

# Forcing Axioms and the Continuum Problem

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In this note, we give a survey on recent developments pertaining to forcing axioms with emphasis on their characterizations and consequences connected to the Continuum Problem. Nonspecialists in set theory and/or students in mathematics are supposed to be the readers, and thus we tried to make this as self-contained as possible.

Due to the limitation on the extent of the article, however, many proofs are simply omitted. This is in particular the case for the results cited in the later sections. Those interested in the technical details may consult with papers and textbooks cited in the references at the end of the article.

In Section 1, we introduce Martin’s axiom formulated in terms of general topology and show how this axiom is used to establish the independence of certain mathematical assertions from the usual axiom system of set theory.

After reviewing some basic facts about forcing in Section 2, we give in Section 3 a characterization of Martin’s axiom in terms of notions connected to the method of forcing. In most of the modern textbooks on set theory, the characterization given in this section is the official definition of Martin’s axiom. In Section 4, we give two further characterizations of Martin’s axiom in terms of forcing: a characterization due to the author and another characterization by J. Bagaria. Both of these characterizations assert a certain absoluteness of generic extension over the universe of set theory, and they suggest that Martin’s Axiom is a very

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natural axiom which might even be considered as a candidate for one of the “true” axioms extending the usual axiom system of set theory.

Finally, in Section 5, several known strengthenings of Martin’s axiom are considered. Some recent results concerning these axioms are also discussed.

Following the request of the editor of the original version of the article, the Japanese original of the present text bore a subtitle which can be translated as “recent developments in set theory”. In a short survey article like this, however, we can mention only a small fraction of the recent explosive developments of set theory. There is also an article by Yo Matsubara [21] in this Journal on topics in set theory, but his article together with the present one still does not cover many branches of set theory such as infinitary combinatorics, set theory of reals, theory of inner models, pcf theory or the theory of large cardinals among others<sup>\*2</sup>. Thus we are trying here merely to describe *some* of the recent topics in *a* branch of set theory. Since most of the recent developments in set theory seem to remain largely unknown to the general audience in mathematics, I do hope very much that expository articles on other fields of set theory will be written in the future as well.

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## 1 Martin’s Axiom

There are several quite different formulations of Martin’s Axiom. In this section, we begin with introducing the axiom as an assertion on topological spaces.

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<sup>\*2</sup>We cannot go into these topics of set theory here but merely suggest some references. For set theory of the reals see e.g. [3]. [13] is a well-written introduction to the pcf theory of S. Shelah ([26]). For the theory of large cardinals, A. Kanamori [16] is a good reference. There are various results which can be classified as infinitary combinatorics and such results are often connected also to other fields of mathematics. Some of such results are to be found in standard textbooks on set theory such as [18] or [14]. The classical theory of inner models (i.e., the theory of the constructible universe) can be learned in [6]. For more recent developments in the theory, the reader may consult, e.g., [19], [32]. The forthcoming [17] and [15] should contain most of the recent topics in all of these subjects.

A topological space  $X = (X, \mathcal{O})$  satisfies the *c.c.c.* (the countable chain condition) if all pairwise disjoint families  $\mathcal{F} \subseteq \mathcal{O}$  of open sets are countable.  $X = (X, \mathcal{O})$  satisfies the *c.c.c.* if and only if  $X$  has the following property, which resembles the definition of a Lindelöf space: For any  $\mathcal{F} \subseteq \mathcal{O}$ , if  $\bigcup \mathcal{F}$  is dense in  $X$ , then there is a countable  $\mathcal{F}' \subseteq \mathcal{F}$  such that  $\bigcup \mathcal{F}'$  is dense in  $X$ <sup>\*3</sup>.

Using this terminology, Martin's Axiom (MA) can be formulated as follows:

(MA) For any *c.c.c.* compact Hausdorff space  $X = (X, \mathcal{O})$ , if  $\mathcal{F} \subseteq \mathcal{O}$  is a family of dense subsets of  $X$  with  $|\mathcal{F}| < 2^{\aleph_0}$ , then the intersection  $\bigcap \mathcal{F}$  of  $\mathcal{F}$  is nonempty.

Here,  $|\mathcal{F}|$  denotes the cardinality (size) of the set  $\mathcal{F}$ , that is, the cardinality of the set  $\Lambda$  if  $\mathcal{F}$  is given as  $\mathcal{F} = \{O_\alpha : \alpha \in \Lambda\}$  where the enumeration  $\alpha \mapsto O_\alpha$  is one-to-one.  $2^{\aleph_0}$  denotes the size of the continuum<sup>\*4</sup>. Hence  $|\mathcal{F}| < 2^{\aleph_0}$  means that the number (cardinality) of elements of  $\mathcal{F}$  is smaller than the cardinality of the set of all real numbers<sup>\*5</sup>.

Remembering that the Continuum Hypothesis (CH) is the assertion  $2^{\aleph_0} = \aleph_1$  = the minimal uncountable cardinality<sup>\*6\*7</sup>, “ $|\mathcal{F}| < 2^{\aleph_0}$ ” is equivalent to “ $\mathcal{F}$  is countable” under CH. By the Baire Category Theorem, it follows that MA holds under CH. By an iterative version of the method of forcing discussed in the next section, we can also prove the following:

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<sup>\*3</sup> For a family  $\mathcal{F}$  of sets, if, for example,  $\mathcal{F} = \{X_\alpha : \alpha \in \Lambda\}$ , we denote by  $\bigcup \mathcal{F}$  the set  $\bigcup_{\alpha \in \Lambda} X_\alpha$  and by  $\bigcap \mathcal{F}$  the set  $\bigcap_{\alpha \in \Lambda} X_\alpha$ . If  $\mathcal{F} = \{a, b\}$  we have  $\bigcup \mathcal{F} = a \cup b$  and  $\bigcap \mathcal{F} = a \cap b$ .

<sup>\*4</sup>In set theory, the set of natural numbers  $\mathbb{N}$  seen as a (transfinite) ordinal is usually denoted by  $\omega$ . It is also denoted by  $\omega_0$  or  $\aleph_0$  when it is seen as the first infinite cardinal. For a cardinal number  $\kappa$ ,  $2^\kappa$  denotes the cardinality of the power set  $\mathcal{P}(\kappa)$  of  $\kappa$ . Thus  $2^{\aleph_0}$  is the cardinality  $|\mathcal{P}(\omega)|$  of the power set (i.e., the set of all subsets)  $\mathcal{P}(\omega)$  of  $\omega$ . Since there is a natural bijection between  $\mathcal{P}(\omega)$  and the set of all reals  $\mathbb{R}$ , we have  $2^{\aleph_0} = |\mathcal{P}(\omega)| = |\mathbb{R}|$ .

