_1$ is paracompact then X itself is also paracompact. L-3

For $\kappa \leq \aleph_1$, $(4.2)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(4.2)_\lambda$ holds for all $\lambda < \kappa$. Let X be as in $(4.2)_\kappa$. We have to show that X is paracompact.

Case 1. κ is regular.

Let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of Lindelöf subspaces of X. By Lemma 4.5, we may assume that each L_α is open. For $\beta < \kappa$, let $X_\beta = \bigcup\{L_\alpha : \alpha < \beta\}$. By $L(X) = \kappa$, we have $X \neq X_\beta$ for every $\beta < \kappa$. We may also assume that the continuously increasing sequence $\langle X_\beta : \beta < \kappa \rangle$ of open set in X is strictly increasing.

Let $S = \{\alpha < \kappa : X_\alpha \neq \overline{X_\alpha}\}$.

Claim 4.6.1. *S is non-stationary in κ.*

$\vdash$ We prove first the following weakening of the claim:

Subclaim 4.6.1.1. *$S \cap E^\kappa_\omega$ is non-stationary in κ.*

$\vdash$ For a contradiction, suppose that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta\in c_\alpha} L_\beta}$.

Now, by FRP, there is $I \in [\kappa]^{\aleph_1}$ such that

(4.3) $\mathrm{cf}(I) = \omega_1\,;$ L-4

(4.4) $h(\alpha) \in I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I\,;$ L-5

(4.5) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I\,;$ L-6

(4.6) if $f : S \cap E^\kappa_\omega \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}(\{\xi^*\})) = \sup(I)$. L-7

Let $Y = \bigcup_{\beta\in I} L_\beta$. Note that, by (4.4), $p_\alpha \in Y$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$.

By $|\,I\,| = \aleph_1$ and since each L_β is open Lindelöf subspace of X, it follows that Y is open and $L(Y) \leq \aleph_1$. Hence, by the assumption on X, Y is a paracompact subspace of X. Thus the open cover $\mathcal{L} = \{L_\beta : \beta \in I\}$ of Y has a locally finite open refinement $\mathcal{E}$. Since each L_β ($\beta \in I$) is Lindelöf, it follows that, for each $\beta \in I$,

(4.7) $\{E \in \mathcal{E} : E \cap L_\beta \neq \emptyset\}$ is countable. L-8

`[Since` $\mathcal{E}$ `is locally finite, for each` $p \in L_\beta$`, there is an open set` O_p `such that` $p \in O_p$ `and` $\{E \in \mathcal{E} : E \cap O_p \neq \emptyset\}$ `is finite. Since` L_β `is open, we may choose` O_p `to be a subset of` L_β`. Since` L_β `is Lindelöf and` $\{O_p : p \in L_\beta\}$ `is an open cover of` L_β `, there is a countable` $A \subseteq L_\beta$ `such that` $\{O_p : p \in A\}$ `already covers` L_β`. We have` $\{E \in \mathcal{E} : E \cap L_\beta \neq \emptyset\} = \{E \in \mathcal{E} : E \cap O_p \neq \emptyset$ `for some` $p \in A\}$`. But the right-side of the equality is easily seen to be countable. ]`

Now, for each $\alpha \in S \cap E^\kappa_\omega \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{L_\beta : \beta \in c_\alpha\}}$, there is $f(\alpha) \in c_\alpha$ such that $E_\alpha \cap L_{f(\alpha)} \neq \emptyset$. Thus, by (4.6), there is a $\xi^* \in I$ such that $\sup(f^{-1}{}''\{\xi^*\}) = \sup(I)$. By (4.7), we have $E \subseteq X_\eta$ for all $E \in \mathcal{E}$ such that $E \cap L_{\xi^*} \neq \emptyset$ for some large enough $\eta \in S \cap E^\kappa_\omega \cap I$ with $f(\eta) = \xi^*$. But, since $\emptyset \neq E_\eta \cap L_{f(\eta)} = E_\eta \cap L_{\xi^*}$ we have $p_\eta \in E_\eta \subseteq X_\eta$. This is a contradiction to the choice of p_η. $\dashv$ (Subclaim 4.6.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E^\kappa_\omega \cap C = \emptyset$ and let

(4.8) $\quad D = \{\alpha \in C : \alpha \setminus S \text{ is cofinal in } \alpha\}$. L-9

Clearly D is also a club subset of κ. So the following subclaim proves the claim.

