Set theoretic reflection principles and topological reflection

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Dow's Metrization Theorem

Theorem 1 (A. Dow, 1988)

For a countably compact topological space X, if all subspaces Y of X cardinality $\leq \aleph_1$ are metrizable then X itself is metrizable.

- (Folklore) Under $\square_{\aleph_1}$ there is a locally countably compact non metrizable space X of cardinality $\aleph_2$ s.t. all $Y \in [X]^{\leq\aleph_1}$ are metrizable.

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Assume Fodor-type Reflection Principle (see the next slide). Then the following reflection theorem on metrizability holds:

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Fodor-type Reflection Principle (FRP)

is the following set theoretic principle introduced in [1]:

For any regular cardinal $\kappa > \aleph_1$, any stationary $S \subseteq E^\kappa_\omega$ and $g : S \to [\kappa]^{\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(I) = \omega_1$; $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

FRP is a strengthening of the Ordinal Reflection Principle (ORP):

For any regular cardinal $\kappa > \aleph_1$ and any stationary $S \subseteq E^\kappa_\omega$, there is an $\alpha < \kappa$ with $\mathrm{cf}(\alpha) = \omega_1$, s.t. $S \cap \alpha$ is stationary in α.

with a side condition which reminds of Fodor's Theorem:

Fodor's Theorem. For any regular cardinal κ and $f : \kappa \to \kappa$ s.t. $f(\alpha) < \alpha$ for all $\alpha < \kappa$, there is a $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in κ.

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Notation:

- For regular κ, FRP(κ) denotes the local version of FRP for this fixed κ.
- For an uncountable cardinal λ, FRP($< \lambda$) denotes the assertion that FRP(κ) holds for every regular $\aleph_1 \leq \kappa < \lambda$.

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Theorem 3 (S.F. et al.[1] + S.F., Sakai, Soukup and Usuba [2])

The reflection theorem on metrizability of locally countably compact spaces in Theorem 2 is equivalent to FRP *over* ZFC.

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Generalization of Theorem 3

Theorem 3 can be obtained by using the characterization of FRP on the next slide and (the proof of) the following result:

Theorem 4 (S.F. et al. [1] + S.F. et al. [2])

FRP *is equivalent to the following assertion:*

> *For a locally separable, countably tight space X, if all subspaces Y of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X itself is meta-Lindelöf.*

- A space X is *countably tight* if, for any $U \subseteq X$ and $x \in \overline{U}$ there is $U' \in [U]^{\aleph_0}$ s.t. $x \in \overline{U'}$.
- A space X is *meta-Lindelöf* if every open cover of X has a point countable refinement which is also an open cover.

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Set-theoretic Characterization of FRP

Theorem 5 (S.F., Sakai, Soukup and Usuba [2])

For any uncountable cardinal λ, $\mathrm{FRP}(<\lambda)$ is equivalent to the following assertion:

- *for any regular $\kappa < \lambda$, stationary $S \subseteq E^{\kappa}_{\omega}$ and a ladder system $g : S \to [\aleph]^{\aleph_0}$, there is an $\alpha < \kappa$ s.t., for any regressive $f : S \cap \alpha \to \alpha$, $\{g(\alpha) \setminus f(\alpha) : \alpha \in S \cap \alpha\}$ is not pairwise disjoint.*

Corollary 6 (reformulation of the key direction of Theorem 5)

Suppose that κ is the minimal cardinal s.t. $\neg\mathrm{FRP}(\kappa)$. Then there are stationary $S \subseteq E^{\kappa}_{\omega}$, and a ladder system $g : S \to [\kappa]^{\aleph_0}$ s.t.

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Further results (1/3)

Theorem 7 (S.F. [3] + S.F., Sakai, Soukup and Usuba [2])

The following assertion is equivalent to FRP*:*

For a T_1 space with point countable base, if all subspaces Y of X of cardinality $\leq \aleph_1$ are left-separated then X itself is left-separated.

[3] S.F., *Left-separated topological spaces under Fodor-type Reflection Principle*, RIMS Kokyuroku No.1619 (2008), 32–42.