<sup>\*5</sup>For the notation “ $\bigcap \mathcal{F}$ ”, see the footnote <sup>\*3</sup>.

<sup>\*6</sup>This is equal to the minimal possible size of an infinite set which cannot be mapped to  $\mathbb{N}$  bijectively.

<sup>\*7</sup>Similarly, we denote by  $\aleph_2$  the smallest cardinal above  $\aleph_1$ , with  $\aleph_3$  the smallest cardinal above  $\aleph_2$ , etc.

**Theorem 1** (Martin–Solovay [20], for a detailed proof, see e.g. [18]) *The axiom system obtained by adding the negation of the Continuum Hypothesis ( $\neg\text{CH}$ ) and MA to the axiom system  $\text{ZFC}$ <sup>\*8</sup> is equiconsistent with ZFC.*

Here, ZFC denotes the axiom system of Zermelo–Fraenkel set theory with the Axiom of Choice. An axiom system  $T$  and another system  $T'$  are equiconsistent if consistency of  $T$  can be concluded from the assumption of the consistency  $T'$  and vice versa.

It is clear that the consistency of the system  $\text{ZFC} + \neg\text{CH} + \text{MA}$  implies the consistency of its subsystem ZFC. So the point here is that the consistency of  $\text{ZFC} + \neg\text{CH} + \text{MA}$  can be derived from the assumption of consistency of ZFC and this can be done using the method of forcing. The assertion of Theorem 1 is often formulated also as “ $\neg\text{CH} + \text{MA}$  is consistent with ZFC”<sup>\*9</sup>.

MA cannot be proved from  $\text{ZFC} + \neg\text{CH}$ . For example, in a Cohen model, which we shall discuss later, it is known that Martin’s axiom does not hold. Thus MA is independent from ZFC or even independent from  $\text{ZFC} + \neg\text{CH}$ . The following consequence of  $\neg\text{CH} + \text{MA}$  is also often used:

( $\text{MA}_{\aleph_1}$ ) For any c.c.c. compact Hausdorff space  $X = (X, \mathcal{O})$ , if  $\mathcal{F} \subseteq \mathcal{O}$  is a family of dense subsets of  $X$  with  $|\mathcal{F}| \leq \aleph_1$ , then the intersection  $\bigcap \mathcal{F}$  of  $\mathcal{F}$  is not empty.

$\text{MA}_{\aleph_1}$  is strictly weaker than  $\neg\text{CH} + \text{MA}$ . But  $\neg\text{CH}$  follows from  $\text{MA}_{\aleph_1}$ :

**Lemma 2**  $\text{MA}_{\aleph_1}$  implies  $\neg\text{CH}$ .

**Proof.** We show the contrapositive. Assume that CH holds. We can then find an enumeration  $r_i, i \in \Lambda$  of the closed interval  $X = [0, 1]$  such that the index set  $\Lambda$  is of cardinality  $\aleph_1$ .  $X$  is a compact Hausdorff space with respect to the canonical

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<sup>\*8</sup>In the following we shall denote this axiom system with  $\text{ZFC} + \neg\text{CH} + \text{MA}$ .

<sup>\*9</sup>In contrast, the statement “ $\text{ZFC} + \neg\text{CH} + \text{MA}$  is consistent” is not correct: Though this is what we actually believe, the exact formulation cannot be done in this way since the consistency of ZFC is unprovable by Gödel’s Incompleteness Theorem (at least by a purely finitary argument). Nevertheless we often argue intuitively and say in such a situation that “ $\text{ZFC} + \neg\text{CH} + \text{MA}$  is consistent”. Further we often leave out mentioning ZFC and say simply that “ $\neg\text{CH} + \text{MA}$  is inconsistent”. But what is meant in such a statement is always the relative consistency in the above sense.

topology on it. It is also easy to check that  $X$  satisfies the c.c.c. For each  $i \in \Lambda$ , let  $D_i = X \setminus \{r_i\}$ . Then each  $D_i$  is a dense subset of  $X$  while  $\bigcap_{i \in \Lambda} D_i = \emptyset$ . This shows that  $\text{MA}_{\aleph_1}$  does not hold.  $\square$

Using  $\neg\text{CH} + \text{MA}$  or  $\text{MA}_{\aleph_1}$  on the one hand and some other principle which is inconsistent with this but is shown to be consistent with ZFC on the other hand, we can prove the independence of a lot of interesting mathematical assertions by purely combinatorial arguments<sup>\*10</sup>.

The independence proof of Souslin's Hypothesis is a typical example of such a proof. The set of reals  $\mathbb{R}$  with the usual ordering  $\leq$  on it is separable and dense in itself<sup>\*11</sup>, connected and without endpoints. Actually, these properties characterize the linearly ordered set  $(\mathbb{R}, \leq)$ .

Souslin's Hypothesis (1920) states that in the characterization of  $(\mathbb{R}, \leq)$  above, separability may be replaced by the c.c.c. For this hypothesis the following (now classical) results are known<sup>\*12</sup>.

**Theorem 3** *Under  $\text{MA}_{\aleph_1}$ , every c.c.c. linearly ordered set which is dense in itself, connected and without endpoints, is order isomorphic to  $(\mathbb{R}, \leq)$ . That is, under  $\text{MA}_{\aleph_1}$ , Souslin's Hypothesis holds.*

On the other hand, from the combinatorial principle  $\diamond_{\aleph_1}$ , which follows from Gödel's  $V = L$  asserting that every set is constructible, that is, obtained by a constructible procedure by transfinite recursion over (possibly any) ordinals, the negation of the assertion of Theorem 3 follows:

**Theorem 4** *Under  $\diamond_{\aleph_1}$ , there exists a c.c.c. connected, dense in itself linear ordering  $(X, \leq)$  without endpoints which is not isomorphic to  $(\mathbb{R}, \leq)$ . In particular,  $(X, \leq)$  is non-separable.*

By the theorems above and by the fact that each of  $\text{MA}_{\aleph_1}$  or  $\diamond_{\aleph_1}$  is consistent over ZFC we conclude:

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<sup>\*10</sup>Though for more sophisticated independence proofs, direct use of a (very often quite involved) forcing argument is inevitable.

<sup>\*11</sup>A linear ordering  $(X, \leq_X)$  is dense in itself if for all  $a, b \in X$  with  $a <_X b$ , there is always  $c \in X$  such that  $a <_X c <_X b$ .