Subclaim 4.6.1.2. *$S \cap D = \emptyset$.*

$\vdash$ For $\alpha \in D \cap E^\kappa_\omega$, we have $\alpha \notin S$ by $D \subseteq C$.

For $\alpha \in D \cap E^\kappa_{>\omega}$, suppose $p \in \overline{X_\alpha}$. By the countable tightness of X there is $\beta < \alpha$ such that $p \in \overline{X_\beta}$. By (4.8), we may assume that $\beta \in E^\kappa_\omega \setminus S$. Thus we have $p \in \overline{X_\beta} = X_\beta \subseteq X_\alpha$. This shows that $X_\alpha = \overline{X_\alpha}$ and hence $\alpha \notin S$.

$\dashv$ (Subclaim 4.6.1.2)

$\dashv$ (Claim 4.6.1)

Now let D be a club subset of κ such that $D \cap S = \emptyset$ and let $\langle \xi_\alpha : \alpha < \kappa \rangle$ be an increasing enumeration of $D \cup \{0\}$. Let $Y_\alpha = X_{\xi_{\alpha+1}} \setminus X_{\xi_\alpha}$ for $\alpha < \kappa$. Then $\{Y_\alpha : \alpha < \kappa\}$ is a partition of X into clopen subspaces. Since each Y_α is the union of $< \kappa$ many Lindelöf spaces, namely $L_\delta \setminus X_{\xi_\alpha}$, $\xi_\alpha \leq \delta < \xi_{\alpha+1}$, we have $L(Y_\alpha) < \kappa$. It follows from the induction hypothesis that each Y_α is paracompact. Hence X itself is also paracompact.

Case 2. κ is singular.

Similarly to Case 1., let $\{L_\alpha : \alpha < \kappa\}$ be a cover of X consisting of open Lindelöf subspaces of X. Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be a continuously and strictly increasing sequence of cardinals cofinal in κ. For $i < \mathrm{cf}(\kappa)$, let $X_i = \bigcup\{L_\alpha : \alpha < \kappa_i\}$. By the induction hypothesis, there is a locally finite open refinement $\mathcal{C}_i$ of the open cover $\{L_\alpha : \alpha < \kappa_i\}$ of X_i for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_\mathcal{C}$ be the intersection relation on $\mathcal{C}$ and $\approx_\mathcal{C}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, we have $|\{C' \in \mathcal{C} : C \approx_\mathcal{C} C'\}| \leq \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_\mathcal{C}$. Then, by Lemma 1.1, each $e \in \mathbb{E}$ has cardinality $\leq \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\leq \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \leq \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact. □ (Theorem 4.6)

In contrast to the reflection theorem in the last section, the following is still open:

Problem 1. *Is the assertion of* Theorem 4.6 *equivalent to* FRP *?*

5 Axiom R-like extension of FRP and a stronger reflection property of paracompactness

FRPR

Similarly to the extension of RP to Axiom R, $\mathsf{FRP}(\kappa)$ for a regular cardinal $\kappa \geq \aleph_2$ can be enhanced with the additional requirement that the reflection point I be an element of a given ω_1-club family $\subseteq [\kappa]^{\aleph_1}$:

$\mathsf{FRP}^R(\kappa)$: For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is $I \in \mathcal{T}$ such that

(5.1) for any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$. FRPR-0

Similarly to FRP, let FRP^R be the axiom asserting that $\mathsf{FRP}^R(\kappa)$ holds for all regular $\kappa \geq \aleph_2$.

Note that we can put the constraints (3.1) and (3.2) on I by thinning out the ω_1-club family $\mathcal{C}$. Thus $\mathsf{FRP}^R(\kappa)$ implies $\mathsf{FRP}(\kappa)$ for all regular $\kappa \geq \aleph_2$. The proof of the implication "$\mathsf{RP}(\kappa) \Rightarrow \mathsf{FRP}(\kappa)$" in [9] can be slightly modified to show the implication "$\mathsf{AR}(\kappa) \Rightarrow \mathsf{FRP}^R(\kappa)$".

equivalence

Lemma A5.1. *For a regular cardinal $\kappa \geq \aleph_2$, $\mathsf{FRP}^R(\kappa)$ is equivalent to the following $\mathsf{FRP}^R_\bullet(\kappa)$:*

$\mathsf{FRP}^R_\bullet(\kappa)$: *For any ω_1-club $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, stationary $S \subseteq E^\kappa_\omega$ and mapping $g : S \to [\kappa]^{\leq\aleph_0}$ there is a continuously increasing sequence $\langle I_\xi : \xi < \omega_1\rangle$ of countable subsets of κ such that*

