Fleissner の Axiom R について

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Let λ be a cardinal.

AR(λ): For any sationary $S \subseteq [\lambda]^{\aleph_0}$ and ω_1-club $T \subseteq [\lambda]^{\aleph_1}$, there is $U \in T$ such that $S \cap [U]^{\aleph_0}$ is stationary in $[U]^{\aleph_0}$.

$U \subseteq [X]^{\aleph_1}$ is said to be ω_1-club if

(1) U is cofinal in $[X]^{\aleph_1}$; and

(2) $\bigcup_{\alpha<\omega_1} u_\alpha \in U$ for any increasing chain $\langle u_\alpha : \alpha < \omega_1 \rangle$ of elements of U with respect to $\subseteq$ of length ω_1.

Definition 1. (W.G. Fleissner)
Axiom R $\Leftrightarrow$ AR(λ) holds for all cardinal $\lambda > \aleph_1$

Remark 1. For cardinals $\aleph_1 < \lambda \leq \lambda'$, AR($\lambda'$) implies AR($\lambda$). ❑

Theorem 2. (W.G. Fleissner) Assume Axiom R. For any topological space X with $X \models$ (i), (ii), (iii) where

(i) T_1, (ii) with a point countable base,
(iii) not left separated,

there is a subspace $Y \subseteq X$ with $|Y| = \aleph_1$ such that $Y \models$ (i), (ii), (iii).

A base $B \subseteq \mathcal{O}$ of a topological space X is point countable if $\{O \in B : x \in O\}$ for each $x \in X$ is countable; A topological space X is left separated if there is an 1-1 mapping $f : X \to \mathrm{On}$ such that $f^{-1}{}''\alpha$ is closed for any $\alpha \in \mathrm{On}$.

A sketch of proof: Assume that there is a topological space $X \models$ (i), (ii), (iii) such that, for every $Y \in [X]^{\aleph_1}$, Y is left separated.
Then $S = \{a \in [X]^{\aleph_0} : a \text{ is not closed}\}$ is stationary. For $Y \in [X]^{\aleph_1}$, we have $|\overline{Y}| = \aleph_1$. Hence $T = \{Y \in [X]^{\aleph_1} : Y \text{ is closed}\}$ is ω_1-club (ω_1-closedness follows from $X \models$(ii)). Thus by Axiom R, there is a closed $Y \in [X]^{\aleph_1}$ such that $\{a \in [Y]^{\aleph_0} : a \text{ is not closed}\}$ is stationary. But from this it follows that Y is not left separated. A constradiction. ❑

Theorem 3. (S.F. and Qi Feng) Assume that Axiom R holds. Then a Boolean algebra B is $\aleph_2$-projectively filtered iff B has the FNP.

A Boolean algebra B is $\aleph_2$-projectively filtered if there is a family $(B_i)_{i\in I}$ of subalgebras of B such that

(3) $I = (I, \leq_I)$ is upward directed partial ordering;

(4) for $i, j \in I$ with $i \leq_I j$ we have always $B_i \leq B_j$;

(5) if $S \subseteq I$ is an increasing sequence with respect to $\leq_I$ with $|\,S\,| < \aleph_2$ then there exists $i^* \in I$ with $i^* = \sup S$;

(6) if $S \subseteq I$, $i^* \in I$ and $i^* = \sup S$, then $B_{i^*} = \bigcup_{i\in S} B_i$;

(7) for every $i \in I$, B_i, is projective;

(8) $B = \bigcup_{i\in I} B_i$.

A Boolean algebra B has the FNP if there is a mapping $f : B \longrightarrow [B]^{<\aleph_0}$ such that, for any a, $b \in B$ with $a \leq_B b$ there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$.

Remark 4. Under the existence of a non-reflecting stationary set $\subseteq E^{\lambda}_{\omega}$ for some $\lambda > \aleph_1$, the assertions of both of Theorem 2. and Theorem 3. are refuted.

Note that, for $\lambda = \mu^{+}$, the existence of a non-reflecting stationary set as above follows from $\square_{\mu}$.

Let λ be a cardinal.

SRP(λ): For any projectively stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ there is a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $M_\alpha \prec \mathcal{H}(\lambda)$, $|M_\alpha| = \aleph_0$ and $M_\alpha \in S$ for all $\alpha < \omega_1$.