<sup>\*12</sup>For more about Souslin's Hypothesis, see e.g. [18]

**Corollary 5** *Souslin's Hypothesis is independent from ZFC.*

Besides Souslin's Hypothesis, there are many mathematical assertions provable under Martin's axiom  $+ \neg\text{CH}$  or  $\text{MA}_{\aleph_1}$  (for more about such assertions, see e.g. [8]).

## 2 Forcing

Historically the forcing method was introduced by P. Cohen to prove the consistency of the negation of the Continuum Hypothesis or the Axiom of Choice in the 1960's. The method was soon generalized by R. Solovay and others into a powerful tool for proving the consistency of various assertions such as  $\text{MA} + \neg\text{CH}$  in the previous section.

The topological definition of MA given in the previous section is mathematically decent, but it seems that the formulation of the axiom gives us no intuition about whether it should be true or not. In terms of forcing, we can give several characterizations of MA in which a better insight into its set-theoretical meaning seems to be obtained. In some sense we can even argue that the characterizations suggest the correctness of the axiom. We shall discuss this in the next sections. In this section, we shall give an overview of the method of forcing. For the technical details omitted here, see e.g. [18] or [14].

Intuitively, forcing is the method for construction of new models of ZFC: Starting from a given model  $M$  of ZFC, we take a so-called generic filter  $G$  outside  $M$  and then add  $G$  to  $M$  to construct the new model  $M \subseteq M[G]$  of ZFC, which is called a generic extension of  $M$ . By constructing a generic extension  $M[G]$  in which a mathematical statement  $\varphi$  holds, we can show the consistency of  $\varphi$  with ZFC.

This can be compared with the construction of a polynomial ring: Starting from a given ring  $R$ , or a model of the axioms of the ring, we choose a variable symbol  $x$  and construct the ring  $R[x] \supseteq R$  consisting of all polynomials with the variable  $x$ , which is again a model of the axioms<sup>\*13</sup>.

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<sup>\*13</sup>A sharper analogy to the a construction of generic extension can be obtained when we consider an infinite number of variable symbols  $x_1, x_2, \dots$ , construct  $R[x_1, x_2, \dots]$  and then take the residue ring  $R[x_1, x_2, \dots]/I$  for some ideal  $I$  over  $R[x_1, x_2, \dots]$ .

For the axiom system ZFC, however, the description of forcing just formulated as above causes some serious problems. Like the example of the ring  $R[x]$ , the construction of a model should be performed inside a set-theoretic universe. But we cannot prove the existence of a model of ZFC from ZFC itself by the Incompleteness Theorem. Hence, to be able to argue as above, we have to start from some strengthening of ZFC (e.g. ZFC + some large cardinal axiom) in which the existence of a model of ZFC is provable. However the primary purpose of forcing is to show the consistency of some mathematical statement  $\varphi$  over ZFC (or over some extension of ZFC) by showing that  $\varphi$  holds in a generic extension  $M[G]$ . Thus the setting would be very clumsy if we were forced to perform the forcing construction in a theory which is stronger than ZFC (or the extension of ZFC in consideration). An alternative would be to treat the “real” universe  $V$  of set theory (i.e. the class of all sets) as the starting model (or the *ground model* in forcing terminology) of the forcing construction. But then the meaning of taking  $G$  outside  $V$  becomes obscure.

To overcome this problematic, we argue as follows<sup>\*14</sup>. Let  $ZFC'$  be an arbitrary (but concretely given) system of axioms consisting of finitely many axioms from ZFC which contains enough axioms needed for the following discussions<sup>\*15</sup>. By Lévy’s Reflection Theorem, the Löwenheim-Skolem Theorem and Mostowski’s Theorem, we can prove in ZFC that, for a finite subsystem  $ZFC'$  of ZFC, there is a countable set  $M$  such that  $M$  together with the element relation  $\in$  restricted to  $M$  (which we shall also denote with  $\in$  for simplicity) forms a model of  $ZFC'$ ; in particular,  $M$  can be chosen to be transitive<sup>\*16</sup>.

For a set-theoretic assertion  $\varphi$ , if we can find some transitive  $M[G]$  such that  $(M[G], \in)$  satisfies<sup>\*17</sup>  $ZFC'$  and  $\varphi$ , we can conclude that the negation  $\neg\varphi$  of  $\varphi$  cannot be proved from  $ZFC'$ : Since  $(M[G], \in)$  satisfies  $ZFC'$ , if  $\neg\varphi$  were provable

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<sup>\*14</sup>This problem can also be solved by arguing with the relation “ $p \Vdash_P \varphi(\dot{x}_1, \dots, \dot{x}_n)$ ” discussed at the beginning of Section 4. For details see [18] Ch.VII §9.

<sup>\*15</sup>Strictly speaking, for the construction of  $M[G]$  from  $M$  below, we have to extend  $ZFC'$  by adding some more (finitely many) axioms of ZFC to a theory  $ZFC''$  where the forcing argument which we describe below is possible in the framework of  $ZFC''$  and then take a model  $M$  of  $ZFC''$ .

<sup>\*16</sup> $M$  is said to be transitive if for all  $x \in M$  and  $y \in x$  we have  $y \in M$ .

<sup>\*17</sup>Here again, we use the symbol “ $\in$ ” to denote the  $\in$ -relation restricted to  $M[G]$ .

from  $ZFC'$  so  $(M[G], \in)$  would satisfy  $\neg\varphi$ . But this is a contradiction. Since  $ZFC'$  was arbitrary except that it contains “enough” axioms of ZFC, we have proved that  $\neg\varphi$  is not provable from  $ZFC$ <sup>\*18</sup>.

Let us look at some more technical details of forcing constructions<sup>\*19</sup>.

For a finite subsystem  $ZFC'$  of ZFC (which is large enough in the sense that it contains all axioms of ZFC necessary for the further arguments), let  $M$  be as above. Since  $M$  satisfies  $ZFC'$  and since  $ZFC'$  contains enough axioms that we need later, we can treat  $M$  as if it were a model of ZFC. Hence, we shall simply say “ $M$  is a model of ZFC” in such a situation. An element  $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}})$  of  $M$  is said to be a forcing notion if  $\mathbb{P}$  is a pseudo-ordering<sup>\*20</sup> with the maximal element  $\mathbb{1}_{\mathbb{P}}$ .