(5.2) $\langle \sup(I_\xi) : \xi < \omega_1\rangle$ *is strictly increasing;* T-0

(5.3) *each I_ξ is closed with respect to g;* T-1

(5.4) $\sup(I_\xi) \in I_{\xi+1}$; T-1-0

(5.5) $\bigcup_{\xi<\omega_1} I_\xi \in \mathcal{T}$ *and* T-1-1

(5.6) $\{\xi < \omega_1 : \sup(I_\xi) \in S$ *and* $g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$ *is stationary in ω_1.* T-2

Proof. First, assume $\mathsf{FRP}^R(\kappa)$. Let $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ be ω_1-club, $S \subseteq E^\kappa_\omega$ be stationary and $g : S \to [\kappa]^{\aleph_0}$. Without loss of generality, we may assume that $g(\alpha) \cap \alpha \neq \emptyset$ for all $\alpha \in S$. Without loss of generality, we may assume that all elements of $\mathcal{T}$ have cofinality ω_1.

Let $I \in \mathcal{T}$ be as in the definition of $\mathsf{FRP}^R(\kappa)$ for these S and g. Then, by (3.2), there is a filtration $\langle I_\xi : \xi < \omega_1\rangle$ of I, that is, a continuously increasing sequence $\langle I_\xi : \xi < \omega_1 \rangle$ of subsets of I of cardinality $< |I|$ with $I = \bigcup_{\xi<\omega_1} I_\xi$, satisfying (5.2), (5.3) and (5.4).

We show that $\langle I_\xi : \xi < \omega_1 \rangle$ satisfies (5.6) as well. Suppose not. Then $\{\xi < \omega_1 : \sup(I_\xi) \notin S$ or $g(\sup(I_\xi)) \cap \sup(I_\xi) \not\subseteq I_\xi\}$ includes a club set $\subseteq \omega_1$. It follows that $S \cap I \setminus S_0$ is non stationary in $\sup(I)$, where

$$S_0 = \{\alpha \in S \cap I : \alpha = \sup(I_\xi) \text{ for some } \xi < \omega_1 \text{ and } g(\alpha) \cap \alpha \not\subseteq I_\xi\}.$$

Let $f : S \cap I \to I$ be defined by

$$\text{(a5.1)}\quad f(\alpha) = \begin{cases} \min((g(\alpha) \cap \alpha) \setminus I_\xi) & \text{if } \alpha \in S_0 \text{ and } \alpha = \sup(I_\xi); \\ \min(g(\alpha)) & \text{otherwise.} \end{cases}$$ T-3

Then f is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. By the choice of I, there is an $\alpha^* \in I$ such that $f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. In particular, $S_0 \cap f^{-1}{}''\{\alpha^*\}$ is stationary in $\sup(I)$. Let $\xi^* \in \omega_1$ be such that $\alpha^* \in I_{\xi^*}$ and let $\beta \in S_0 \cap f^{-1}{}''\{\alpha^*\}$ be such that $\beta > \sup(I_{\xi^*})$. Let $\eta < \omega_1$ be such that $\beta = \sup(I_\eta)$. Then $\alpha^* \in I_{\xi^*} \subseteq I_\eta$. Since $\beta \in S_0$, we have $f(\beta) \notin I_\eta$ by the definition (a5.1) of f. It follows that $f(\beta) \neq \alpha^*$. This is a contradiction to the choice of β.

Now, assume $\mathsf{FRP}^R_\bullet(\kappa)$. Suppose that $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ is ω_1-club, $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\aleph_0}$. Let $\langle I_\xi : \xi < \omega_1 \rangle$ be as in the definition of $\mathsf{FRP}^R_\bullet(\kappa)$ and let $I = \bigcup_{\xi<\omega_1} I_\xi$.