For any $X \supseteq \omega_1$ $S \subseteq [X]^{\aleph_0}$ is projectively stationary if, for any stationary $T \subseteq \omega_1$, $\{x \in S : x \cap \omega_1 \in T\}$ is a stationary subset of $[X]^{\aleph_0}$.

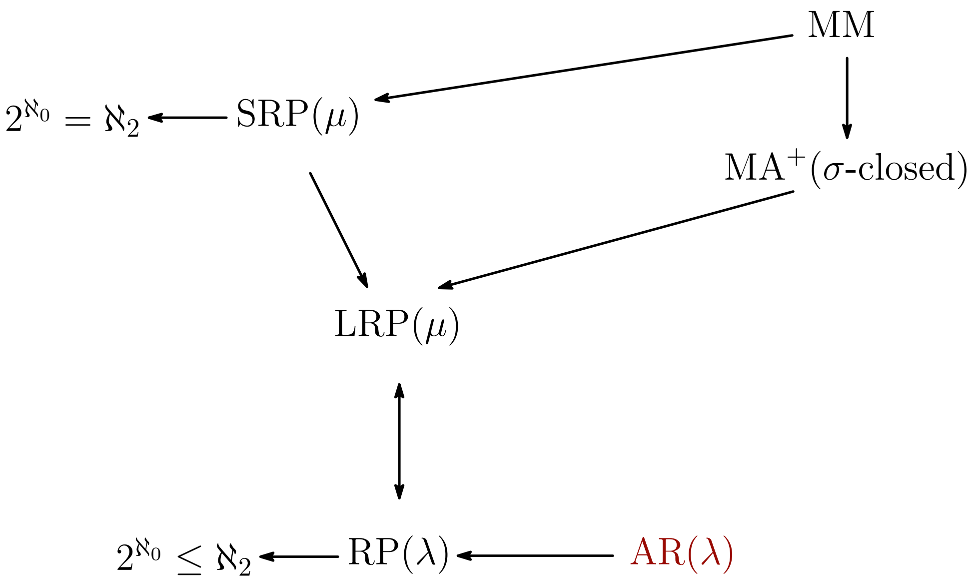
LRP(λ): For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$, there is a continuously increasing chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of $\mathcal{H}(\lambda)$ such that $\{\alpha < \omega_1 : M_\alpha \in S\}$ is stationary.

RP(λ): For any stationary $S \subseteq [\lambda]^{\aleph_0}$ and $X \in [\lambda]^{\aleph_1}$, there is $Y \in [\lambda]^{\aleph_1}$ with $X \subseteq Y$ such that $S \cap [Y]^{\aleph_0}$ is stationary in $[Y]^{\aleph_0}$.

It is immediate from the definition that

Lemma 5. AR(λ) implies RP(λ).

$\lambda = |\,\mathcal{H}(\mu)\,|$, $\lambda > \aleph_1$



Definition 2. SRP $\Leftrightarrow$ SRP(λ) for all $\lambda > \aleph_1$

Definition 3. RP $\Leftrightarrow$ RP(λ) for all $\lambda > \aleph_1$

.... SRP($\aleph_2$) implies that NS_{ω_1} is $\aleph_2$-saturated (S. Todorčević).

.... SRP($\aleph_2$) implise $2^{\aleph_0} = \aleph_2$ (S. Todorčević).

.... SRP implies $L(\mathbb{R}) \models \mathrm{AD}$ (J.R. Steel and A.S. Zoble)

— RP implies that every stationary preserving p.o. is semistationary (S. Shelah + H. Sakai).

— Chang's conj. follows from the conclusion of the fact above(S. Shelah).

— RP implies $2^{\aleph_0} \leq \aleph_2$. RP (actually also AR) is consistent with CH.

Problem 1 Is $\mathrm{AR}(\lambda)$ strictly stronger than $\mathrm{RP}(\lambda)$?

Lemma 6. For $\lambda > \omega_1$ the following are equivalent:

(a) $\mathrm{RP}(\lambda)$.