$D \subseteq \mathbb{P}$  is dense in  $\mathbb{P}$  if for every  $p \in \mathbb{P}$  there is a  $q \in D$  such that  $q \leq_{\mathbb{P}} p$ . A subset  $G$  of  $\mathbb{P}$  which need not be an element of  $M$  is said to be a filter over  $\mathbb{P}$  if  $G$  is closed upward and for any  $p, q \in G$  there is an  $r \in G$  such that  $r \leq_{\mathbb{P}} p, q$ <sup>\*21</sup>. A filter  $G$  over  $\mathbb{P}$  is  $M$ -generic if we have  $D \cap G \neq \emptyset$  for any dense  $D \subseteq \mathbb{P}$  with  $D \in M$ . In some of the literature such a  $G$  is also called a  $\mathbb{P}$ -generic filter over  $M$ , or  $(\mathbb{P}, M)$ -generic filter.

**Lemma 6** *For any forcing notion  $\mathbb{P} \in M$  and  $p \in \mathbb{P}$ , there is an  $M$ -generic filter  $G$  over  $\mathbb{P}$  such that  $p \in G$ .*

**Proof.** Since  $M$  is countable, there are at most countably many  $D \in M$  which

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<sup>\*18</sup>Some readers may find this assertion of the form “we proved that ... is not provable” strange: with “... is not provable” we mean that “there is no formal proof of ... from the axioms of ZFC” while the claim “we proved that ...” is considered to be a statement in ‘Metamathematics’ about the formal deduction system of predicate logic with the axiom system ZFC. Analyzing the argument above more closely, we see that the conclusion we actually obtained here is the following metamathematical statement: “given a proof of  $\neg\varphi$  from ZFC, we can rearrange the proof to obtain a proof of the contradiction from ZFC alone”.

<sup>\*19</sup>Formally the arguments below are proofs from the axiom system ZFC of corresponding formal assertions (or intuitively they are done in the “universe” of set theory that these formal assertions describe).

<sup>\*20</sup> $(\mathbb{P}, \leq_{\mathbb{P}})$  is a pseudo-ordering if  $\leq_{\mathbb{P}}$  is transitive and  $p \leq_{\mathbb{P}} p$  holds for every  $p \in \mathbb{P}$ .

<sup>\*21</sup>If there exists  $r \in \mathbb{P}$  with  $r \leq_{\mathbb{P}} p, q$ , we say that  $p$  and  $q$  are *compatible* in  $\mathbb{P}$ ; otherwise  $p$  and  $q$  are *incompatible* in  $\mathbb{P}$ . Thus this condition simply means that elements of  $G$  are pairwise compatible.



are dense subsets of  $\mathbb{P}$ . Hence we can enumerate such sets as  $D_0, D_1, D_2, \dots$ . Thus we can construct a sequence  $p_0, p_1, p_2, \dots$  of elements of  $\mathbb{P}$  inductively such that

- (1)  $p \geq_{\mathbb{P}} p_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} p_2 \geq_{\mathbb{P}} \dots$ , and
- (2)  $p_i \in D_i$  for all  $i \in \mathbb{N}$ .

((2) is possible by the density of  $D_i$ .) Now let  $G = \{q \in \mathbb{P} : p_i \leq_{\mathbb{P}} q \text{ for some } i \in \omega\}$ . Then  $G$  is an  $M$ -generic filter over  $\mathbb{P}$  containing  $p$ .  $\square$

In the proof above, we have  $p_i \in M$  for all  $i \in \omega$ . However, the enumeration  $\langle D_i : i \in \mathbb{N} \rangle$  in general is not in  $M$  ( $M$  might “think” that there are uncountably many dense subsets of  $\mathbb{P}$ !). Thus the inductive construction of  $p_0, p_1, p_2, \dots$  using this enumeration cannot be done in general inside  $M$ . Hence,  $G$  need not be an element of  $M$ . Actually, if, for every  $p \in \mathbb{P}$ , there exist incompatible  $q, r \leq_{\mathbb{P}} p$ , we can show that any  $M$ -generic filter over  $M$  is not an element of  $M$ .

For  $M, \mathbb{P}, G$  as above, we can construct a countable transitive model  $N$  of ZFC such that: (a)  $M \subseteq N$  and  $G \in N$ ; (b) the ordinals in the sense of  $M$  coincide with ordinals in the sense of  $N$ ; (c)  $N$  is the minimal transitive model of ZFC satisfying (a) and (b) (with respect to  $\subseteq$ ).

Since  $N$  as above is unique by (c), we denote it by  $M[G]$  and call a *generic extension* of  $M$  by the generic set  $G$ .

The crucial point here is that most of the properties of  $N$  we want to know can be decided already inside  $M$  in the following sense: We can define names of each element of  $M[G]$ , or  $\mathbb{P}$ -names, independent of  $G$ . More precisely, we can introduce the class  $M^{\mathbb{P}} \subseteq M$  of  $\mathbb{P}$ -names in  $M$  as a definable class in  $M$  (that is, definable with the parameter  $\mathbb{P}$ ). Given an  $M$ -generic filter  $G$  over  $\mathbb{P}$ , we can define, for each  $\mathbb{P}$ -name  $\dot{x}$ , the interpretation  $\dot{x}^G$  of  $\dot{x}$  such that  $M[G] = \{\dot{x}^G \mid \dot{x} \in M^{\mathbb{P}}\}$ . For each  $x \in M$ , we have a standard  $\mathbb{P}$ -name  $\check{x} \in M^{\mathbb{P}}$  such that  $\check{x}^G = x$  for any generic filter  $G$ . If no confusion can arise, we often denote  $\check{x}$  simply by  $x$ .

For any  $p \in \mathbb{P}$ , for any formula  $\varphi$  representing a set-theoretic assertion with  $n$  free variables and for any  $\dot{x}_1, \dots, \dot{x}_n \in M^{\mathbb{P}}$ , let us denote with

$$p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)$$

the assertion

“ $\varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$  holds in  $M[G]$  for any  $M$ -generic filter over  $\mathbb{P}$  with  $p \in G$ ”<sup>\*22</sup>.

Then the relation “ $p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)$ ” on  $p$  and  $\dot{x}_1, \dots, \dot{x}_n$  is definable in  $M$ . The largest element  $\mathbb{1}_{\mathbb{P}}$  of  $\mathbb{P}$  is contained in every  $M$ -generic filter over  $\mathbb{P}$ . Hence, if  $M \models “\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)”$ , then we have  $M[G] \models \varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$  for any generic extension over  $\mathbb{P}$ . “ $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \dots$ ” is also denoted by “ $\Vdash_{\mathbb{P}} \dots$ ”.