We claim that this I satisfies the conditions in the definition of $\mathsf{FRP}(\kappa)$. It is clear that I satisfies (3.1) and (3.2). To see that it also satisfies (3.3), suppose that $f : S \cap I \to \kappa$ is regressive and $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$. Let $S_1 = \{\xi \in \omega_1 : f(\sup(I_\xi)) \in I_\xi\}$. Then we have

$$S_1 \supseteq \{\xi \in \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$

and thus S_1 is stationary by the choice of I. For each $\xi \in S_1$, let

$$h(\xi) = \min\{\eta < \omega_1 : f(\sup(I_\xi)) \in I_\eta\}.$$

Then the mapping $h : S_1 \to \omega_1$ is regressive. Thus, by Fodor's theorem, there is a stationary $S_2 \subseteq S_1$ such that $h''S_2 = \{\eta^*\}$ for some $\eta^* \in \omega_1$. Since I_{η^*} is countable, there is a stationary $S_3 \subseteq S_2$ such that, for any $\xi \in S_3$, $f(\sup(I_\xi)) = \alpha^*$ for some fixed $\alpha^* \in I_{\eta^*}$. It follows that $f^{-1}{}''\{\alpha^*\} \supseteq \{\sup(I_\xi) : \xi \in S_3\}$ is stationary in $\sup(I)$. ❑ (Lemma A 5.1)

AR implies FRPR

Theorem A.5.2. *For any regular cardinal $\kappa > \aleph_1$, $\mathsf{AR}(\kappa)$ implies $\mathsf{FRP}^R(\kappa)$.*

Proof. By Lemma A 5.1, it is enough to show that $\mathsf{AR}(\kappa)$ implies $\mathsf{FRP}^R_\bullet(\kappa)$. Suppose that $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$, $S \subseteq E^\kappa_\omega$ is stationary and $g : S \to [\kappa]^{\leq\aleph_0}$. Let

$$\text{(5.7)}\quad S_0 = \{a \in [\kappa]^{\aleph_0} : \sup(a) \in S \setminus a \text{ and } g(\sup(a)) \cap \sup(a) \subseteq a\}.$$ c-2-0

Claim A 5.2.1. *S_0 is a stationary subset of $[\kappa]^{\aleph_0}$.*

$\vdash$ Suppose that $C \subseteq [\kappa]^{\aleph_0}$ is a club. We show that $C \cap S_0 \neq \emptyset$.

By Kueker's theorem, there is a mapping $s : \kappa^{<\omega} \to \kappa$ such that $C \supseteq C(s) = \{a \in [\kappa]^{\aleph_0} : s''a^{<\omega} \subseteq a\}$. Let $D = \{\alpha < \kappa : s''\alpha^{<\omega} \subseteq \alpha\}$. Since κ is regular, D is a club subset of κ. So there is an $\alpha^* \in S \cap D$. Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence of ordinals such that $\alpha^* = \sup_{n\in\omega} \alpha_n$. Let a^* be the closure of $a_0 = \{\alpha_n : n \in \omega\} \cup (g(\alpha^*) \cap \alpha^*)$ with respect to s. Since a_0 is cofinal in α^* and $\alpha^* \in D$, we have $\sup(a^*) = \alpha^*$. Hence $a^* \in S_0$. By the definition of a^*, we also have $a^* \in C(s) \subseteq C$.

$\dashv$ (Claim A 5.2.1)

Let $\mathcal{T}_0 = \{X \in \mathcal{T} : \mathrm{cf}(X) = \omega_1$ and X is closed with respect to $g\}$. Then $\mathcal{T}_0$ is still ω_1-club. By $\mathsf{AR}(\kappa)$, there is $I \in \mathcal{T}_0$ such that

(a5.2) $\mathrm{cf}(I) = \omega_1$; c-3

(a5.3) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$; c-6

(a5.4) $S_0 \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$. c-4

Let $\langle I_\xi : \xi < \omega_1 \rangle$ be a filtration of I such that each I_ξ is closed with respect to g (this is possible by (a5.3)) and $\langle \sup(I_\xi) : \xi < \omega_1 \rangle$ is strictly increasing (possible by (a5.2)).

Let

$$S_1 = \{\xi < \omega_1 : \xi \text{ is a limit and } I_\xi \in S_0\} \text{ and}$$
$$S_2 = \{\xi < \omega_1 : g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi)\}.$$

By the definition (5.7) of S_0, we have $S_2 \supseteq S_1$ and S_1 is a stationary subset of ω_1 by (a5.4). Thus S_2 is stationary as well. ❑ (Theorem A 5.2)

Corollary A 5.3. Axiom R *implies* FRP^R. ❑

A straight forward modification of Theorem 3.4 in [9] shows also that $\mathsf{FRP}^R(\kappa)$ is preserved in generic extensions by c.c.c. forcing.

Shelah proved that SCH follows from a weakening of RP ([17]). Since RP also implies $2^{\aleph_0} \leq \aleph_2$ (Todorcevic, see [13]), it follows that, under RP, we have $\mathrm{cf}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ for all cardinal κ with $cf(\kappa) = \omega$. Thus the assumption FRP^R + (5.8) of Theorem 5.1 below is a consequence of Axiom R. This assumption is also still much weaker than Axiom R, since it is easy to see that this is still preserved in extensions by c.c.c. forcing.