(b) For every stationary $S \subseteq [\lambda]^{\aleph_0}$, there is $X \in [\lambda]^{\aleph_1}$ such that $\omega_1 \subseteq X$ and that $S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}$.

(c) For every stationary $S \subseteq [\lambda]^{\aleph_0}$, $\{X \in [\lambda]^{\aleph_1} : S \cap [X]^{\aleph_0}$ is stationary in $[X]^{\aleph_0}\}$ is stationary in $[\lambda]^{\aleph_1}$.

Proposition 7. (B. König) $\mathrm{AR}(\omega_2)$ is equivalent to the following assertion:

(9) For every stationary $S \subseteq [\omega_2]^{\aleph_0}$, there is $\alpha < \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$ such that $S \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$.

Proposition 7. (B. König) $\mathrm{AR}(\omega_2)$ is equivalent to the following assertion:

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Proof. $\mathrm{AR}(\omega_2) \Rightarrow$ (9): Easy to see by letting $T = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$.

(9) $\Rightarrow \mathrm{AR}(\omega_2)$: Assume that (9) holds. Suppose that $S \subseteq [\omega_2]^{\aleph_0}$ is stationary and $T \subseteq [\omega_2]^{\aleph_1}$ is ω_1-club.

Let $\tilde{T} \subseteq \omega_2$ be the closure of $T \cap \omega_2$. Since T is ω_1-club, we have

(10) $\tilde{T} = (T \cap \omega_2) \cup \{\alpha < \omega_2 : \alpha \text{ is an } \omega \text{ limit of elements of } T \cap \omega_2\}$.

Let $f : [\omega_2]^{<\omega} \to \omega_2$ be such that $C(f) = \{\alpha < \omega_2 : f''[\alpha]^{<\omega} \subseteq \alpha\} \subseteq \tilde{T}$ and let $S' = \{x \in [\omega_2]^{\aleph_0} : x \in S,\ f''[x]^{<\omega} \subseteq x\}$. Then $S' \subseteq S$ and S' is still stationary. Hence, by (9), there is $\alpha < \omega_2$ such that $cf(\alpha) = \omega_1$ and $S' \cap [\alpha]^{\aleph_0}$ is stationary in $[\alpha]^{\aleph_0}$. Since $S' \cap [\alpha]^{\aleph_0}$ is cofinal in $[\alpha]^{\aleph_0}$, $f''[\alpha]^{<\aleph_0} \subseteq \alpha$ and hence $\alpha \in \tilde{T}$. By $\mathrm{cf}(\alpha) = \omega_1$ and (10), it follows that $\alpha \in T$. ❑ **(Proposition 7.)**

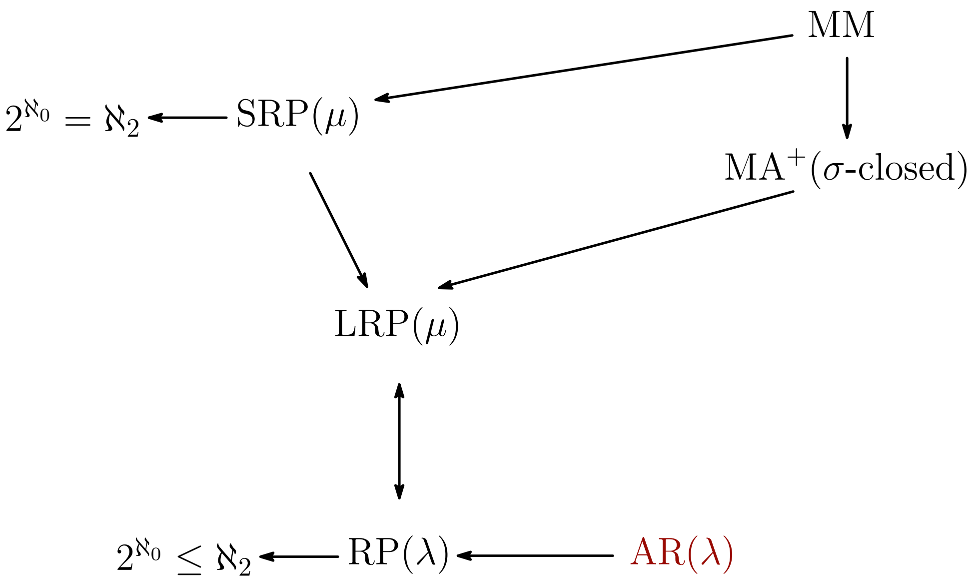
For a cardinal λ

$\mathrm{RP}^+(\lambda)$: For any stationary $S \subseteq [\lambda]^{\aleph_0}$ and any regular $\theta \geq \lambda$, if $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}(\theta), \in \rangle$, then there is an $(\mathcal{M}, \omega_1)$-w.i.a. chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $\{\alpha < \omega_1 : \lambda \cap M_\alpha \in S\}$ is stationary.