- A space X is *left-separated* if there is a well-ordering $<$ of X s.t. each initial segment of X w.r.t. $<$ is a closed subset of X.
- W. Fleissner (1986) proved that the assertion of Theorem 7 follows from Axiom R and it is refuted under $\neg$ ORP.

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Some applications

- The topological (graph-theoretic, Boolean algebraic) reflection principles mentioned above are all equivalent to each other over ZFC.
- All these reflection principles imply Ordinal Reflection Principle.

Theorem 9 (S.F. et al. [1])

FRP *is preserved by c.c.c. generic extension.*

- Hence, all reflection principles above impose almost no restriction on the size of continuum. **cf.:** Under slightly stronger reflection principles, the continuum is $\leq \aleph_2$ (S. Todorcevic).
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Theorem 10 (S.F. and Rinot [4])

FRP *implies Shelah's Strong Hypothesis.*

- *Shelah's Strong Hypothesis* (SSH) is the assertion equivalent to the following:

> For every uncountable cardinal κ of countable cofinality, we have $cf([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$.

- By the characterization above of SSH, Singular Cardinal Hypothesis (SCH) follows from SSH.

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Suppose that SSH does not hold. Then there is a better scale $\langle\langle\lambda_i : i < \omega\rangle, \langle f_\alpha : \alpha < \lambda^+\rangle\rangle$ for a cardinal λ with $\mathrm{cf}(\lambda) = \omega_1$.

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Then g together with E is a counterexample to $\mathrm{FRP}(\lambda^+)$. □

- Theorem 10 suggests that SSH should be also regarded as a reflection principle. We can in fact characterize SSH in terms of the following topological reflection principle:

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SSH as a topological reflection theorem

Theorem 11 (S.F. and Rinot [4])

SSH *is equivalent to the following assertion:*

For any countably tight space X, if X is $<\aleph_1$-thin then X is thin.

- A topological space X is *thin* if $|\overline{D}| \leq |D|^+$ holds for all $D \subseteq X$.
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Open problems:

- Are there any natural topological assertions which are equivalent to Axiom R (RP, WRP etc. resp.) ?
- For each topological theorem independent from ZFC, provide a set-theoretic principle characterizing the theorem (Topological Reverse Mathematics)!

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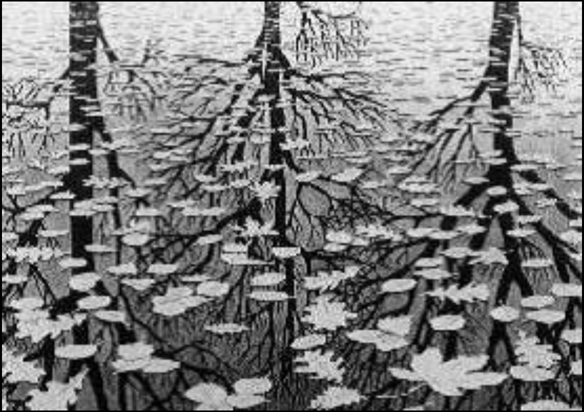
¡Gracias por su atención!

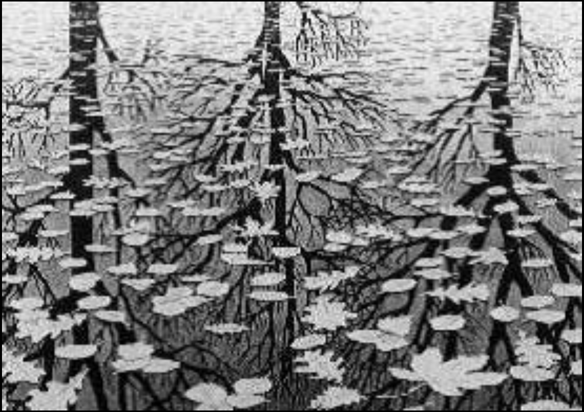


A part of: M.C. Escher, "Three Worlds" (1955)

These slides and their printer friendly version are linked to:
http://kurt.scitec.kobe-u.ac.jp/~fuchino/

ご静聴ありがとうございました!

¡Gracias por su atención!



A part of: M.C. Escher, "Three Worlds" (1955)

These slides and their printer friendly version are linked to:
http://kurt.scitec.kobe-u.ac.jp/~fuchino/