Balogh proved the following theorem under Axiom R (Theorem 1.4 in [2]). LR-2

Theorem 5.1. *Assume* FRP^R *and*

(5.8) $\{\kappa < \lambda \,:\, \mathrm{cf}([\kappa]^{\aleph_0}) = \kappa\}$ *is cofinal in* λ *for any singular cardinal* λ. *-0

Suppose that X *is a countably tight locally Lindelöf space such that*

(5.9) *for all open subspaces* Y *of* X *with* $L(Y) \leq \aleph_1$, *we have* $L(\overline{Y}) \leq \aleph_1$ *and* LR-2-0

(5.10) *every clopen subspace* Y *of* X *with* $L(Y) \leq \aleph_1$ *is paracompact.* LR-2-1

Then X *itself is paracompact.*

Proof of Theorem 5.1: The proof is a modification of the proof of Theorem 4.6.

It is enough to prove that the following $(5.11)_\kappa$ holds for all cardinal κ by induction on κ:

$(5.11)_\kappa$ For any countably tight and locally Lindelöf space X with $L(X) = \kappa$, if X satisfies (5.9) and (5.10), then X is paracompact. LR-3

For $\kappa \leq \aleph_1$, $(5.11)_\kappa$ trivially holds. So assume that $\kappa > \aleph_1$ and that $(5.11)_\lambda$ holds for all $\lambda < \kappa$. Let X be a countably tight and locally Lindelöf space with $L(X) = \kappa$ such that X satisfies (5.9) and (5.10). We have to show that X is paracompact.

By Lemma 4.5, and since X is locally Lindelöf and $L(X) = \kappa$, there is a cover $\{L_\alpha \,:\, \alpha < \kappa\}$ of X consisting of open Lindelöf subspaces.

Let

$$\mathcal{T} = \{I \in [\kappa]^{\aleph_1} \,:\, \textstyle\bigcup_{\alpha \in I} L_\alpha \ \text{ is a clopen subspace of } X\}.$$

By (5.9) and since X is countably tight, it is easy to see that $\mathcal{T}$ is ω_1-club.

Case 1. κ is regular.

For $\beta < \kappa$, let $X_\beta = \bigcup\{L_\alpha \,:\, \alpha < \beta\}$. By induction hypothesis we may also assume that $X \neq X_\beta$ for every $\beta < \kappa$ and that the sequence $\langle X_\beta \,:\, \beta < \kappa\rangle$ is strictly increasing.

Let $S = \{\alpha < \kappa \,:\, X_\alpha \neq \overline{X_\alpha}\}$.

Claim 5.1.1. S *is non-stationary in* κ.

$\vdash$ We prove first the following weakening of the claim:

Subclaim 5.1.1.1. $S \cap E^\kappa_\omega$ *is non-stationary in* κ.

$\vdash$ For a contradiction, suppose that $S \cap E^\kappa_\omega$ were stationary. For each $\alpha \in S \cap E^\kappa_\omega$, let $p_\alpha \in \overline{X_\alpha} \setminus X_\alpha$ and let $h(\alpha) \in \kappa$ be such that $p_\alpha \in L_{h(\alpha)}$. Since X is countably tight, there is $c_\alpha \in [\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{\bigcup_{\beta \in c_\alpha} L_\beta}$.

Now, by FRP^R, there is $I \in \mathcal{T}$ such that

(5.12) $\mathrm{cf}(I) = \omega_1$; LR-4

(5.13) $h(\alpha) \in I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$; LR-5

(5.14) $c_\alpha \subseteq I$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$; LR-6

(5.15) if $f : S \cap E^\kappa_\omega \cap I \to \kappa$ is such that $f(\alpha) \in c_\alpha$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$, then there is $\xi^* \in I$ with $\sup(f^{-1}{''}\{\xi^*\}) = \sup(I)$. LR-7

Let $Y = \bigcup_{\beta \in I} L_\beta$. Note that, by (5.13), $p_\alpha \in Y$ for all $\alpha \in S \cap E^\kappa_\omega \cap I$.