We can also construct a standard  $\mathbb{P}$ -name  $\dot{G}$  such that  $\dot{G}^G = G$  holds for any  $M$ -generic filter  $G$  over  $\mathbb{P}$ . For any  $p, q \in \mathbb{P}$ , we have  $M \models “p \Vdash_{\mathbb{P}} q \in \dot{G}”$  if and only if, for any  $r \leq_{\mathbb{P}} p$  there is an  $s \leq_{\mathbb{P}} r$  such that  $s \leq_{\mathbb{P}} q$ .

The following property is also often used: for an  $M$ -generic filter  $G$  over  $\mathbb{P}$ , if  $M[G] \models \varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$ , then there is a  $p \in G$  such that  $M \models “p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)”$ . The following lemma follows directly from this.

**Lemma 7** (1) *For any  $p \in \mathbb{P}$ , formula  $\varphi$  and  $\dot{x}_1, \dots, \dot{x}_n \in M^{\mathbb{P}}$ , there is a  $q \leq_{\mathbb{P}} p$  such that either  $M \models “q \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)”$  or  $M \models “q \Vdash_{\mathbb{P}} \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)”$  holds.*

(2) *For any  $p \in \mathbb{P}$ , formula  $\varphi$  and  $\dot{x}_2, \dots, \dot{x}_n \in M^{\mathbb{P}}$ , if  $M \models “p \Vdash_{\mathbb{P}} \exists x \varphi(x, \dot{x}_2, \dots, \dot{x}_n)”$ , then there is a  $q \leq_{\mathbb{P}} p$  and an  $\dot{x}_1 \in M^{\mathbb{P}}$  such that  $M \models “q \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n)”$ .*

(3) *For any  $p \in \mathbb{P}$ ,  $X \in M$  and  $\mathbb{P}$ -name  $\dot{x}$ , if  $M \models “p \Vdash_{\mathbb{P}} \dot{x} \in X”$  then there is a  $q \leq_{\mathbb{P}} p$  and an  $x \in X$  such that  $M \models “q \Vdash_{\mathbb{P}} \dot{x} = x”$ .*

A forcing notion  $\mathbb{P}$  satisfies the c.c.c. (countable chain condition) if any uncountable  $X \subseteq \mathbb{P}$  contains compatible elements. The following lemma shows a reason for the importance of c.c.c. forcing notions in the theory of forcing.

**Lemma 8** *Let  $\mathbb{P} \in M$  be a forcing notion such that  $M \models “\mathbb{P}$  satisfies the c.c.c.”. For any  $M$ -generic filter  $G$  over  $\mathbb{P}$ , if  $\kappa$  is a cardinal in  $M$ , then  $\kappa$  remains a cardinal in  $M[G]$ .<sup>\*23</sup>*

**Proof.** We show here that  $\aleph_1$  in  $M$  (or  $(\aleph_1)^M$ ) is also  $\aleph_1$  in  $M[G]$ . For a regular cardinal  $\kappa$  the proof is almost identical with this special case; the case for singular cardinals follows from the case of regular cardinals.

Suppose for a contradiction that  $(\aleph_1)^M$  is not a cardinal in  $M[G]$ . Then there is a surjection  $f \in M[G]$  from  $\mathbb{N}$  onto  $(\aleph_1)^M$ . Let  $\dot{f}$  be a  $\mathbb{P}$ -name of  $f$ ; that is, let

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<sup>\*22</sup>In the following, the assertion “ $\varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$  holds in  $M[G]$ ” will be denoted by  $M[G] \models \varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$ . More generally, for a structure  $A$  and  $a_1, \dots, a_n \in A$ ,  $A \models \psi(a_1, \dots, a_n)$  denotes the assertion “ $\psi$  holds in  $A$  for  $a_1, \dots, a_n$ ”.

<sup>\*23</sup>We say also that  $\mathbb{P}$  preserves  $\kappa$  (as a cardinal).

$\dot{f} \in M^{\mathbb{P}}$  be such that  $f = \dot{f}^G$ . Then there is a  $p_0 \in G$  such that  $p_0 \Vdash_{\mathbb{P}} \text{“}\dot{f} \text{ is a surjection from } \mathbb{N} \text{ onto } (\aleph_1)^M \text{”}$ .

For the rest of the proof, we argue in  $M$ : By Lemma 7 (2) and (3), there is a  $D_n \subseteq \mathbb{P}$  for each  $n \in \mathbb{N}$  such that: (a) each  $p \in D_n$  is either incompatible with  $p_0$  or such that there is  $k_p \in \mathbb{N}$  such that  $p \Vdash_{\mathbb{P}} \text{“}\dot{f}(n) = k_p \text{”}$  (i.e.,  $p$  decides the value of  $\dot{f}(n)$ ); (b) elements of  $D_n$  are pairwise incompatible; (c)  $D_n$  is maximal with respect to  $\subseteq$  among those subsets of  $\mathbb{P}$  satisfying (a) and (b).

By the c.c.c. of  $\mathbb{P}$  and (b), each  $D_n$  is countable. Hence: (d)  $E_n = \{k_p : p \in D_n\}$  is countable. By the maximality of  $D_n$ ,  $\{p \in \mathbb{P} : p \leq_{\mathbb{P}} q \text{ for some } q \in D_n\}$  is a dense subset of  $\mathbb{P}$ . It follows that (e)  $f(n) = \dot{f}^G(n) \in E_n$ .

Since both  $D_n$  and  $E_n$  for  $n \in \mathbb{N}$  are (uniformly) definable  $M$  (and since  $M$  satisfies ZFC), it follows that  $\langle D_n : n \in \mathbb{N} \rangle, \langle E_n : n \in \mathbb{N} \rangle \in M$ . Hence  $E = \bigcup \{E_n : n \in \mathbb{N}\}$  is an element of  $M$  as well. In particular, we have  $E \subsetneq (\aleph_1)^M$ . By (e), it follows that  $f[\mathbb{N}] \subseteq E$ . But this contradicts the assumption that  $f$  is a surjection on  $(\aleph_1)^M$ .  $\square$

