

Set theoretic reflection principles and topological reflection

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Theorem 1 (A. Dow, 1988)

*For a countably compact topological space X ,
if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X
itself is metrizable.*

► (Folklore) Under \square_{\aleph_1} there is a locally countably compact non metrizable space X of cardinality \aleph_2 s.t. all $Y \in [X]^{\leq \aleph_1}$ are metrizable.

Theorem 2 (S.F., Juhász, Soukup, Szentmiklóssy, Usuba [1])

Assume *Fodor-type Reflection Principle* (see the next slide). Then the following reflection theorem on metrizability holds:

For a locally countably compact topological space X ,
if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X
itself is metrizable.

[1] S.F., István Juhász, Lajos Soukup, Zoltán Szentmiklóssy and Toshimichi Usuba, *Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness*, Top. and its Appl., 157(8), (2010).

► The reflection theorem on metrizability in above was proved under Axiom R by Zoltan Balogh (posthumous, 2002).

Fodor-type Reflection Principle (FRP)

is the following set theoretic principle introduced in [1]:

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E_\omega^\kappa$ and $g : S \rightarrow [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- ▶ $\text{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- ▶ for any $f : S \cap I \rightarrow \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1} \parallel \{\xi^*\}$ is stationary in $\text{sup}(I)$.

FRP is a strengthening of the Ordinal Reflection Principle (ORP):

For any regular cardinal $\kappa > \aleph_1$ and any stationary $S \subseteq E_\omega^\kappa$, there is an $\alpha < \kappa$ with $\text{cf}(\alpha) = \omega_1$, s.t. $S \cap \alpha$ is stationary in α .

with a side condition which reminds of Fodor's Theorem:

Fodor's Theorem. For any regular cardinal κ and $f : \kappa \rightarrow \kappa$ s.t. $f(\alpha) < \alpha$ for all $\alpha < \kappa$, there is a $\xi^* < \kappa$ s.t. $f^{-1} \parallel \{\xi^*\}$ is stationary in κ .

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Notation:

- ▶ For regular κ , $\text{FRP}(\kappa)$ denotes the local version of FRP for this fixed κ .
- ▶ For an uncountable cardinal λ , $\text{FRP}(< \lambda)$ denotes the assertion that $\text{FRP}(\kappa)$ holds for every regular $\aleph_1 \leq \kappa < \lambda$.

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Theorem 2 (S.F., Juhász, Soukup, Szentmiklóssy and Usuba [1])

Assume that FRP holds. Then the following reflection theorem on metrizability holds:

For a locally countably compact topological space X , if all subspaces Y of X of cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

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For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E_\omega^\kappa$ and $g : S \rightarrow [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

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Theorem 3 (S.F. et al.[1] + S.F., Sakai, Soukup and Usuba [2])

*The reflection theorem on metrizability of locally countably compact spaces in Theorem 2 is **equivalent to** FRP over ZFC.*

[2] S.F., Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, *More about Fodor-type Reflection Principle*, preprint.

Theorem 3 can be obtained by using the characterization of FRP on the next slide and (the proof of) the following result:

Theorem 4 (S.F. et al. [1] + S.F. et al. [2])

FRP is equivalent to the following assertion:

For a locally separable, countably tight space X , if all subspaces Y of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X itself is meta-Lindelöf.

- ▶ A space X is *countably tight* if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.
- ▶ A space X is *meta-Lindelöf* if every open cover of X has a point countable refinement which is also an open cover.

Theorem 5 (S.F., Sakai, Soukup and Usuba [2])

For any uncountable cardinal λ , $\text{FRP}(< \lambda)$ is equivalent to the following assertion:

- ▶ for any regular $\kappa < \lambda$, stationary $S \subseteq E_\omega^\kappa$ and a ladder system $g : S \rightarrow [\mathbb{N}]^{\aleph_0}$, there is an $\alpha < \kappa$ s.t., for any regressive $f : S \cap \alpha \rightarrow \alpha$, $\{g(\alpha) \setminus f(\alpha) : \alpha \in S \cap \alpha\}$ is not pairwise disjoint.

Corollary 6 (reformulation of the key direction of Theorem 5)

Suppose that κ is the minimal cardinal s.t. $\neg \text{FRP}(\kappa)$. Then there are stationary $S \subseteq E_\omega^\kappa$, and a ladder system $g : S \rightarrow [\kappa]^{\aleph_0}$ s.t.