By $I \in \mathcal{T}$ and since each L_β is open Lindelöf subspace of X, it follows that Y is clopen and $L(Y) \leq \aleph_1$. Hence, by (5.10), Y is a paracompact subspace of X. The rest of this case can be treated exactly as the Case 1 in the proof of Theorem 4.6. Thus the open cover $\{L_\beta : \beta \in I\}$ of Y has a locally finite open refinement $\mathcal{E}$. Since each L_β ($\beta \in I$) is Lindelöf, it follows that

(a5.5) $\{E \in \mathcal{E} : E \cap L_\beta \neq \emptyset\}$ is countable. LR-8

Now, for each $\alpha \in S \cap E^\kappa_\omega \cap I$, let $E_\alpha \in \mathcal{E}$ be such that $p_\alpha \in E_\alpha$. Since $p_\alpha \in \overline{\bigcup\{L_\beta : \beta \in c_\alpha\}}$, there is $f(\alpha) \in c_\alpha$ such that $E_\alpha \cap L_{f(\alpha)} \neq \emptyset$. Thus, by (5.15), there is a $\xi^* \in I$ such that $\sup(f^{-1}{''}\{\xi^*\}) = \sup(I)$. By (a5.5), there is $\eta \in S \cap E^\kappa_\omega \cap I$ such that $f(\eta) = \xi^*$ and $E \subseteq X_\eta$ for all $E \in \mathcal{E}$ such that $E \cap L_{\xi^*} \neq \emptyset$. But, since $\emptyset \neq E_\eta \cap L_{f(\eta)} = E_\eta \cap L_{\xi^*}$, we have $p_\eta \in E_\eta \subseteq X_\eta$. This is a contradiction to the choice of p_η. $\dashv$ (Subclaim 5.1.1.1)

Let C be a club subset of κ consisting of limit ordinals such that $S \cap E^\kappa_\omega \cap C = \emptyset$ and let

(5.16) $D = \{\alpha \in C : \alpha \setminus S$ is cofinal in $\alpha\}$. LR-9

Clearly D is also a club subset of κ. So the following subclaim proves the claim.

Subclaim 5.1.1.2. *$S \cap D = \emptyset$.*

$\vdash$ For $\alpha \in D \cap E^\kappa_\omega$, we have $\alpha \notin S$ by $D \subseteq C$.

For $\alpha \in D \cap E^\kappa_{>\omega}$, suppose $p \in \overline{X_\alpha}$. By the countable tightness of X, there is $\beta < \alpha$ such that $p \in \overline{X_\beta}$. By (5.16), we may assume that $\beta \in E^\kappa_\omega \setminus S$. Thus we have $p \in \overline{X_\beta} = X_\beta \subseteq X_\alpha$. This shows that $X_\alpha = \overline{X_\alpha}$ and hence $\alpha \notin S$.

$\dashv$ (Subclaim 5.1.1.2)

$\dashv$ (Claim 5.1.1)

Now let D be a club subset of κ such that $D \cap S = \emptyset$ and let $\langle \xi_\alpha : \alpha < \kappa \rangle$ be an increasing enumeration of $D \cup \{0\}$. Let $Y_\alpha = X_{\xi_{\alpha+1}} \setminus X_{\xi_\alpha}$ for $\alpha < \kappa$. Then $\{Y_\alpha : \alpha < \kappa\}$ is a partition of X into clopen subspaces. Since each Y_α is the union of $< \kappa$ many Lindelöf spaces, namely $L_\delta \setminus X_{\xi_\alpha}$, $\xi_\alpha \leq \delta < \xi_{\alpha+1}$, we have $L(Y_\alpha) < \kappa$. It follows from the induction hypothesis that each Y_α is paracompact. Hence X itself is also paracompact.

Case 2. κ is singular.

Let θ be a sufficiently large cardinal. Let $\mathcal{L} = \{L_\alpha : \alpha < \kappa\}$. The singularity of κ is not yet necessary in the following claim:

LR-9-0

Claim 5.1.2. *If $M \prec \mathcal{H}(\theta)$ is such that*

(5.17) $\omega_1 \subseteq M$; LR-10

(5.18) $X, \mathcal{L} \in M$; LR-10-0

(5.19) M *is* `internally cofinal`, LR-11

then $Z = \bigcup(\mathcal{L} \cap M)$ is a clopen subspace of X.