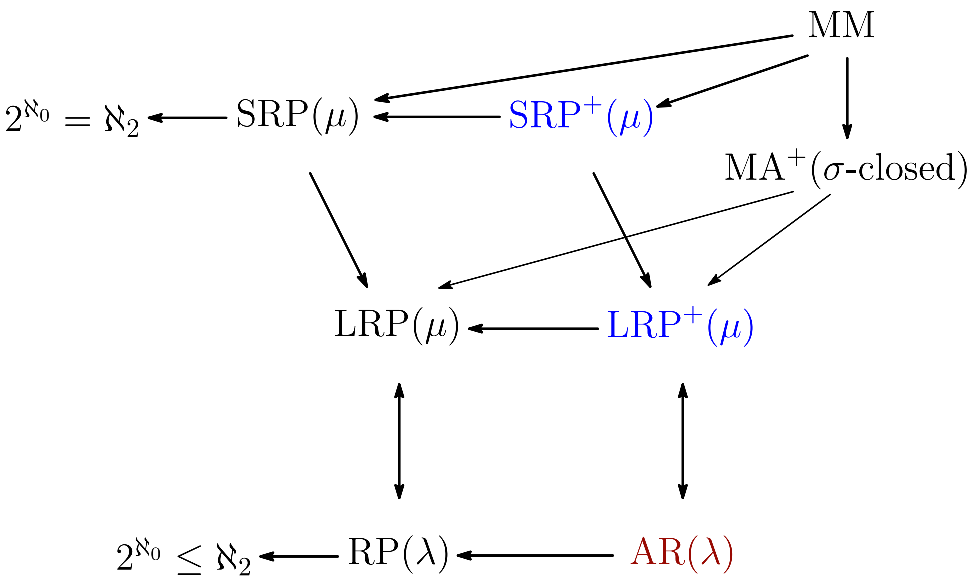
Suppose that κ and λ are cardinals and $\mathcal{M}$ is a countable expansion of $\langle \mathcal{H}(\lambda), \in \rangle$. $M \subseteq \mathcal{H}(\lambda)$ is $(\mathcal{M}, \kappa)$-weakly internally approachable (or $(\mathcal{M}, \kappa)$-w.i.a. for short) if M is the union of a continuously increasing chain $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $\mathcal{M}$ of length κ such that $|\, M_\alpha \,| < \kappa$ and $M_\alpha \in M_{\alpha+1}$ for all successor ordinal $\alpha < \kappa$.

Proposition 8. (1) $\mathrm{RP}^+(\lambda)$ implies $\mathrm{AR}(\lambda)$.
(2) If λ is such that there is a cofinal subset of $[\lambda]^{\aleph_0}$ of cardinality λ, then $\mathrm{RP}^+(\lambda)$ and $\mathrm{AR}(\lambda)$ are equivalent.

$\lambda = |\,\mathcal{H}(\mu)\,|$, $\mu > \aleph_1$



$\lambda = |\,\mathcal{H}(\mu)\,|$, $\mu > \aleph_1$



LRP$^+(\lambda)$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathcal{M}$ of $\mathcal{H}(\lambda)$, there is an $(\mathcal{M}, \omega_1)$-w.i.a. chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $\{\alpha < \omega_1 : M_\alpha \in S\}$ is stationary.

SRP$^+(\lambda)$: For any projectively stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ and any countable expansion $\mathcal{M}$ of $\mathcal{H}(\lambda)$, there is an $(\mathcal{M}, \omega_1)$-w.i.a. chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ such that $M_\alpha \in S$ for all $\alpha < \omega_1$.

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