- ▶ for any $\alpha < \kappa$ there is a regressive $f : S \cap \alpha \rightarrow \alpha$ s.t. $\{g(\alpha) \setminus f(\alpha) : \alpha \in S \cap \alpha\}$ is pairwise disjoint.

Theorem 7 (S.F. [3] + S.F., Sakai, Soukup and Usuba [2])

The following assertion is equivalent to FRP:

For a T_1 space with point countable base, if all subspaces Y of X of cardinality $\leq \aleph_1$ are left-separated then X itself is left-separated.

[3] S.F., *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kokyuroku No.1619 (2008), 32–42.

- ▶ A space X is *left-separated* if there is a well-ordering $<$ of X s.t. each initial segment of X w.r.t. $<$ is a closed subset of X .
- ▶ W. Fleissner (1986) proved that the assertion of Theorem 7 follows from Axiom R and it is refuted under \neg ORP.

Theorem 8 (S.F. et al. [2])

The following assertion is equivalent to FRP:

For a countably tight space of local density \aleph_1 , if all subspaces Y of X of cardinality $\leq \aleph_1$ are collectionwise Hausdorff then X itself is collectionwise Hausdorff.

- ▶ A space X is *of local density* κ if for every $p \in X$ there is $Y \in [X]^{\leq \kappa}$ s.t. $p \in \text{int}(\overline{Y})$.
- ▶ A space X is *collectionwise* Hausdorff if any closed discrete subset D of X can be simultaneously separated by disjoint open sets, i.e., if, for any closed and discrete $D \subseteq X$, there is a family \mathcal{U} of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$.
- ▶ Fleissner (1986) proved also the assertion in Theorem 8 under Axiom R.

- ▶ FRP is also equivalent to the reflection theorems in terms of:
 - ▷ countable coloring number of infinite graphs (S.F. et al.[2]);
and
 - ▷ openly generatedness of Boolean algebras (S.F. and Rinot [4]).

[4] S.F. and Assaf Rinot, *Openly generated Boolean algebras and the Fodor-type Reflection Principle*, submitted.

- ▶ The topological (graph-theoretic, Boolean algebraic) reflection principles mentioned above are all equivalent to each other over ZFC.
- ▶ All these reflection principles imply Ordinal Reflection Principle.

Theorem 9 (S.F. et al. [1])

FRP is preserved by c.c.c. generic extension.

- ▶ Hence, all reflection principles above impose almost no restriction on the size of continuum. **cf.:** Under slightly stronger reflection principles, the continuum is $\leq \aleph_2$ (S. Todorćević).
- ▶ But these reflection principles (or equivalently FRP) do have certain effect on cardinal arithmetic.

Theorem 10 (S.F. and Rinot [4])

FRP implies Shelah's Strong Hypothesis.

► *Shelah's Strong Hypothesis* (SSH) is the assertion equivalent to the following:

For every uncountable cardinal κ of countable cofinality, we have $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$.

► By the characterization above of SSH, *Singular Cardinal Hypothesis* (SCH) follows from SSH.

Theorem 10 (S.F. and Rinot [4])

FRP implies Shelah's Strong Hypothesis.

(Very Rough) Sketch of the Proof:

Suppose that SSH does not hold. Then there is a better scale $\langle \langle \lambda_i : i < \omega \rangle, \langle f_\alpha : \alpha < \lambda^+ \rangle \rangle$ for a cardinal λ with $\text{cf}(\lambda) = \omega_1$.

Let $\varphi : {}^\omega \lambda \rightarrow \lambda$ be a 1-1 mapping, $E = E_\omega^{\lambda^+} \setminus \lambda$ and let $g : E \rightarrow [\lambda^+]^{\aleph_0}$ be s.t. $g(\alpha) = \{\varphi(f_\alpha \upharpoonright n) : n \in \omega\}$.

Then g together with E is a counterexample to $\text{FRP}(\lambda^+)$. □

► Theorem 10 suggests that SSH should be also regarded as a reflection principle. We can in fact characterize SSH in terms of the following topological reflection principle:

Theorem 11 (S.F. and Rinot [4])

SSH is equivalent to the following assertion:

For any countably tight space X , if X is $< \aleph_1$ -thin then X is thin.

▶ A topological space X is *thin* if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$.

▶ A topological space X is *$< \kappa$ -thin* if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$ of cardinality $< \kappa$.

Open problems:

▶ Are there any natural topological assertions which are equivalent to Axiom R (RP, WRP etc. resp.) ?

▶ For each topological theorem independent from ZFC, provide a set-theoretic principle characterizing the theorem (Topological Reverse Mathematics)!

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¡Gracias por su atención!

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