$\vdash$ Z is an open subspace of X as the union of open subspaces $\mathcal{L} \cap M$. Thus it is enough to show that X is closed. Suppose $x \in \overline{Z}$. By the countable tightness of X, there is $c \in [\mathcal{L} \cap M]^{\aleph_0}$ such that $x \in \overline{\bigcup c}$. By (5.19), there is $c' \in [\mathcal{L} \cap M]^{\aleph_0} \cap M$ such that $c \subseteq c'$. By (5.9) and by the elementarity of M, we have

$$M \models \exists d \in [\mathcal{L}]^{\aleph_1}(\overline{\bigcup c'} \subseteq \bigcup d).$$

Let $d \in [\mathcal{L}]^{\aleph_1} \cap M$ be such that $\overline{\bigcup c'} \subseteq \bigcup d$. By (5.17), we have $d \subseteq M$. Thus there is an $L^* \in d = d \cap M$ such that $x \in L^* \subseteq \bigcup d \subseteq \bigcup(\mathcal{L} \cap M)$.

$\dashv$ (Claim 5.1.2)

Let $\langle M_i : i < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ such that, for $i < \mathrm{cf}(\kappa)$,

(5.20) $|\,M_i\,| < \kappa$; LR-12

(5.21) $\omega_1 \subseteq M_i$; LR-13

(5.22) $X, \mathcal{L} \in M_i$; LR-14

(5.23) M_i is `internally cofinal` and LR-15

(5.24) $\kappa \subseteq \bigcup_{i<\mathrm{cf}(\kappa)} M_i$. LR-16

We can construct such a sequence in particular with the property (5.23) by the assumption on the cardinal arithmetic.

Let $X_i = \bigcup(\mathcal{L} \cap M_i)$ for $i < \mathrm{cf}(\kappa)$. By Claim 5.1.2, each X_i is a clopen subspace of X. Since $L(X_i) \leq |\,M_i\,| < \kappa$, each X_i is paracompact by induction hypothesis . Note that we need here the closedness of X_i so that (5.9) holds for X_i.

$\mathcal{L} \cap M_i$ has a locally finite open refinement $\mathcal{C}_i$ for each $i < \mathrm{cf}(\kappa)$. Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\kappa)} \mathcal{C}_i$.

Let $\sim_{\mathcal{C}}$ be the intersection relation on $\mathcal{C}$ and $\approx_{\mathcal{C}}$ be its transitive closure. Since each $\mathcal{C}_i$ is locally finite and each $C \in \mathcal{C}_i$ is Lindelöf, $|\{C' \in \mathcal{C}_i : C' \approx_{\mathcal{C}_i} C\}| \leq \aleph_0$ for all $i < \mathrm{cf}(\kappa)$. Hence $|\{C' \in \mathcal{C} : C \approx_{\mathcal{C}} C'\}| \leq \mathrm{cf}(\kappa) < \kappa$ for all $C \in \mathcal{C}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx_{\mathcal{C}}$. Then, by Lemma 1.1, each $e \in \mathbb{E}$ has cardinality $\leq \mathrm{cf}(\kappa)$.

$\mathcal{P} = \{\bigcup e : e \in \mathbb{E}\}$ is a partition of X into clopen subspaces. Since each $Y \in \mathcal{P}$ is the union of $\leq \mathrm{cf}(\kappa)$ many Lindelöf subspaces, we have $L(Y) \leq \mathrm{cf}(\kappa) < \kappa$. It follows that each $Y \in \mathcal{P}$ is paracompact by the induction hypothesis and hence X is also paracompact. ❑ (Theorem 5.1)

Though we presently do not know if $\mathsf{FRP}^R(\kappa)$ is equivalent to $\mathsf{FRP}(\kappa)$ for all regular κ, it *is* the case for many instances of κ:

Theorem 5.2. *Suppose that κ is regular and*

(5.25) $cf([\lambda]^{\aleph_0}, \subseteq) < \kappa$ *for all* $\lambda < \kappa$. R-0

Then we have $\mathsf{FRP}^R(\kappa) \Leftrightarrow \mathsf{FRP}(\kappa)$.

Proof. It is enough to show the direction "$\Leftarrow$".

Assume that κ is a regular cardinal $> \aleph_1$ with (5.25) and $\mathsf{FRP}(\kappa)$ holds. Let $S \subseteq E^\kappa_\omega$ be stationary, $g : S \to [\kappa]^{\aleph_0}$ and $\mathcal{T} \subseteq [\kappa]^{\aleph_1}$ be ω_1-club. We want to show that there is $I \in \mathcal{T}$ such that I satisfies (5.1).