The historical forcing notion P. Cohen invented (together with the archetype of modern forcing to prove the independence of the Continuum Hypothesis) is a typical example of the c.c.c. forcing notion. For any sets  $X$  and  $Y$ , let  $\text{Fn}(X, Y) = \{f : f \text{ is a finite partial function from } X \text{ to } Y\}$  with the ordering defined by  $f \leq g \Leftrightarrow f \text{ extends } g \text{ for } f, g \in \text{Fn}(X, Y)$ . We can show that  $(\text{Fn}(X, Y), \leq)$  satisfies the c.c.c. for any set  $X$  and any countable  $Y$ . Let  $X = \aleph_2 \times \mathbb{N}$  in  $M$  (that is,  $X = (\aleph_2)^M \times \mathbb{N}$ ),  $Y = \mathbb{N} = \mathbb{N}^M$  and  $\mathbb{P} = (\text{Fn}(X, Y), \leq)$ . Translated in the modern terminology,  $\mathbb{P}$  corresponds to the forcing Cohen used in his seminal work and is called today the Cohen forcing. Let  $G$  be an  $M$ -generic filter over  $\mathbb{P} = (\text{Fn}(X, Y), \leq)$ . By genericity of  $G$ ,  $g = \bigcup G$  is a mapping from  $\aleph_2^M \times \mathbb{N}$  to  $\mathbb{N}$  and the  $g_\alpha$ 's for  $\alpha \in (\aleph_2)^M$  defined by  $g_\alpha : \mathbb{N} \rightarrow \mathbb{N}; n \mapsto g(\alpha, n)$  are pairwise distinct. Hence, by Lemma 8, we have  $\text{“}2^{\aleph_0} \geq (\aleph_2)^M = \aleph_2 \text{”}$  in  $M[G]$ . In other words,  $\neg\text{CH}$  holds in  $M[G]$ . Combining this with the argument before footnote \*18, we conclude that  $\text{ZFC} + \neg\text{CH}$  is consistent provided that ZFC itself is consistent.

### 3 Martin's Axiom as a forcing axiom

Let  $\mathbb{P}$  be a forcing notion and  $\mathcal{D}$  be a family of dense subsets of  $\mathbb{P}$ .  $G \subseteq \mathbb{P}$  is said to be a  $\mathcal{D}$ -generic filter if  $G$  is a filter (in the sense of Section 2) and  $G \cap D \neq \emptyset$  for all  $D \in \mathcal{D}$ . In this terminology, the  $M$ -generic filter in Section 2 is nothing but a  $\mathcal{D}$ -generic filter for  $\mathcal{D} = \{D : D \text{ is a dense subset of } \mathbb{P} \text{ and } D \in M\}$ .

Using this notion, MA can be characterized as the following assertion on forcing notions (for more on classical characterizations of MA including the one we are going to consider below, see e.g. [18] or [8]).

**Theorem 9** *MA is equivalent with the following assertion:*

- (A) *For any c.c.c. forcing notion  $\mathbb{P}$  and for any family  $\mathcal{D}$  of dense subsets of  $\mathbb{P}$  with cardinality less than  $2^{\aleph_0}$ , there is a  $\mathcal{D}$ -generic filter over  $\mathbb{P}$ .*

**Proof.** MA  $\Rightarrow$  (A): Suppose that  $\mathbb{P}$  is a c.c.c. forcing notion and  $\mathcal{D}$  is a family of dense subsets of  $\mathbb{P}$  of cardinality less than  $2^{\aleph_0}$ . Let  $B$  be a Boolean algebra such that there is an order-preserving mapping  $f : \mathbb{P} \rightarrow B \setminus \{0\}$  which also preserves incompatibility and  $f[\mathbb{P}]$  is a dense subset of  $B \setminus \{0_B\}$ . Let  $X = (X, \mathcal{O})$  be the canonical Boolean space dual to  $B$ . That is,  $X$  consists of ultrafilters over  $B$  and, letting  $O_b = \{F \in X : b \in F\}$  for each  $b \in B \setminus \{0\}$ , the topology  $\mathcal{O}$  on  $X$  is generated from  $\mathcal{B} = \{O_b : b \in B \setminus \{0\}\}$ .  $X$  is then a compact Hausdorff space satisfying the c.c.c. For each  $D \in \mathcal{D}$ ,  $C_D = \bigcup \{O_b : b \in f[D]\}$  is a dense open subset of  $X$ . Hence, by MA, there is an  $F^* \in X$  such that

$$(1) \quad F^* \in C_D \quad (\Leftrightarrow \quad F^* \cap f[D] \neq \emptyset) \quad \text{for all } D \in \mathcal{D}.$$

Let  $G = f^{-1}[F^*]$ . By the assumption on  $f$ ,  $G$  is a filter over  $\mathbb{P}$  and, by (1), it is  $\mathcal{D}$ -generic.

(A)  $\Rightarrow$  MA: Suppose that  $(X, \mathcal{O})$  is a c.c.c. compact Hausdorff space and  $\mathcal{D}$  a family of open dense subsets of  $X$  of cardinality less than  $2^{\aleph_0}$ . Let  $\mathbb{P} = \{A \subseteq X : A \text{ is a nonempty closed subset of } X\}$  be the forcing notion equipped with the partial ordering defined by  $A \leq_{\mathbb{P}} B \Leftrightarrow A \subseteq B$  for  $A, B \in \mathbb{P}$ . For each  $D \in \mathcal{D}$ ,  $E_D = \{A \in \mathbb{P} : A \subseteq D\}$  is dense in  $\mathbb{P}$ . Hence, for  $\mathcal{E} = \{E_D : D \in \mathcal{D}\}$ , there is an  $\mathcal{E}$ -generic filter  $G$  over  $\mathbb{P}$  by (A). Since  $G$  is a family of closed subsets of  $X$  and, being a filter over  $\mathbb{P}$ , it has the finite intersection property by the definition of  $\leq_{\mathbb{P}}$ , hence, by compactness of  $X$ , we have  $\bigcap G \neq \emptyset$ . Let  $x \in \bigcap G$ . Since  $x$  belongs to

all  $D \in \mathcal{D}$ , we have  $\bigcap \mathcal{D} \neq \emptyset$ . □

Note that, while an  $M$ -generic filter cannot be taken as an element of  $M$  in general, we are talking here about a  $\mathcal{D}$ -generic filter which does exist (in the real universe of set theory).

Similarly to Theorem 9, we can show that  $\text{MA}_{\aleph_1}$  is equivalent to the following:

(A') For any c.c.c. forcing notion  $\mathbb{P}$  and for any family  $\mathcal{D}$  of dense subsets of  $\mathbb{P}$  with  $|\mathcal{D}| \leq \aleph_1$ , there is a  $\mathcal{D}$ -generic filter over  $\mathbb{P}$ .

Assertions of the form similar to (A) or (A') are often called forcing axioms. Besides MA and  $\text{MA}_{\aleph_1}$ , the PFA and MM we will be talking about in Section 5 are examples of important forcing axioms.

## 4 Martin's axiom as generic absoluteness

For a forcing notion  $\mathbb{P}$  in a countable transitive model  $M$  and a formula  $\varphi$  in the language of set theory, we saw in Section 2 that the relation  $p \Vdash_{\mathbb{P}} \varphi(\dot{x}_1, \dots, \dot{x}_n)$  for  $p \in \mathbb{P}$  and  $\mathbb{P}$ -names  $\dot{x}_1, \dots, \dot{x}_n \in M^{\mathbb{P}}$  is definable in  $M$ . The same definition can be carried out in the universe  $V$  of all sets starting from an arbitrary forcing notion  $\mathbb{P}$ . Of course, we cannot consider  $V[G]$  in a strict sense;  $V$  consists of *all* sets so that we cannot take a set  $G$  outside  $V$ . However, by rephrasing " $\Vdash_{\mathbb{P}} \varphi$ " as " $\varphi$  holds in  $V[G]$ ", we can still argue as if " $V[G]$ " would exist.

Using this, we can give the following characterization of MA:

**Theorem 10** (S. Fuchino [9]) MA is equivalent to the following assertion:

(B) For every c.c.c. forcing notion  $\mathbb{P}$  and mathematical structures  $A, B$ , if  $A$  has cardinality less than  $2^{\aleph_0}$  and  $\Vdash_{\mathbb{P}}$  " $A$  is embeddable into  $B$ ", then  $A$  is actually embeddable into  $B$ .

In other words, Martin's axiom asserts that (non)embeddability of a structure of cardinality less than  $2^{\aleph_0}$  into another structure does not change its validity in generic extensions of the universe  $V$  by any c.c.c. forcing notion. Thus, the characterization shows that Martin's axiom is an assertion on the absoluteness of  $V$  with respect to the embeddability of structures. Seen in this way MA corresponds

with our intuition about  $V$  that it is so rich that it contains everything which exists in any (reasonable) imaginary extension of  $V$ .

Similarly,  $\text{MA}_{\aleph_1}$  is shown to be equivalent with the following:

(B') For any c.c.c. forcing notion  $\mathbb{P}$  and mathematical structures  $A, B$ , if  $A$  has cardinality  $\aleph_1$  and  $\Vdash_{\mathbb{P}}$  “ $A$  is embeddable into  $B$ ”, then  $A$  is really embeddable into  $B$ .

For another characterization of Martin’s axiom in terms of absoluteness, we need some notions from mathematical logic and axiomatic set theory.

A logical formula (in the language of set theory) in which quantifiers appear only in the bounded form of “ $\forall x \in y$ ” or “ $\exists x \in y$ ” is called a  $\Sigma_0$ -formula.

The formula

$$(\forall x \in f)(\forall x' \in f)(\exists y \in x)(\exists u \in y)(\exists v \in y)(\forall y' \in x')(\forall u' \in y')(\forall v' \in y') \\ (x = (u, v) \wedge u \in \mathbb{N} \wedge v \in \mathbb{N} \wedge (x' = (u', v') \wedge u = u' \rightarrow v = v'))$$

asserting that  $f$  is a mapping from  $\mathbb{N}$  to  $\mathbb{N}$  is an example of a  $\Sigma_0$ -formula<sup>\*24</sup>.

Many predicates used in everyday mathematics can be expressed in  $\Sigma_0$ -formulas; the assertion that  $f$  is a continuous function from  $\mathbb{Q}$  to  $\mathbb{Q}$ <sup>\*25</sup> is another example of properties expressible in  $\Sigma_0$ -formula<sup>\*26</sup>, as well as the assertion “ $x$  is an ordinal number”.

A formula obtained from a  $\Sigma_0$ -formula by attaching (finitely many) “unbounded” quantifiers of the form “ $\exists x$ ” before it is called a  $\Sigma_1$ -formula. Note that, by this definition,  $\Sigma_0$ -formulas are also  $\Sigma_1$ -formulas.

Since many mathematical predicates are expressible in  $\Sigma_0$ -formulas, existence assertions in mathematics are often expressible in  $\Sigma_1$ -formulas.

A set  $x$  is said to be hereditarily countable if  $x$  is countable and every set  $y$  accessible by a decreasing sequence  $x \ni x_1 \ni \dots \ni x_n \ni y$  with respect to the  $\in$ -relation starting from  $x$  is countable.

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<sup>\*24</sup>Provided that  $\mathbb{N}$  is allowed as a parameter.

<sup>\*25</sup>Note that such  $f$  codes a continuous function from  $\mathbb{R}$  to  $\mathbb{R}$ .

<sup>\*26</sup>Also here,  $\mathbb{N}$  should be used as a parameter and  $f$  is a variable in this formula. We will not give this formula explicitly since it will become quite long and complicated, and also its exact form depends on how we introduce  $\mathbb{Q}$  or  $\mathbb{R}$  formally.

More generally, for a cardinal  $\kappa$ ,  $x$  is hereditarily of cardinality strictly less than  $\kappa$  if  $x$  is of cardinality strictly less than  $\kappa$  and every set  $y$  accessible by a decreasing sequence  $x \ni x_1 \ni \cdots \ni x_n \ni y$  starting from  $x$  is of cardinality strictly less than  $\kappa$ .

Thus a set is hereditary countable if and only if it is hereditary of cardinality strictly less than  $\aleph_1$ .

Let us denote by ZF the axiom system obtained from ZFC by discarding the Axiom of Choice. The following theorem is of importance:

**Theorem 11** (Lévy-Shoenfield Absoluteness Lemma<sup>\*27</sup>) *Suppose that  $V$  and  $W$  are transitive classes satisfying (a large enough finite subsystem of) ZF with  $V \subseteq W$ . If  $a \in V$  is hereditarily countable in the sense of  $V$  and  $\varphi$  is a  $\Sigma_1$ -formula, then  $\varphi(a)$  holds in  $V$  (i.e.  $V \models \varphi(a)$ ) if and only if  $\varphi(a)$  holds in  $W$  (i.e.  $W \models \varphi(a)$ ).*

An interesting application of this theorem is as follows. Suppose that a mathematical assertion can be formulated as a  $\Sigma_1$ -formula  $\varphi$  with a real  $a$  as a parameter<sup>\*28</sup>. If this assertion can be proved using the Continuum Hypothesis or Axiom of Choice, then it is actually provable without using them: if the assertion  $\varphi(a)$  cannot be proved without one of these axioms, the axiom system ZF + the negation of the assertion ( $\neg\varphi(a)$ ) is consistent. Working in (a model  $V$  of) ZF +  $\neg\varphi(a)$ ,  $L[a]$  (the class of all sets definable from the parameter  $a$  by transfinite induction) satisfies the axioms of ZFC as well as the Continuum Hypothesis. Hence  $L[a] \models \varphi(a)$ . But since  $L[a] \subseteq V$  and  $V \models \neg\varphi(a)$ , this is a contradiction to Theorem 11.