Let θ be sufficiently large and let $\mathcal{M}^* = \langle \mathcal{H}(\theta), S, g, \mathcal{T}, \ldots, \trianglelefteq, \in \rangle$ and let $\mathcal{M} \prec \mathcal{M}^*$ be the union of the continuously increasing chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $\mathcal{M}^*$ such that

(5.26) $|M_\alpha| < \kappa$ for all $\alpha < \kappa$; R-1

(5.27) $M_{\alpha+1}$ is `internally cofinal` for all $\alpha < \kappa$; R-2

(5.28) $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$ and R-3

(5.29) $\kappa \subseteq \mathcal{M}$. R-4

Note that (5.27) is possible by (5.25). Let $C = \{\alpha \in \kappa : \kappa \cap M_\alpha = \alpha\}$. Since C is club in κ, $S_0 = S \cap C$ is stationary. Applying $\mathsf{FRP}(\kappa)$ to S_0 and $g \restriction S_0$ we obtain $I_0 \in [\lambda]^{\aleph_1}$ such that, letting $\alpha_0 = \sup(I_0)$, (5.30) $\mathrm{cf}(\alpha_0) = \omega_1$; R-5

(5.31) $g(\alpha) \subseteq I_0$ for all $\alpha \in I \cap S_0$; R-6

(5.32) for any regressive $f : S_0 \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S_0 \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I_0)$. R-6-0

Since $S_0 \cap \alpha_0$ is cofinal in α_0 by (5.33), we have $\alpha_0 \in C$. By (5.30) and (5.27) it follows that

(5.33) M_{α_0} is `internally cofinal`. R-7

`Suppose that` $x \in [M_{\alpha_0}]^{\aleph_0}$ `there is` $\alpha < \alpha_0$ `such that` $x \in [M_\alpha]^{\aleph_0}$. `Since` $M_\alpha \subseteq M_{\alpha+1}$ `and` $M_{\alpha+1}$ `is internally cofinal there is` $y \in [M_{\alpha+1}]^{\alpha_0} \cap M \subseteq [M_{\alpha_0}]^{\aleph_0} \cap M_{\alpha_0}$ `such that` $x \subseteq y$.

Let $\langle N_\alpha : \alpha < \omega_1 \rangle$ be a continuously increasing sequence of elementary submodels of M_{α_0} such that

(5.34) $|\, N_\alpha \,| = \aleph_0$ for every $\alpha < \omega_1$; R-8-0

(5.35) there is a countable set $x_\alpha \in N_{\alpha+1}$ such that $N_\alpha \subseteq x_\alpha$ for every $\alpha < \omega_1$ and R-8

(5.36) $I_0 \subseteq \bigcup_{\alpha<\omega_1} N_\alpha$. R-9

The condition (5.35) is realizable by (5.33). Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and $I = \kappa \cap N$. Then $I_0 \subseteq I$ by (5.36). So $|\, I \,| = \aleph_1$ by (5.34). Since $N \subseteq M_{\alpha_0}$, we have $\sup(I) = \alpha_0$.

Thus the following claim implies that this I is as in the definition of $\mathsf{FRP}^R(\kappa)$ for S, g and $\mathcal{T}$.

Claim 5.2.1. *$I \in \mathcal{T}$.*

$\vdash$ For $\alpha < \omega_1$ there is $A_\alpha \in \mathcal{T} \cap N_{\alpha+1}$ such that

(5.37) $\bigcup(\mathcal{T} \cap N_\alpha) \subseteq A_\alpha$ R-10

by (5.35) and elementarity. $\langle A_\alpha : \alpha < \omega_1 \rangle$ is then an increasing sequence in $\mathcal{T}$. Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. By the ω_1-clubness of $\mathcal{T}$, we have $A \in \mathcal{T}$. By (5.37) and (5.35), we have $I \cap N_\alpha \subseteq A_\alpha \subseteq I$ for all $\alpha < \omega_1$. By (5.36), it follows that $A = I$. $\dashv$ (Claim 5.2.1)

□ (Theorem 5.2)

By the theorem above we have $\mathsf{FRP}^R(\aleph_n) \Leftrightarrow \mathsf{FRP}(\aleph_n)$ for all $n \in \omega \setminus 1$. Thus the test question in this connection would be the following:

P-0

Problem 2. *Is* $\mathsf{FRP}^R(\aleph_{\omega+1})$ *equivalent to* $\mathsf{FRP}(\aleph_{\omega+1})$ *?*

The following problem is also still open:

P-1

Problem 3. *Does* (5.8) *follow from* FRP *or* FRP^R *?*

Meanwhile it is proved that (5.8) does follow from FRP. See [10].

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