Martin's axiom can be characterized as the following extension of the Lévy-Shoenfield Absoluteness Lemma:

**Theorem 12** (J. Bagaria [2]) MA is equivalent to the following assertion:

- (C) For a c.c.c. forcing notion  $\mathbb{P}$ , for any set  $a$  hereditarily of cardinality strictly less than  $2^{\aleph_0}$  and  $\Sigma_1$ -formula  $\varphi$ , we have  $\varphi(a)$  if and only if  $\Vdash_{\mathbb{P}} \varphi(a)$ .

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<sup>\*27</sup>See e.g. [4] or [10] for a proof. An older version of [14] also contained this lemma and a proof.

<sup>\*28</sup>A real is hereditarily countable, if  $\mathbb{R}$  is introduced in a standard manner.

In (C) above, we can see more clearly the feature of MA as an absoluteness assertion. Moreover, this characterization tells also that MA is a natural extension of (a part of) the Lévy-Shoenfield Absoluteness Lemma.

Similarly,  $MA_{\aleph_1}$  is shown to be equivalent to the following assertion:

(C') For a c.c.c. forcing notion  $\mathbb{P}$ , for any  $a$  hereditarily of cardinality strictly less than  $\aleph_2$  and any  $\Sigma_1$ -formula  $\varphi$ ,  $\varphi(a)$  is equivalent to  $\Vdash_{\mathbb{P}} \varphi(a)$ .

## 5 Strengthenings of Martin's Axiom

By replacing the condition “c.c.c.” in the assertion (A') introduced after Theorem 9 by weaker properties, we obtain several other important axioms. Among the properties of forcing notions considered in this context are the properness and the stationary preserving. We do not give the precise definition of these notions here, since this would require some additional preparations<sup>\*29</sup>. Instead we will content ourselves with noting that the forcing notions with these properties preserve  $\aleph_1$  (see the footnote \*23) and, for any forcing notion  $\mathbb{P}$ , we have

(\*)  $\mathbb{P}$  has the c.c.c.  $\Rightarrow$   $\mathbb{P}$  is proper  $\Rightarrow$   $\mathbb{P}$  is stationary preserving.

The assertions obtained by replacing “c.c.c.” in (A') by “proper” and “stationary preserving” are called Proper Forcing Axiom (PFA) and Martin's Maximum (MM) respectively. The “Maximum” in the naming of MM indicates that this axiom is the strongest among a certain class of axioms of similar form for which inconsistency cannot be proved immediately. By (\*) above, we have<sup>\*30</sup>  $MM \Rightarrow PFA \Rightarrow MA_{\aleph_1}$ . In contrast to Martin's Axiom, the consistency of PFA or MM cannot be proved from ZFC alone. Starting from ZFC + an axiom guaranteeing the existence of a fairly large cardinal we can show the consistency of MM, and hence also PFA([7]<sup>\*31</sup>). Under MM or PFA, we can prove a number of interesting results otherwise unprovable. Worth noticing is the fact that these axioms decide the size of the continuum (see, e.g., [5], [7], [14]). In particular, PFA implies that the size

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<sup>\*29</sup>For the definition of these notions and their basic properties, see e.g. [14].

<sup>\*30</sup>Actually, we have  $PFA \Rightarrow MA$  since PFA implies  $2^{\aleph_0} = \aleph_2$  ([29]).

<sup>\*31</sup>More exactly, in ZFC with the axiom asserting the existence of a supercompact cardinal we can construct a generic extension satisfying PFA or MM.



of the continuum is equal to  $\aleph_2$  (i.e.  $2^{\aleph_0} = \aleph_2$ ). In contrast, MA imposes almost no restriction on the size of the continuum except that it is regular and that most of the known cardinal invariants of reals are made to be equal to it.

The assertion (B') after Theorem 10 with the c.c.c. there replaced with proper (stationary preserving respectively) gives a characterization of PFA (MM respectively). However it is known that (C') after Theorem 12 with corresponding replacements gives axioms strictly weaker than PFA and MM. These axioms are called the Bounded Proper Forcing Axiom (BPFA) and the Bounded Martin's Maximum (BMM) respectively. In particular, it is known that BPFA is equiconsistent with the existence of a certain type of large cardinal which is fairly small compared with the known lower bound of the consistency strength of PFA<sup>\*32</sup> ([12]).

BPFA and BMM are thus axioms with natural characterization parallel to (C') and, in the case of BPFA, with consistency strength exactly known and tame enough. In this sense, these axioms can be seen as good candidates for "true" axioms.

S. Todorćević proved that BMM implies  $2^{\aleph_0} = \aleph_2$  [28]. T. Miyamoto then proved that the Bounded Semi-Proper Forcing Axiom (BSPFA), which lies between BPFA and BMM, together with the existence of a measurable cardinal implies the same cardinal equation [24]. Todorćević's and Miyamoto's proofs not only show the equation  $2^{\aleph_0} = \aleph_2$  but they actually show that very strong combinatorial principles called  $\theta_{AC}$  and  $\theta_{AC}^*$  can be derived from BMM and BSPFA + measurable cardinal respectively and that the equation  $2^{\aleph_0} = \aleph_2$  follows from these principles.

Interestingly, Miyamoto's result recalls Gödel's "prophecy" that large cardinals may have some contribution to the solution of the Continuum Problem ([11]).

More recently, Justin Moore proved that the equation  $2^{\aleph_0} = \aleph_2$  already follows from BPFA without any further assumptions ([25]). This improves the result by Miyamoto and Todorćević considerably concerning this cardinal equation. We should note, however, that this does not mean that the results by Todorćević and Miyamoto are completely covered by Moore's because of the principles  $\theta_{AC}$  and  $\theta_{AC}^*$  mentioned above.

As discussed above, BPFA is a quite natural axiom and thus we can interpret Moore's result as a strong circumstantial evidence suggesting that the size of

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<sup>\*32</sup>For the notion of consistency strength, see e.g. [16] or [21].

continuum should be  $\aleph_2$ .

J.R. Steel proved that PFA implies Projective Determinacy (and more see [27]) which is a very strong principle implying that almost sets of reals appearing in everyday mathematics are free from set-theoretic pathologies. So, in particular, such sets are all Lebesgue measurable, have the Baire property, etc. under PFA. Some other deep results in connection with forcing axioms are to be found in H. Woodin [30], [31].

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